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Analytical solutions using a higher-order refined theory for the static analysis of antisymmetric angle-ply composite and sandwich plates

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Abstract

Analytical formulations and solutions to the static analysis of simply supported anti-symmetric angle-ply composite and sandwich plates hitherto not reported in the literature based on a higher-order refined theory already reported in the literature are presented. The theoretical model presented herein incorporates laminate deformations, which account for the effect of transverse shear deformation and a non-linear variation of in-plane displacements with respect to the thickness coordinate. The transverse displacement is assumed to be constant throughout the thickness. The equations of equilibrium are obtained using principle of minimum potential energy. Solutions are obtained in closed form using Navier's technique by solving the boundary value problem. Accuracy of the theoretical formulations and the solution method is first ascertained by comparing the results with that already reported in the literature. After establishing the accuracy of the solutions, numerical results with real properties are presented for the multilayer antisymmetric angle-ply composite and sandwich plates, which will serve as a benchmark for future investigations.

Keywords: Static analysis; Higher-order theory; Shear deformation; Angle-ply plates; Analytical solutions

1. Introduction

Structural elements made up of fibre reinforced composite material are being increasingly used in the aeronautical and aerospace industries as well as in other fields of modern technology, primarily due to their high strength-to-weight and stiffness-to-weight ratios and also due to their anisotropic material properties that can be tailored through variation of the fibre orientation and stacking sequence. During the last few decades, numerous investigators have used a variety of methods for the analysis of laminated composite and sandwich plates. These include analytical and numerical methods. Review of literature with many citations up to the year 1989 can be found in the articles by Soni and Pagano [1], Herakovich [2] and Pagano [3]. A complete review of various shear deformation theories for the analysis of multilayer composite plates and shells is available in the review articles by Noor and Burton [4,5] and Noor *et al.* [6]. A selective review of the various analytical and numerical methods used for the stress analysis of laminated composite and sandwich plates was presented by Kant and Swaminathan [7]. Analytical formulations, solutions and comparison of numerical results for the buckling, free vibration, stress analyses of cross ply composite and sandwich plates based on the higher-order refined theories already reported in the literature by Kant [8], Pandya and Kant [9–13] and Kant and Manjunatha [14] were presented recently by Kant and Swaminathan [15–18]. In this paper, analytical formulations developed and solutions obtained for the first time is presented for the static analysis of antisymmetric angle-ply laminated composite and sandwich plates using a higher-order refined theory already reported in the literature. Correctness of the solutions is first established and then benchmark results with real properties are presented for the antisymmetric angle-ply composite and sandwich plates.

2. Theoretical formulation

2.1. Displacement model

In order to approximate the three-dimensional elasticity problem to a two-dimensional plate problem, the

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displacement components $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ at any point in the plate space are expanded in Taylor's series in terms of the thickness coordinate. The elasticity solution indicates that the transverse shear stresses vary parabolically through the plate thickness. This requires the use of a displacement field in which the in-plane displacements are expanded as cubic functions of the thickness coordinate. The variation of transverse displacement component $w(x,y,z)$ is assumed constant through the plate thickness and thus setting $\varepsilon_z = 0$, then the displacement field may be expressed as [13]

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) + z^2u_0^*(x,y) + z^3\theta_x^*(x,y) \\ v(x,y,z) &= v_0(x,y) + z\theta_y(x,y) + z^2v_0^*(x,y) + z^3\theta_y^*(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{1}$$

The parameters u_0, v_0 are the in-plane displacements and w_0 is the transverse displacement of a point (x,y) on the middle plane. The functions θ_x, θ_y are rotations of the normal to the middle plane about y- and x-axes respectively. u_0^*, v_0^*, θ_x^*, θ_y^* are the higher-order terms in the Taylor's series expansion and they represent higher-order transverse cross sectional deformation modes.

In this paper the analytical formulations and solution method followed using the higher-order refined theory given by Eq. (1) is presented in detail. The geometry of a two-dimensional laminated composite and sandwich plates with positive set of coordinate axes and the physical middle plane displacement terms are shown in Figs. 1 and 2 respectively. By substitution of the displacement relations given by Eq. (1) into the strain-displacement equations of the classical theory of elasticity, the following relations are obtained:

$$\begin{aligned} \varepsilon_x &= \varepsilon_{x0} + z\kappa_x + z^2\varepsilon_{x0}^* + z^3\kappa_x^* \\ \varepsilon_y &= \varepsilon_{y0} + z\kappa_y + z^2\varepsilon_{y0}^* + z^3\kappa_y^* \\ \gamma_{xy} &= \varepsilon_{xy0} + z\kappa_{xy} + z^2\varepsilon_{xy0}^* + z^3\kappa_{xy}^* \\ \gamma_{yz} &= \phi_y + z\kappa_{yz} + z^2\phi_y^* \\ \gamma_{xz} &= \phi_x + z\kappa_{xz} + z^2\phi_x^* \end{aligned} \tag{2}$$

where

$$\begin{aligned} (\varepsilon_{x0}, \varepsilon_{y0}, \varepsilon_{xy0}) &= \left(\frac{\partial u_0}{\partial x}, \frac{\partial v_0}{\partial y}, \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) \\ (\varepsilon_{x0}^*, \varepsilon_{y0}^*, \varepsilon_{xy0}^*) &= \left(\frac{\partial u_0^*}{\partial x}, \frac{\partial v_0^*}{\partial y}, \frac{\partial u_0^*}{\partial y} + \frac{\partial v_0^*}{\partial x}\right) \\ (\kappa_x, \kappa_y, \kappa_{xy}) &= \left(\frac{\partial \theta_x}{\partial x}, \frac{\partial \theta_y}{\partial y}, \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \\ (\kappa_x^*, \kappa_y^*, \kappa_{xy}^*) &= \left(\frac{\partial \theta_x^*}{\partial x}, \frac{\partial \theta_y^*}{\partial y}, \frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}\right) \\ (\kappa_{xz}, \kappa_{yz}) &= (2u_0^*, 2v_0^*) \\ (\phi_x, \phi_x^*, \phi_y, \phi_y^*) &= \left(\theta_x + \frac{\partial w_0}{\partial x}, 3\theta_x^*, \theta_y + \frac{\partial w_0}{\partial y}, 3\theta_y^*\right) \end{aligned} \tag{3}$$

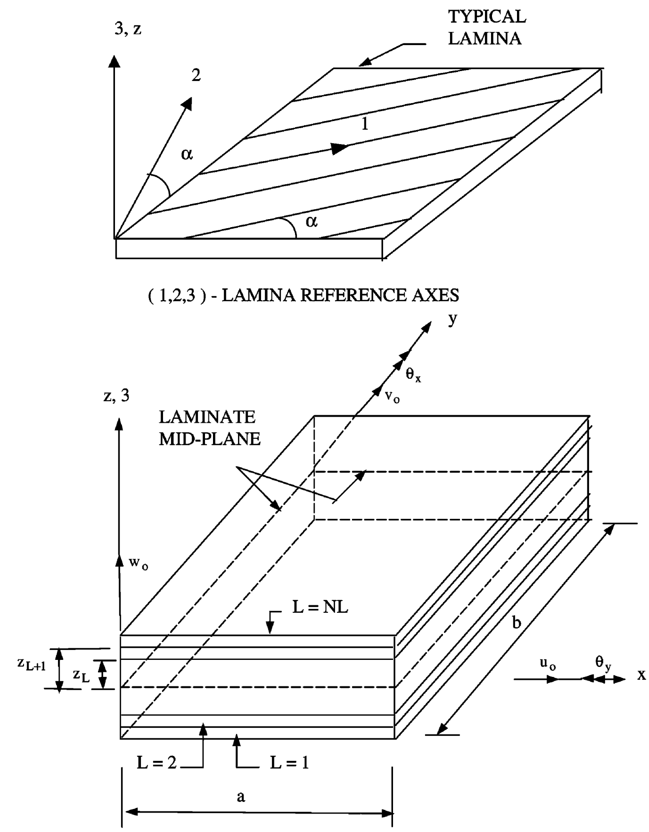

Fig. 1. Laminate geometry with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

2.2. Constitutive equations

Each lamina in the laminate is assumed to be in a three-dimensional stress state so that the constitutive relation neglecting the normal stress σ_3, normal strain ε_3 and retaining a 6×6 matrix for a typical lamina L with reference to the fibre-matrix coordinate axes (1–2–3) can be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{Bmatrix}^L = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 0 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}^L \tag{4}$$

where (σ_1, σ_2, τ_{12}, τ_{23}, τ_{13}) are the stresses and (ε_1, ε_2, γ_{12}, γ_{23}, γ_{13}) are the linear strain components referred to the lamina coordinates (1–2–3) and the C_{ij}s are the elastic constants or the elements of stiffness matrix of the Lth lamina with reference to the fibre axes (1–2–3). In the laminate coordinates (x,y,z) the stress–strain relations for the Lth lamina can be written as

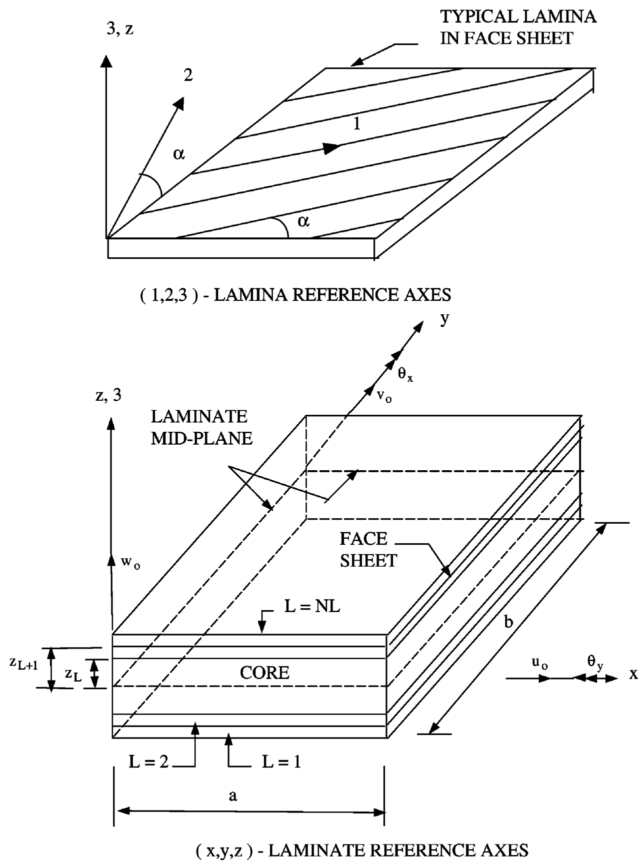

Fig. 2. Geometry of a sandwich plate with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ 0 \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^L = \begin{bmatrix} Q_{11} & Q_{12} & 0 & Q_{14} & 0 & 0 \\ & Q_{22} & 0 & Q_{24} & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & Q_{44} & 0 & 0 \\ & \text{symmetric} & & & Q_{55} & Q_{56} \\ & & & & & Q_{66} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0 \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^L \tag{5}$$

where (σ_x, σ_y, τ_{xy}, τ_{yz}, τ_{xz}) are the stresses and (ε_x, ε_y, γ_{xy}, γ_{yz}, γ_{xz}) are the strains with respect to the laminate axes. Q_{ij}s are the transformed elastic constants or stiffness matrix with respect to the laminate axes x, y, z. The elements of matrices $[C]$ and $[Q]$ are defined in Appendices A and B.

2.3. Governing equations of equilibrium

The equations of equilibrium for the stress analysis are obtained using the principle of minimum potential energy (PMPE). In analytical form it can be written as follows [19]:

$$\delta(U + V) = 0 \tag{6}$$

where U is the total strain energy due to deformations, V is the potential of the external loads, and $U + V = \Pi$ is the total potential energy and δ denotes the variational symbol. Substituting the appropriate energy expression in the above equation, the final expression can thus be written as

$$\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_A (\sigma_x\, \delta\varepsilon_x + \sigma_y\, \delta\varepsilon_y + \tau_{xy}\, \delta\gamma_{xy} + \tau_{yz}\, \delta\gamma_{yz} + \tau_{xz}\, \delta\gamma_{xz})\, dA\, dz - \int_A p_z^+\, \delta w^+\, dA \right] = 0 \tag{7}$$

where $w^+ = w_0$ is the transverse displacement of any point on the top surface of the plate and p_z^+ is the transverse load applied at the top surface of the plate. Using Eqs. (1)–(3) in Eq. (7) and integrating the resulting expression by parts, and collecting the coefficients of δu_0, δv_0, δw_0, $\delta\theta_x$, $\delta\theta_y$, δu_0^*, δv_0^*, $\delta\theta_x^*$, $\delta\theta_y^*$ the following equations of equilibrium are obtained:

$$\begin{aligned}
\delta u_0 &: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\
\delta v_0 &: \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \\
\delta w_0 &: \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z^+ = 0 \\
\delta\theta_x &: \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \\
\delta\theta_y &: \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \\
\delta u_0^* &: \frac{\partial N_x^*}{\partial x} + \frac{\partial N_{xy}^*}{\partial y} - 2S_x = 0 \\
\delta v_0^* &: \frac{\partial N_y^*}{\partial y} + \frac{\partial N_{xy}^*}{\partial x} - 2S_y = 0 \\
\delta\theta_x^* &: \frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3Q_x^* = 0 \\
\delta\theta_y^* &: \frac{\partial M_y^*}{\partial y} + \frac{\partial M_{xy}^*}{\partial x} - 3Q_y^* = 0
\end{aligned} \tag{8}$$

and the boundary conditions are of the form.

On the edge $x =$ constant

$$\begin{aligned}
&u_0 = \bar{u}_0 \text{ or } N_x = \bar{N}_x, \quad v_0 = \bar{v}_0 \text{ or } N_{xy} = \bar{N}_{xy}, \\
&w_0 = \bar{w}_0 \text{ or } Q_x = \bar{Q}_x, \quad u_0^* = \bar{u}_0^* \text{ or } N_x^* = \bar{N}_x^*, \\
&v_0^* = \bar{v}_0^* \text{ or } N_{xy}^* = \bar{N}_{xy}^*, \quad \theta_x = \bar{\theta}_x \text{ or } M_x = \bar{M}_x, \\
&\theta_y = \bar{\theta}_y \text{ or } M_{xy} = \bar{M}_{xy}, \quad \theta_x^* = \bar{\theta}_x^* \text{ or } M_x^* = \bar{M}_x^*, \\
&\theta_y^* = \bar{\theta}_y^* \text{ or } M_{xy}^* = \bar{M}_{xy}^*
\end{aligned} \tag{9}$$

On the edge $y = $ constant

$$u_0 = \bar{u}_0 \text{ or } N_{xy} = \bar{N}_{xy}, \quad v_0 = \bar{v}_0 \text{ or } N_y = \bar{N}_y,$$
$$w_0 = \bar{w}_0 \text{ or } Q_y = \bar{Q}_y, \quad u_0^* = \bar{u}_0^* \text{ or } N_{xy}^* = \bar{N}_{xy}^*,$$
$$v_0^* = \bar{v}_0^* \text{ or } N_y^* = \bar{N}_y^*, \quad \theta_x = \bar{\theta}_x \text{ or } M_{xy} = \bar{M}_{xy},$$
$$\theta_y = \bar{\theta}_y \text{ or } M_y = \bar{M}_y, \quad \theta_x^* = \bar{\theta}_x^* \text{ or } M_{xy}^* = \bar{M}_{xy}^*,$$
$$\theta_y^* = \bar{\theta}_y^* \text{ or } M_y^* = \bar{M}_y^* \tag{10}$$

where the stress resultants are defined by

$$\begin{bmatrix} M_x & M_x^* \\ M_y & M_y^* \\ M_{xy} & M_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{ZL}^{Z_{L+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} [z \quad z^3]\, dz \tag{11}$$

$$\begin{bmatrix} Q_x & Q_x^* \\ Q_y & Q_y^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{ZL}^{Z_{L+1}} \begin{Bmatrix} \tau_{xy} \\ \tau_{yz} \end{Bmatrix} [1 \quad z^2]\, dz \tag{12}$$

$$\begin{bmatrix} N_x & N_x^* \\ N_y & N_y^* \\ N_{xy} & N_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{ZL}^{Z_{L+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} [1 \quad z^2]\, dz \tag{13}$$

$$\begin{bmatrix} S_x \\ S_y \end{bmatrix} = \sum_{L=1}^{NL} \int_{ZL}^{Z_{L+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} [z \quad z^3]\, dz \tag{14}$$

the resultants in Eqs. (11)–(14) can be related to the total strains in Eq. (2) by the following equations:

$$\begin{Bmatrix} N_x \\ N_y \\ N_x^* \\ N_y^* \\ M_x \\ M_y \\ M_x^* \\ M_y^* \end{Bmatrix} = [A] \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0^*}{\partial x} \\ \frac{\partial v_0^*}{\partial y} \\ \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x^*}{\partial x} \\ \frac{\partial \theta_y^*}{\partial y} \end{Bmatrix} + [A'] \begin{Bmatrix} \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} \\ \frac{\partial u_0^*}{\partial y} \\ \frac{\partial v_0^*}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x^*}{\partial y} \\ \frac{\partial \theta_y^*}{\partial x} \end{Bmatrix},$$

$$\begin{Bmatrix} N_{xy} \\ N_{xy}^* \\ M_{xy} \\ M_{xy}^* \end{Bmatrix} = [B'] \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0^*}{\partial x} \\ \frac{\partial v_0^*}{\partial y} \\ \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x^*}{\partial x} \\ \frac{\partial \theta_y^*}{\partial y} \end{Bmatrix} + [B] \begin{Bmatrix} \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} \\ \frac{\partial u_0^*}{\partial y} \\ \frac{\partial v_0^*}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x^*}{\partial y} \\ \frac{\partial \theta_y^*}{\partial x} \end{Bmatrix} \tag{15}$$

$$\begin{Bmatrix} Q_x \\ Q_x^* \\ S_x \end{Bmatrix} = [D] \begin{Bmatrix} \theta_x \\ \frac{\partial w_0}{\partial x} \\ \theta_x^* \\ u_0^* \end{Bmatrix} + [D'] \begin{Bmatrix} \theta_y \\ \frac{\partial w_0}{\partial y} \\ \theta_y^* \\ v_0^* \end{Bmatrix},$$

$$\begin{Bmatrix} Q_y \\ Q_y^* \\ Sy \end{Bmatrix} = [E'] \begin{Bmatrix} \theta_x \\ \frac{\partial w_0}{\partial x} \\ \theta_x^* \\ u_0^* \end{Bmatrix} + [E] \begin{Bmatrix} \theta_y \\ \frac{\partial w_0}{\partial y} \\ \theta_y^* \\ v_0^* \end{Bmatrix} \tag{16}$$

where the matrices $[A]$, $[A']$, $[B]$, $[B']$, $[D]$, $[D']$, $[E]$, $[E']$ are the matrices of plate stiffnesses whose elements are defined in Appendix C.

3. Analytical solutions

Here the exact solutions of Eqs. (8)–(16) for anti-symmetric angle-ply plates are considered. Assuming that the plate is simply supported with SS-2 boundary conditions [22] in such a manner that tangential displacement is admissible, but the normal displacement is not, the following boundary conditions are appropriate:

At edges $x = 0$ and $x = a$:

$$u_0 = 0, \quad w_0 = 0, \quad \theta_y = 0, \quad M_x = 0, \quad u_0^* = 0, \quad \theta_y^* = 0,$$
$$M_x^* = 0, \quad N_{xy} = 0, \quad N_{xy}^* = 0 \tag{17}$$

At edges $y = 0$ and $y = b$:

$$v_0 = 0, \quad w_0 = 0, \quad \theta_x = 0, \quad M_y = 0, \quad v_0^* = 0, \quad \theta_x^* = 0,$$
$$M_y^* = 0, \quad N_{xy} = 0, \quad N_{xy}^* = 0 \tag{18}$$

following Navier's solution procedure [20–22], the solution to the displacement variables satisfying the above boundary conditions can be expressed in the following forms:

$$u_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{0_{mn}} \sin \alpha x \cos \beta y$$

$$v_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{0_{mn}} \cos \alpha x \sin \beta y$$

$$w_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{0_{mn}} \sin \alpha x \sin \beta y$$

$$\theta_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{x_{mn}} \cos \alpha x \sin \beta y$$

$$\theta_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{y_{mn}} \sin \alpha x \cos \beta y$$

$$u_0^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{0_{mn}}^* \sin \alpha x \cos \beta y$$

$$v_0^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{0_{mn}}^* \cos \alpha x \sin \beta y$$

$$\theta_x^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{x_{mn}}^* \cos \alpha x \sin \beta y$$

$$\theta_y^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{y_{mn}}^* \sin \alpha x \cos \beta y$$

and the loading term is expanded as

$$p_z^+ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{z_{mn}} \sin \alpha x \cos \beta y \tag{19}$$

where $\alpha = \frac{m\pi}{a}$ and $\beta = \frac{n\pi}{b}$

Substituting Eqs. (17)–(19) into Eq. (8) and collecting the coefficients one obtains

$$[X]_{9\times 9}\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_x \\ \theta_y \\ u_0^* \\ v_0^* \\ \theta_x^* \\ \theta_y^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ p_z^+ \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{9\times 1} \tag{20}$$

for any fixed values of m and n. The elements of coefficient matrix $[X]$ are given in Appendix D.

4. Numerical results and discussion

In this section, various numerical examples solved are described and discussed for establishing the accuracy of the theory for the stress analysis of antisymmetric angle-ply laminated composite and sandwich plates. For all the problems a simply supported plate with SS-2 boundary conditions is considered for the analysis. The transverse loading considered is sinusoidal. Results are obtained in closed-form using *Navier's* solution technique for the above geometry and loading and the accuracy of the solution is established by comparing the results with the solutions wherever available in the literature.

The following sets of data are used in obtaining numerical results:

Material 1 [23]

$E_1 = 40 \times 10^6$ psi (276 GPa),

$E_2 = E_3 = 1 \times 10^6$ psi (6.895 GPa),

$G_{12} = G_{13} = 0.5 \times 10^6$ psi (3.45 GPa),

$G_{23} = 0.6 \times 10^6$ psi (4.12 GPa),

$\upsilon_{12} = \upsilon_{23} = \upsilon_{13} = 0.25$

Material 2
Glass epoxy

$E_1 = 5.6 \times 10^6$ psi (38.61 GPa),

$E_2 = 1.2 \times 10^6$ psi (8.27 GPa),

$E_3 = 1.3 \times 10^6$ psi (8.96 GPa),

$G_{12} = 0.60 \times 10^6$ psi (4.14 GPa),

$G_{13} = 0.60 \times 10^6$ psi (4.14 GPa),

$G_{23} = 0.50 \times 10^6$ psi (3.45 GPa),

$\upsilon_{12} = 0.26, \quad \upsilon_{13} = 0.26, \quad \upsilon_{23} = 0.34$

Material 3
Face sheets (*graphite epoxy T300/934*)

$E_1 = 19 \times 10^6$ psi (131 GPa),

$E_2 = 1.5 \times 10^6$ psi (10.34 GPa), $\quad E_2 = E_3$,

$G_{12} = 1 \times 10^6$ psi (6.895 GPa),

$G_{13} = 0.90 \times 10^6$ psi (6.205 GPa),

$G_{23} = 1 \times 10^6$ psi (6.895 GPa), $\quad \upsilon_{12} = 0.22$,

$\upsilon_{13} = 0.22, \quad \upsilon_{23} = 0.49$

Core (*isotropic*)

$E_1 = E_2 = E_3 = 2G = 1000$ psi (6.90×10^{-3} GPa),

$G_{12} = G_{13} = G_{23} = 500$ psi (3.45×10^{-3} GPa),

$\upsilon_{12} = \upsilon_{13} = \upsilon_{23} = 0$

Results reported in tables and plots are using the following nondimensional form:

$$\bar{u} = u\left(\frac{100h^3E_2}{P_0a^4}\right), \quad \bar{v} = v\left(\frac{100h^3E_2}{P_0a^4}\right),$$

$$\bar{w} = w\left(\frac{100h^3E_2}{P_0a^4}\right), \quad \bar{\sigma}_x = \sigma_x\left(\frac{h^2}{P_0a^2}\right),$$

$$\bar{\sigma}_y = \sigma_y\left(\frac{h^2}{P_0a^2}\right), \quad \bar{\tau}_{xy} = \tau_{xy}\left(\frac{h^2}{P_0a^2}\right)$$

Unless otherwise specified within the table(s) the locations (i.e. x-, y-, and z-coordinates) for maximum values of displacements and stresses for the present evaluations are as follows:

In-plane displacement (u) : $(0, b/2, \pm h/2)$
In-plane displacement (v) : $(a/2, 0, \pm h/2)$
Transverse displacement (w) : $(a/2, b/2, 0)$
In-plane normal stress (σ_x) : $(a/2, b/2, \pm h/2)$
In-plane normal stress (σ_y) : $(a/2, b/2, \pm h/2)$
In-plane shear stress (τ_{xy}) : $(0, 0, \pm h/2)$

Example 1. A simply supported two and four layered square and two layered rectangular antisymmetric angle-ply $(\theta/-\theta/\ldots)$ composite plates under sinusoidal transverse load is considered. The layers are of equal thickness. Material set 1 is used. The numerical values of maximum transverse deflection $\bar{w}$ for the square and rectangular plates are given in Tables 1 and 2. The results are compared with the values reported by Ren [23]. For all the values of a/h ratio and the fibre orientation, the values of transverse deflection predicted by the higher-order theory considered in the present investigation is slightly lower. In the case of a square plate, the difference between the results computed using the present theory and the results reported by Ren is very less in

Table 1
Non-dimensionalized deflection in a simply supported anti-symmetric ($\theta/-\theta/\ldots$) square laminate under sinusoidal transverse load

θ	a/h	Theory	$\bar{w}$	
			$n^a=2$	$n^a=4$
15°	4	Present	1.4596	1.2869
		Ren[b]	1.4989	1.3050
	10	Present	0.6374	0.4446
		Ren[b]	0.6476	0.4505
	100	Present	0.4679	0.2667
		Ren[b]	0.4680	0.2668
30°	4	Present	1.3775	1.0605
		Ren[b]	1.4865	1.0943
	10	Present	0.6432	0.3454
		Ren[b]	0.6731	0.3543
	100	Present	0.4972	0.2048
		Ren[b]	0.4975	0.2049
45°	4	Present	1.3175	0.9814
		Ren[b]	1.4471	1.0160
	10	Present	0.6084	0.3114
		Ren[b]	0.6427	0.3201
	100	Present	0.4682	0.1820
		Ren	0.4685	0.1821

[a] Number of layers.
[b] See [23].

the case of four layered plate as compared to two layered plate. For a thick rectangular plate ($a/h = 4$), the difference in values increases with the increase in fibre orientation up to $\theta = 45°$ and decreases with θ greater than 45°. Both for the square and rectangular thin laminates ($a/h = 100$), for all the values of a/h ratio and fibre orientation θ considered, the present results are in good agreement with that reported by Ren. The

Table 2
Non-dimensionalized deflection in a simply supported two layered anti-symmetric ($\theta/-\theta$) rectangular ($b = 3a$) laminate under sinusoidal transverse load

θ	a/h	Theory	$\bar{w}$
15°	4	Present	2.1245
		Ren[a]	2.1922
	10	Present	1.0146
		Ren[a]	1.0272
	100	Present	0.8019
		Ren[a]	0.8020
30°	4	Present	2.6980
		Ren[a]	2.8881
	10	Present	1.5388
		Ren[a]	1.5787
	100	Present	1.3158
		Ren[a]	1.3163
45°	4	Present	3.6716
		Ren[a]	3.9653
	10	Present	2.3323
		Ren[a]	2.3953
	100	Present	2.0679
		Ren[a]	2.0686
60°	4	Present	5.3528
		Ren[a]	5.6283
	10	Present	3.8013
		Ren[a]	3.8621
	100	Present	3.4864
		Ren[a]	3.4868
75°	4	Present	7.8743
		Ren[a]	8.0042
	10	Present	6.4668
		Ren[a]	6.4962
	100	Present	6.1831
		Ren[a]	6.1836

[a] See [23].

Table 3
Non-dimensionalized in-plane stresses in a simply supported two ($n = 2$) and four ($n = 4$) layered anti-symmetric ($\theta/-\theta\ldots$) square laminate under sinusoidal transverse load

θ	a/h	$\bar{\sigma}_x$		$\bar{\sigma}_y$		$\bar{\tau}_{xy}$	
		$n=2$	$n=4$	$n=2$	$n=4$	$n=2$	$n=4$
15°	4	0.8274	0.7261	0.1122	0.1181	−0.0835	−0.0858
	10	0.6420	0.4730	0.0782	0.0584	−0.0753	−0.0663
	100	0.6002	0.4173	0.0682	0.0448	−0.0727	−0.0588
30°	4	0.7727	0.5806	0.2978	0.2340	−0.1726	−0.1615
	10	0.4979	0.3009	0.1885	0.1141	−0.1897	−0.1266
	100	0.4395	0.2439	0.1652	0.0893	−0.1962	−0.1179
45°	4	0.5351	0.4046	0.5351	0.4046	−0.2310	−0.1911
	10	0.3123	0.1885	0.3123	0.1885	−0.2465	−0.1488
	100	0.2652	0.1443	0.2652	0.1443	−0.2531	−0.1395
60°	4	0.3246	0.2340	0.7997	0.5807	−0.1726	−0.1615
	10	0.1935	0.1141	0.5029	0.3009	−0.1897	−0.1266
	100	0.1652	0.0894	0.4395	0.2439	−0.1962	−0.1179
75°	4	0.1378	0.1181	0.8435	0.7261	−0.0835	−0.0858
	10	0.0811	0.0584	0.6448	0.4730	−0.0753	−0.0663
	100	0.0682	0.0448	−0.6002	0.4173	−0.0727	−0.0588

Table 4
Non-dimensionalized in-plane stresses in a simply supported two layered antisymmetric ($\theta/-\theta$) rectangular ($b = 3a$) laminate under sinusoidal transverse load

θ	a/h	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$
15°	4	1.1110	0.0934	−0.0443
	10	0.9002	0.0725	−0.0443
	100	0.8542	0.0681	−0.0448
30°	4	1.1889	0.3872	−0.1271
	10	0.9193	0.2966	−0.1676
	100	0.8641	0.2782	−0.1803
45°	4	0.9998	0.8794	−0.2816
	10	0.7437	0.6476	−0.3563
	100	0.6904	0.5998	−0.3765
60°	4	0.6559	1.2389	−0.4419
	10	0.4987	0.9451	−0.5130
	100	0.4649	0.8830	−0.5304
75°	4	0.4665	1.2976	−0.3093
	10	0.4114	1.2487	−0.3339
	100	0.3994	1.2399	−0.3399

Table 5
Non-dimensionalized transverse deflection and in-plane stresses in a simply supported four layered anti-symmetric ($\theta/-\theta/\theta/-\theta$) square laminate under sinusoidal transverse load

θ	a/h	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$
15°	4	2.2963	0.3399	0.1253	−0.1017
	10	1.5462	0.3205	0.1077	−0.0922
	100	1.4018	0.3170	0.1040	−0.0902
30°	4	2.1180	0.2615	0.1510	−0.1321
	10	1.3793	0.2333	0.1308	−0.1214
	100	1.2381	0.2278	0.1267	−0.1193
45°	4	2.0392	0.1945	0.1945	−0.1468
	10	1.3040	0.1681	0.1681	−0.1348
	100	1.1638	0.1628	0.1628	−0.1325
60°	4	2.1180	0.1510	0.2615	−0.1321
	10	1.3793	0.1308	0.2333	−0.1214
	100	1.2381	0.1267	0.2278	−0.1193
75°	4	2.2963	0.1253	0.3399	−0.1017
	10	1.5462	0.1077	0.3205	−0.0922
	100	1.4018	0.1040	0.3170	−0.0902

numerical values of nondimensionalized in-plane stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$ and $\bar{\tau}_{xy}$ computed using the present theory for two and four layered square and for two layered rectangular plate for various a/h ratios and fibre orientations are given in Tables 3 and 4. The variation of maximum nondimensionalized transverse displacement $\bar{w}$ of a two and four layered plate with a/h ratio equal to 10 for varying fibre orientation is given in Fig. 3.

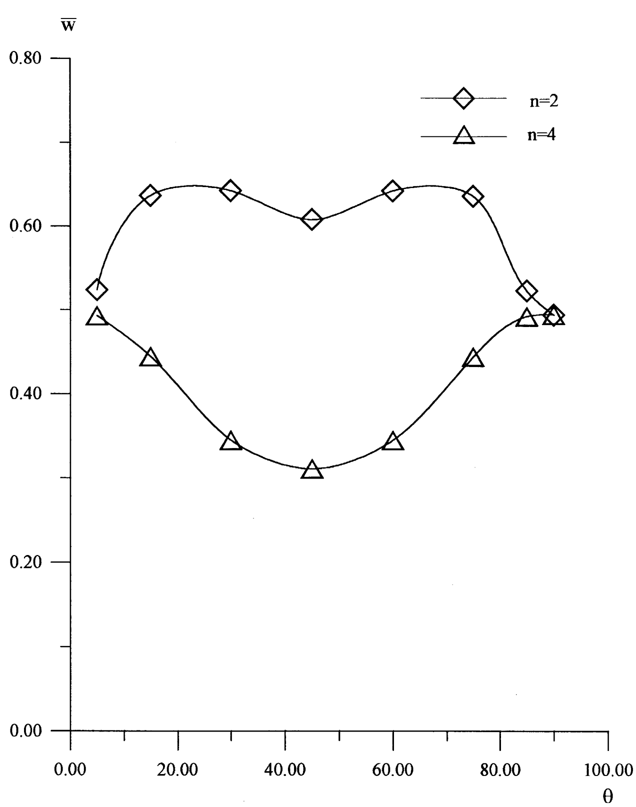

Fig. 3. Variation of transverse displacement ($\bar{w}$) for various angle of orientation (θ) in a two and four layered simply supported anti-symmetric angle-ply plate subjected to transverse sinusoidal load for a/h ratio 10.

Example 2. A simply supported four layered anti-symmetric angle-ply ($\theta/-\theta/\theta/-\theta$) square composite plate with layers of equal thickness and with real material properties under sinusoidal transverse load is considered. Material set 2 is used. The nondimensionalized maximum values of transverse displacement $\bar{w}$, in-plane stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$ and $\bar{\tau}_{xy}$ for various values of side-to-thickness ratio and angle of orientation are given in Table 5. For a plate with a/h ratio equal to 10, the through the thickness variation of the nondimensionalized in-plane stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$ and the nondimensionalized in-plane displacements $\bar{u}$ and $\bar{v}$ are given in Figs. 4–8.

Example 3. In order to study the flexural behaviour of laminated sandwich plate, a five layered plate ($\theta/-\theta$/core/$\theta/-\theta$) with isotropic core and antisymmetric angle-ply face sheets is considered. Material set 3 is used. The ratio of the thickness of core to thickness of the face sheet t_c/t_f considered equal to 10. The nondimensionalized maximum values of transverse displacement $\bar{w}$, in-plane stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$ and $\bar{\tau}_{xy}$ for various values of side-to-thickness ratio and angle of orientation are given in Table 6. The through the thickness variation of the nondimensionalized in-plane stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$ and the nondimensionalized in-plane displacements $\bar{u}$ and $\bar{v}$ are shown in Figs. 9–13. The

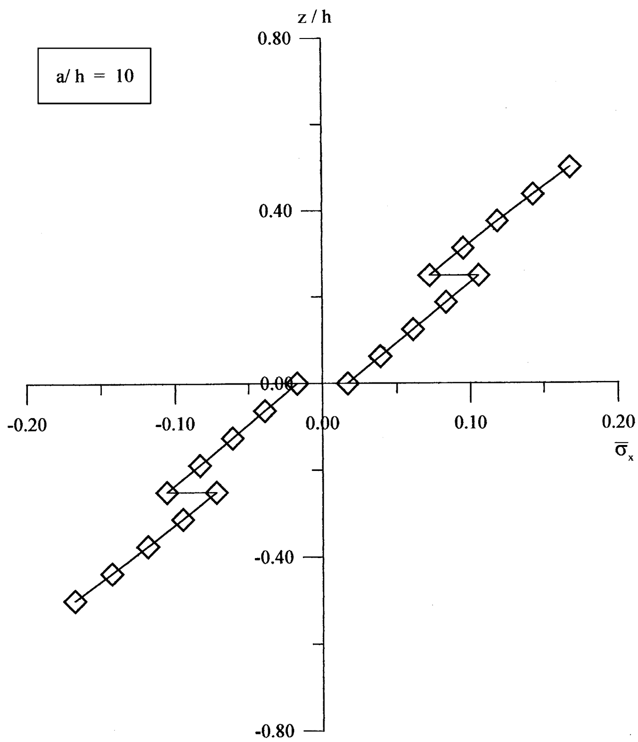

Fig. 4. Variation of non-dimensionalized in-plane normal stress ($\bar{\sigma}_x$) through the thickness (z/h) of a four layered (45°/−45°/45°/−45°) simply supported anti-symmetric angle-ply composite plate under sinusoidal transverse load.

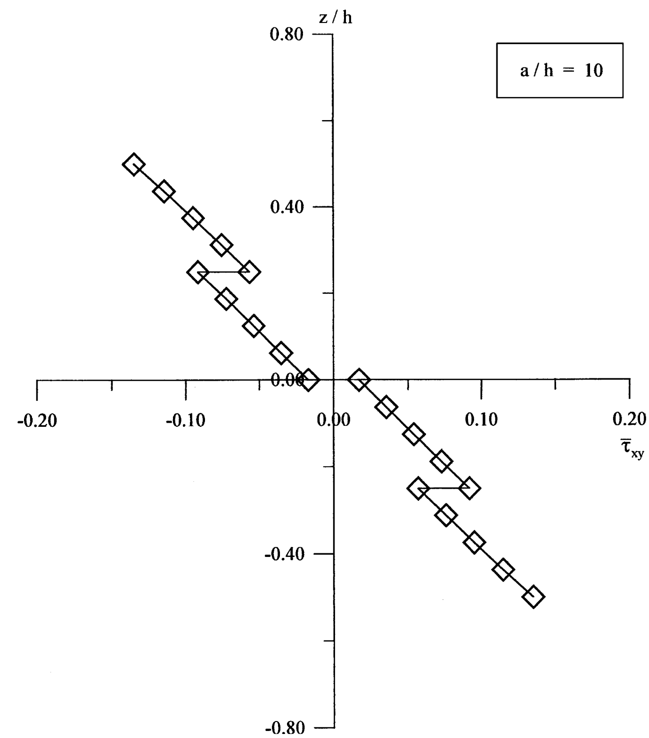

Fig. 6. Variation of non-dimensionalized in-plane shear stress ($\bar{\tau}_{xy}$) through the thickness (z/h) of a four layered (45°/−45°/45°/−45°) simply supported anti-symmetric angle-ply composite plate under sinusoidal transverse load.

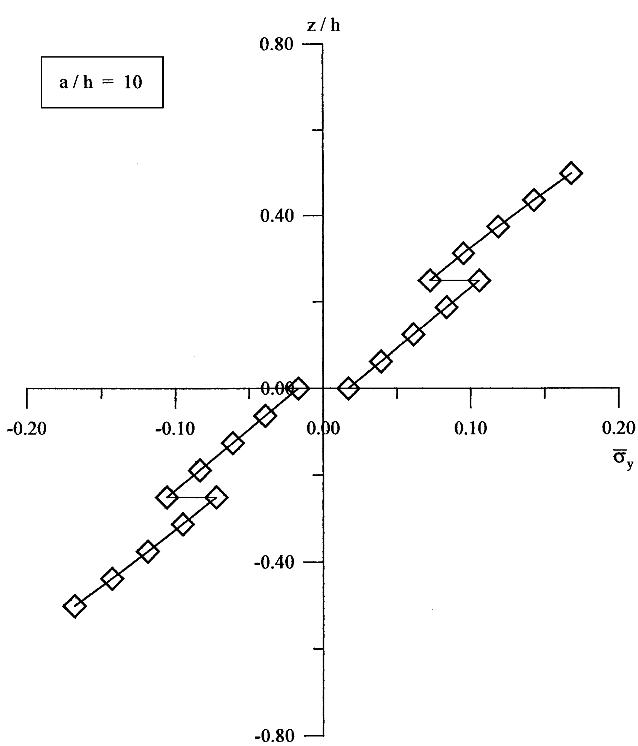

Fig. 5. Variation of non-dimensionalized in-plane normal stress ($\bar{\sigma}_y$) through the thickness (z/h) of a four layered (45°/−45°/45°/−45°) simply supported anti-symmetric angle-ply composite plate under sinusoidal transverse load.

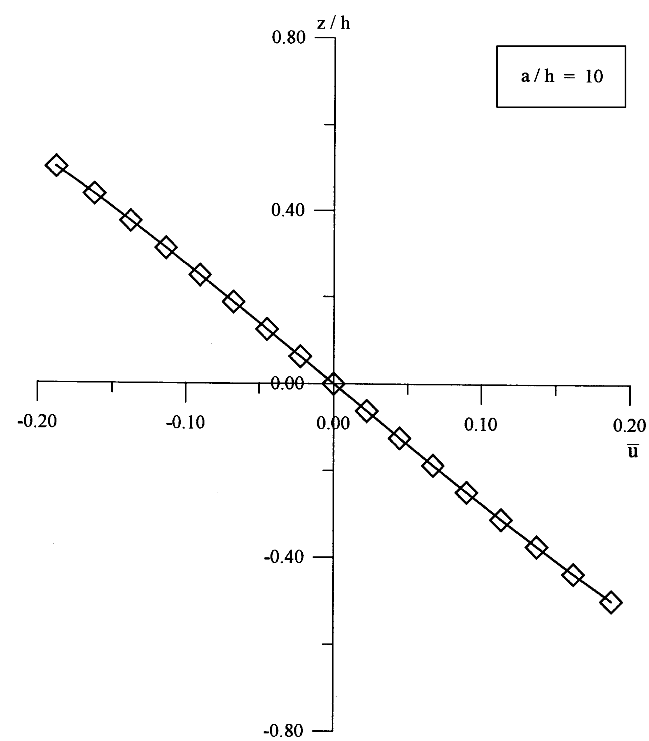

Fig. 7. Variation of non-dimensionalized in-plane displacement ($\bar{u}$) through the thickness (z/h) of a four layered (45°/−45°/45°/−45°) simply supported anti-symmetric angle-ply composite plate under sinusoidal transverse load.

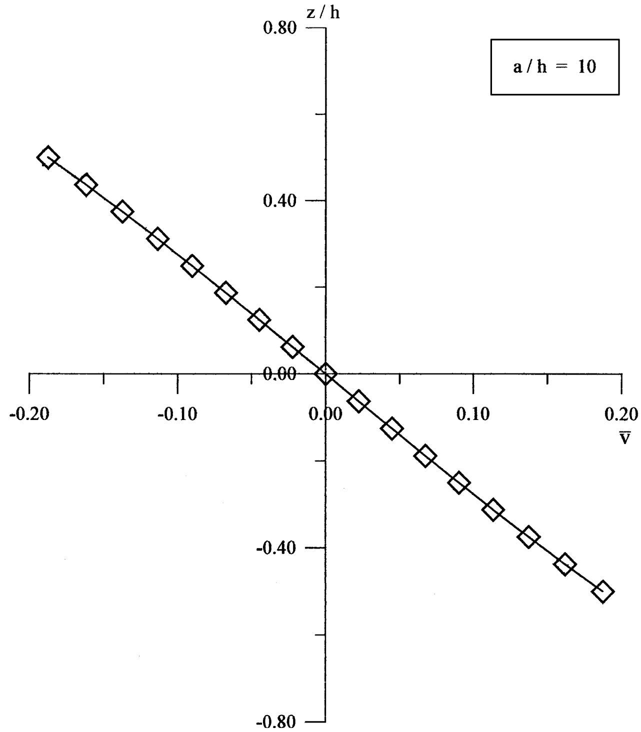

Fig. 8. Variation of non-dimensionalized in-plane displacement ($\bar{v}$) through the thickness (z/h) of a four layered (45°/−45°/45°/−45°) simply supported anti-symmetric angle-ply composite plate under sinusoidal transverse load.

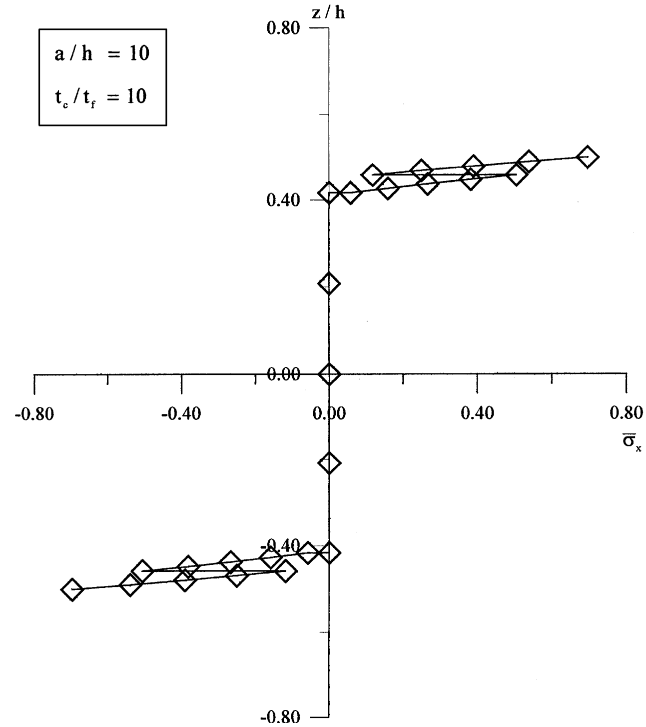

Fig. 9. Variation of non-dimensionalized in-plane normal stress ($\bar{\sigma}_x$) through the thickness (z/h) of a five layered (45°/−45°/core/45°/−45°) simply supported anti-symmetric angle-ply sandwich plate under sinusoidal transverse load.

Table 6
Non-dimensionalized transverse deflection and in-plane stresses in a simply supported five layered anti-symmetric ($\theta/-\theta$/core/$\theta/-\theta$) square sandwich plate under sinusoidal transverse load

θ	a/h	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$
15°	4	97.9539	4.6390	1.0314	−1.1546
	10	19.6701	1.2712	0.4183	−0.4158
	100	1.7176	0.8814	0.1547	−0.2001
30°	4	98.1091	3.2410	1.5440	−1.8205
	10	18.5049	0.8971	0.5039	−0.6308
	100	1.3658	0.5677	0.2403	−0.3049
45°	4	97.5559	2.2653	2.2665	−2.1437
	10	18.0605	0.6972	0.6978	−0.6577
	100	1.2404	0.3653	0.3653	−0.3419
60°	4	98.1077	1.5433	3.2427	−1.8196
	10	18.5050	0.5036	0.8979	−0.6304
	100	1.3658	0.2403	0.5677	−0.3049
75°	4	97.9538	1.0310	4.6390	−1.1541
	10	19.6700	0.4181	1.2712	−0.4856
	100	1.7276	0.1547	0.8814	−0.2001

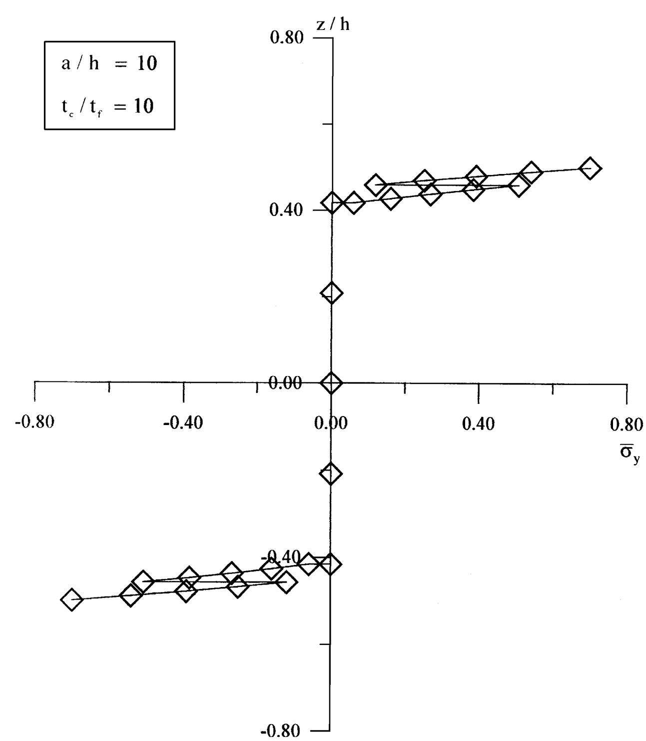

Fig. 10. Variation of non-dimensionalized in-plane normal stress ($\bar{\sigma}_y$) through the thickness (z/h) of a five layered (45°/−45°/core/45°/−45°) simply supported anti-symmetric angle-ply sandwich plate under sinusoidal transverse load.

through-the-thickness variation of in-plane stresses implies that the contribution of core in resisting the in-plane stresses is almost negligible and the through-the-thickness variation of in-plane displacements plotted in

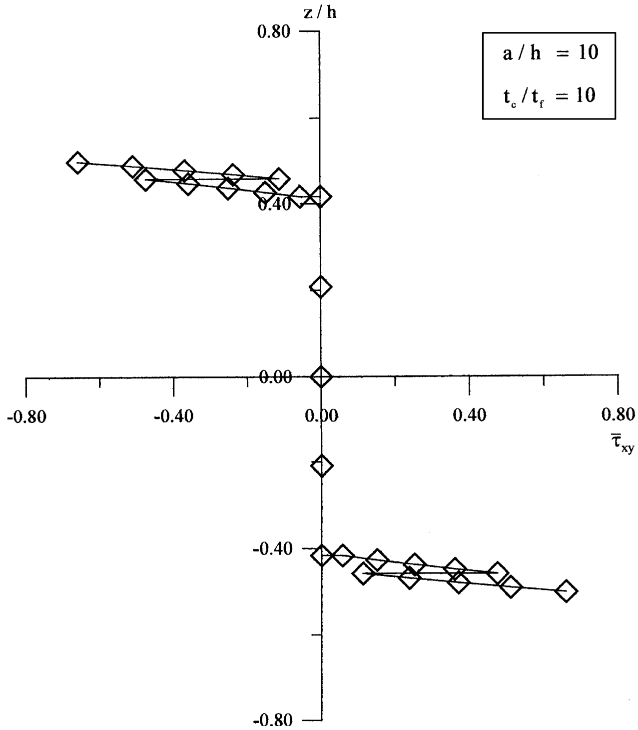

Fig. 11. Variation of non-dimensionalized in-plane shear stress ($\bar{\tau}_{xy}$) through the thickness (z/h) of a five layered (45°/−45°/core/45°/−45°) simply supported anti-symmetric angle-ply sandwich plate under sinusoidal transverse load.

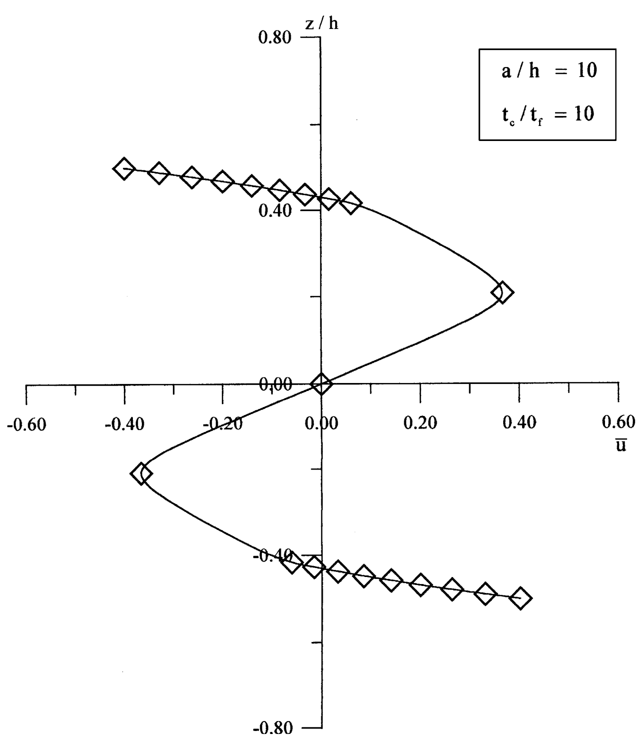

Fig. 12. Variation of non-dimensionalized in-plane displacement ($\bar{u}$) through the thickness (z/h) of a five layered (45°/−45°/core/45°/−45°) simply supported anti-symmetric angle-ply sandwich plate under sinusoidal transverse load.

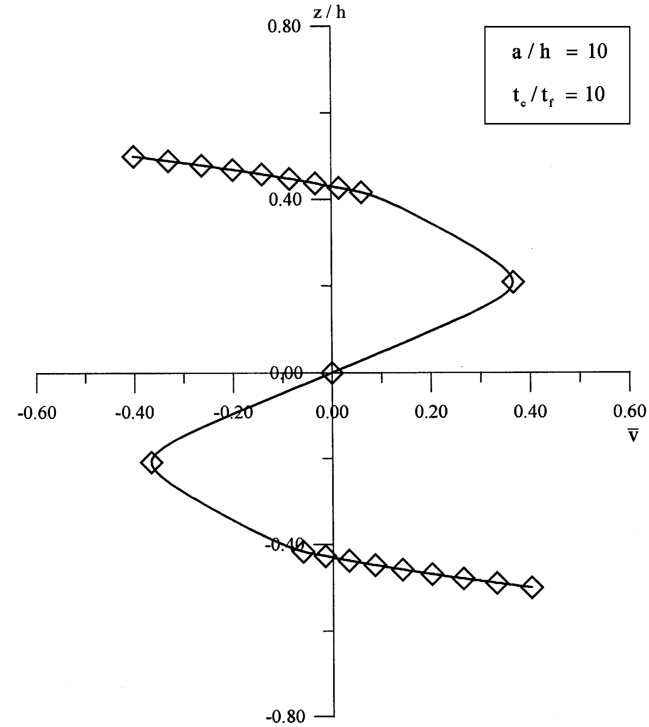

Fig. 13. Variation of non-dimensionalized in-plane displacement ($\bar{v}$) through the thickness (z/h) of a five layered (45°/−45°/core/45°/−45°) simply supported anti-symmetric angle-ply sandwich plate under sinusoidal transverse load.

Figs. 12 and 13 clearly indicates that the variation is cubic nature.

5. Conclusions

Analytical formulations and solutions to the static analysis of simply supported angle-ply composite and sandwich plates hitherto not reported in the literature based on a higher-order refined theory already reported in the literature are presented. The accuracy of the theoretical formulations and the solution method is established by comparing the results generated in the present investigation using the higher-order refined theory with the solutions already available in the literature. After ascertaining the accuracy, new results for multilayered sandwich plates with antisymmetric angle-ply face sheets are presented which will serve as a benchmark for future investigations.

Appendix A. Coefficients of [*C*] matrix

$$C_{11} = \frac{E_1}{(1 - \upsilon_{12}\upsilon_{21})}, \quad C_{12} = \frac{\upsilon_{21}E_1}{(1 - \upsilon_{12}\upsilon_{21})},$$

$$C_{22} = \frac{E_2}{(1 - \upsilon_{12}\upsilon_{21})}, \quad C_{44} = G_{12}, \quad C_{55} = G_{23}, \quad C_{66} = G_{13}$$

and

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \upsilon_{21}\frac{\sigma_2}{E_2}, \quad \varepsilon_2 = \frac{\sigma_2}{E_2} - \upsilon_{12}\frac{\sigma_1}{E_1}$$

$$\gamma_{12} = \frac{\tau_{12}}{G_{12}}, \quad \gamma_{23} = \frac{\tau_{23}}{G_{23}}, \quad \gamma_{13} = \frac{\tau_{13}}{G_{13}}$$

$$\frac{\upsilon_{12}}{E_1} = \frac{\upsilon_{21}}{E_2}, \quad \frac{\upsilon_{31}}{E_3} = \frac{\upsilon_{13}}{E_1}, \quad \frac{\upsilon_{32}}{E_3} = \frac{\upsilon_{23}}{E_2}$$

Appendix B. Coefficients of [Q] matrix

$$Q_{11} = C_{11}c^4 + 2(C_{12} + 2C_{44})s^2c^2 + C_{22}s^4$$

$$Q_{12} = C_{12}(c^4 + s^4) + (C_{11} + C_{22} - 4C_{44})s^2c^2$$

$$Q_{14} = (C_{11} - C_{12} - 2C_{44})sc^3 + (C_{12} - C_{22} + 2C_{44})cs^3$$

$$Q_{22} = C_{11}s^4 + C_{22}c^4 + (2C_{12} + 4C_{44})s^2c^2$$

$$Q_{24} = (C_{11} - C_{12} - 2C_{44})s^3c + (C_{12} - C_{22} + 2C_{44})c^3s$$

$$Q_{44} = (C_{11} - 2C_{12} + C_{22} - 2C_{44})c^2s^2 + C_{44}(c^4 + s^4)$$

$$Q_{55} = C_{55}c^2 + C_{66}s^2$$

$$Q_{56} = (C_{66} - C_{55})cs$$

$$Q_{66} = C_{55}s^2 + C_{66}c^2$$

and

$$Q_{ij} = Q_{ji}, \; i,j = 1\text{–}6$$

where

$$c = \cos\alpha \quad \text{and} \quad s = \sin\alpha$$

Appendix C. Elements of [A], [B], [A′], [B′], [D], [D′], [E], [E′] matrices

$$[A] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{11}H_1 & Q_{12}H_1 & Q_{11}H_3 & Q_{12}H_3 & Q_{11}H_2 & Q_{12}H_2 & Q_{11}H_4 & Q_{12}H_4 \\ Q_{12}H_1 & Q_{22}H_1 & Q_{12}H_3 & Q_{22}H_3 & Q_{12}H_2 & Q_{22}H_2 & Q_{12}H_4 & Q_{22}H_4 \\ Q_{11}H_3 & Q_{12}H_3 & Q_{11}H_5 & Q_{12}H_5 & Q_{11}H_4 & Q_{12}H_4 & Q_{11}H_6 & Q_{12}H_6 \\ Q_{12}H_3 & Q_{22}H_3 & Q_{12}H_5 & Q_{22}H_5 & Q_{12}H_4 & Q_{22}H_4 & Q_{12}H_6 & Q_{22}H_6 \\ Q_{11}H_2 & Q_{12}H_2 & Q_{11}H_4 & Q_{12}H_4 & Q_{11}H_3 & Q_{12}H_3 & Q_{11}H_5 & Q_{12}H_5 \\ Q_{12}H_2 & Q_{22}H_2 & Q_{12}H_4 & Q_{22}H_4 & Q_{12}H_3 & Q_{22}H_3 & Q_{12}H_5 & Q_{22}H_5 \\ Q_{11}H_4 & Q_{12}H_4 & Q_{11}H_6 & Q_{12}H_6 & Q_{11}H_5 & Q_{12}H_5 & Q_{11}H_7 & Q_{12}H_7 \\ Q_{12}H_4 & Q_{22}H_4 & Q_{12}H_6 & Q_{22}H_6 & Q_{12}H_5 & Q_{22}H_5 & Q_{12}H_7 & Q_{22}H_7 \end{bmatrix}$$

$$[B] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{44}H_1 & Q_{44}H_1 & Q_{44}H_3 & Q_{44}H_3 & Q_{44}H_2 & Q_{44}H_2 & Q_{44}H_4 & Q_{44}H_4 \\ Q_{44}H_3 & Q_{44}H_3 & Q_{44}H_5 & Q_{44}H_5 & Q_{44}H_4 & Q_{44}H_4 & Q_{44}H_6 & Q_{44}H_6 \\ Q_{44}H_2 & Q_{44}H_2 & Q_{44}H_4 & Q_{44}H_4 & Q_{44}H_3 & Q_{44}H_3 & Q_{44}H_5 & Q_{44}H_5 \\ Q_{44}H_4 & Q_{44}H_4 & Q_{44}H_6 & Q_{44}H_6 & Q_{44}H_5 & Q_{44}H_5 & Q_{44}H_7 & Q_{44}H_7 \end{bmatrix}$$

$$[A'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{14}H_1 & Q_{14}H_1 & Q_{14}H_3 & Q_{14}H_3 & Q_{14}H_2 & Q_{14}H_2 & Q_{14}H_4 & Q_{14}H_4 \\ Q_{24}H_1 & Q_{24}H_1 & Q_{24}H_3 & Q_{24}H_3 & Q_{24}H_2 & Q_{24}H_2 & Q_{24}H_4 & Q_{24}H_4 \\ Q_{14}H_3 & Q_{14}H_3 & Q_{14}H_5 & Q_{14}H_5 & Q_{14}H_4 & Q_{14}H_4 & Q_{14}H_6 & Q_{14}H_6 \\ Q_{24}H_3 & Q_{24}H_3 & Q_{24}H_5 & Q_{24}H_5 & Q_{24}H_4 & Q_{24}H_4 & Q_{24}H_6 & Q_{24}H_6 \\ Q_{14}H_2 & Q_{14}H_2 & Q_{14}H_4 & Q_{14}H_4 & Q_{14}H_3 & Q_{14}H_3 & Q_{14}H_5 & Q_{14}H_5 \\ Q_{24}H_2 & Q_{24}H_2 & Q_{24}H_4 & Q_{24}H_4 & Q_{24}H_3 & Q_{24}H_3 & Q_{24}H_5 & Q_{24}H_5 \\ Q_{14}H_4 & Q_{14}H_4 & Q_{14}H_6 & Q_{14}H_6 & Q_{14}H_5 & Q_{14}H_5 & Q_{14}H_7 & Q_{14}H_7 \\ Q_{24}H_4 & Q_{24}H_4 & Q_{24}H_6 & Q_{24}H_6 & Q_{24}H_5 & Q_{24}H_5 & Q_{24}H_7 & Q_{24}H_7 \end{bmatrix}$$

$$[B'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{14}H_1 & Q_{24}H_1 & Q_{14}H_3 & Q_{24}H_3 & Q_{14}H_2 & Q_{24}H_2 & Q_{14}H_4 & Q_{24}H_4 \\ Q_{14}H_3 & Q_{24}H_3 & Q_{14}H_5 & Q_{24}H_5 & Q_{14}H_4 & Q_{24}H_4 & Q_{14}H_6 & Q_{24}H_6 \\ Q_{14}H_2 & Q_{24}H_2 & Q_{14}H_4 & Q_{24}H_4 & Q_{14}H_3 & Q_{24}H_3 & Q_{14}H_5 & Q_{24}H_5 \\ Q_{14}H_4 & Q_{24}H_4 & Q_{14}H_6 & Q_{24}H_6 & Q_{14}H_5 & Q_{24}H_5 & Q_{14}H_7 & Q_{24}H_7 \end{bmatrix}$$

$$[D] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{66}H_1 & Q_{66}H_1 & 3Q_{66}H_3 & 2Q_{66}H_2 \\ Q_{66}H_3 & Q_{66}H_3 & 3Q_{66}H_5 & 2Q_{66}H_4 \\ Q_{66}H_2 & Q_{66}H_2 & 3Q_{66}H_4 & 2Q_{66}H_3 \end{bmatrix}$$

$$[D'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{56}H_1 & Q_{56}H_1 & 3Q_{56}H_3 & 2Q_{56}H_2 \\ Q_{56}H_3 & Q_{56}H_3 & 3Q_{56}H_5 & 2Q_{56}H_4 \\ Q_{56}H_2 & Q_{56}H_2 & 3Q_{56}H_4 & 2Q_{56}H_3 \end{bmatrix}$$

$$[E] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{55}H_1 & Q_{55}H_1 & 3Q_{55}H_3 & 2Q_{55}H_2 \\ Q_{55}H_3 & Q_{55}H_3 & 3Q_{55}H_5 & 2Q_{55}H_4 \\ Q_{55}H_2 & Q_{55}H_2 & 3Q_{55}H_4 & 2Q_{55}H_3 \end{bmatrix}$$

$$[E'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{56}H_1 & Q_{56}H_1 & 3Q_{56}H_3 & 2Q_{56}H_2 \\ Q_{56}H_3 & Q_{56}H_3 & 3Q_{56}H_5 & 2Q_{56}H_4 \\ Q_{56}H_2 & Q_{56}H_2 & 3Q_{56}H_4 & 2Q_{56}H_3 \end{bmatrix}$$

Appendix D. Coefficients of matrix [X]

$X_{1,1} = \alpha^2 A_{1,1} + \beta^2 B_{1,1}, \quad X_{1,2} = \alpha\beta(A_{1,2} + B_{1,2})$

$X_{1,3} = 0, \quad X_{1,4} = \alpha\beta(A'_{1,5} + B'_{1,5}), \quad X_{1,5} = A'_{1,6}\alpha^2 + B'_{1,6}\beta^2$

$X_{1,6} = A_{1,3}\alpha^2 + B_{1,3}\beta^2, \quad X_{1,7} = \alpha\beta(A_{1,4} + B_{1,4})$

$X_{1,8} = \alpha\beta(A'_{1,7} + B'_{1,7}), \quad X_{1,9} = A'_{1,8}\alpha^2 + B'_{1,8}\beta^2$

$X_{2,1} = \alpha\beta(A_{2,1} + B_{1,1})$

$X_{2,2} = A_{2,2}\beta^2 + B_{1,2}\alpha^2, \quad X_{2,3} = 0, \quad X_{2,4} = A'_{2,5}\beta^2 + B'_{1,5}\alpha^2$

$X_{2,5} = \alpha\beta(A'_{2,6} + B'_{1,6}), \quad X_{2,6} = \alpha\beta(A_{2,3} + B_{1,3})$

$X_{2,7} = A_{2,4}\beta^2 + B_{1,4}\alpha^2, \quad X_{2,8} = A'_{2,7}\beta^2 + B'_{1,7}\alpha^2$

$X_{2,9} = \alpha\beta(A'_{2,8} + B'_{1,8})$

$X_{3,1} = 0, \quad X_{3,2} = 0, \quad X_{3,3} = D_{1,2}\alpha^2 + E_{1,2}\beta^2$

$X_{3,4} = D_{1,1}\alpha, \quad X_{3,5} = E_{1,1}\beta, \quad X_{3,6} = E'_{1,4}\beta$

$X_{3,7} = D'_{1,4}\alpha, \quad X_{3,8} = D_{1,3}\alpha, \quad X_{3,9} = E_{1,3}\beta$

$X_{4,1} = \alpha\beta(A'_{5,1} + B'_{3,1}), \quad X_{4,2} = A'_{5,2}\alpha^2 + B'_{3,2}\beta^2$

$X_{4,3} = D_{1,2}\alpha, \quad X_{4,4} = A_{5,5}\alpha^2 + B_{3,5}\beta^2 + D_{1,1}$

$X_{4,5} = \alpha\beta(A_{5,6} + B_{3,6}), \quad X_{4,6} = \alpha\beta(A'_{5,3} + B'_{3,3})$

$X_{4,7} = A'_{5,4}\alpha^2 + B'_{3,4}\beta^2 + D'_{1,4}, \quad X_{4,8} = A_{5,7}\alpha^2 + B_{3,7}\beta^2 + D_{1,3}$

$X_{4,9} = \alpha\beta(A_{5,8} + B_{3,8})$

$X_{5,1} = A'_{6,1}\beta^2 + B'_{3,1}\alpha^2, \quad X_{5,2} = \alpha\beta(A'_{6,2} + B'_{3,2})$

$X_{5,3} = E_{1,2}\beta, \quad X_{5,4} = \alpha\beta(A_{65} + B_{35})$

$X_{5,5} = A_{66}\beta^2 + B_{36}\alpha^2 + E_{11}, \quad X_{5,6} = A'_{6,3}\beta^2 + B'_{3,3}\alpha^2 + E'_{1,4}$

$X_{5,7} = \alpha\beta(A'_{6,4} + B'_{3,4}), \quad X_{5,8} = \alpha\beta(A_{6,7} + B_{3,7})$

$X_{5,9} = A_{6,8}\beta^2 + B_{3,8}\alpha^2 + E_{1,3}$

$X_{6,1} = A_{3,1}\alpha^2 + B_{2,1}\beta^2, \quad X_{6,2} = \alpha\beta(A_{3,2} + B_{2,2})$

$X_{6,3} = 2D'_{3,2}\beta, \quad X_{6,4} = \alpha\beta(A'_{3,5} + B'_{2,5})$

$X_{6,5} = A'_{3,6}\alpha^2 + B'_{2,6}\beta^2 + 2D'_{3,1}, \quad X_{6,6} = A_{3,3}\alpha^2 + B_{2,3}\beta^2 + 2D_{3,4}$

$X_{6,7} = A_{3,4}\alpha\beta + B_{2,4}\alpha\beta, \quad X_{6,8} = \alpha\beta(A'_{3,7} + B'_{2,7})$

$X_{6,9} = A'_{3,8}\alpha^2 + B'_{2,8}\beta^2 + 2D'_{3,3}$

$X_{7,1} = \alpha\beta(A_{4,1} + B_{2,1})$

$X_{7,2} = A_{4,2}\beta^2 + B_{2,2}\alpha^2, \quad X_{7,3} = 2E'_{3,2}\alpha, \quad X_{7,4} = A'_{4,5}\beta^2 + B'_{2,5}\alpha^2 + 2E'_{3,1}$

$X_{7,5} = \alpha\beta(A'_{4,6} + B'_{2,6}), \quad X_{7,6} = \alpha\beta(A_{4,3} + B_{2,3})$

$X_{7,7} = A_{4,4}\beta^2 + B_{2,4}\alpha^2 + 2E_{3,4}, \quad X_{7,8} = A'_{4,7}\beta^2 + B'_{2,7}\alpha^2 + 2E'_{3,3}$

$X_{7,9} = \alpha\beta(A'_{4,8} + B'_{2,8})$

$X_{8,1} = \alpha\beta(A'_{7,1} + B'_{4,1}), \quad X_{8,2} = A'_{7,2}\alpha^2 + B'_{4,2}\beta^2$

$X_{8,3} = 3D'_{2,2}\alpha, \quad X_{8,4} = A_{7,5}\alpha^2 + B_{4,5}\beta^2 + 3D_{2,1}$

$X_{8,5} = \alpha\beta(A_{7,6} + B_{4,6}), \quad X_{8,6} = \alpha\beta(A'_{7,3} + B'_{4,3})$

$X_{8,7} = A'_{7,4}\alpha^2 + B'_{4,4}\beta^2 + 3D'_{2,4}, \quad X_{8,8} = A_{7,7}\alpha^2 + B_{4,7}\beta^2 + 3D_{2,3}$

$X_{8,9} = \alpha\beta(A_{7,8} + B_{4,8})$

$X_{9,1} = A'_{8,1}\beta^2 + B'_{4,1}\alpha^2, \quad X_{9,2} = \alpha\beta(A'_{8,2} + B'_{4,2}), \quad X_{9,3} = 3E_{2,2}\beta$

$X_{9,4} = \alpha\beta(A_{8,5} + B_{4,5}), \quad X_{9,5} = A_{8,6}\beta^2 + B_{4,6}\alpha^2 + 3E_{2,1}$

$X_{9,6} = A'_{8,3}\beta^2 + B'_{4,3}\alpha^2 + 3E'_{2,4}, \quad X_{9,7} = \alpha\beta(A'_{8,4} + B'_{4,4})$

$X_{9,8} = \alpha\beta(A'_{8,7} + B'_{4,7}), \quad X_{9,9} = A_{8,8}\beta^2 + B_{4,8}\alpha^2 + 2E_{2,3}$

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