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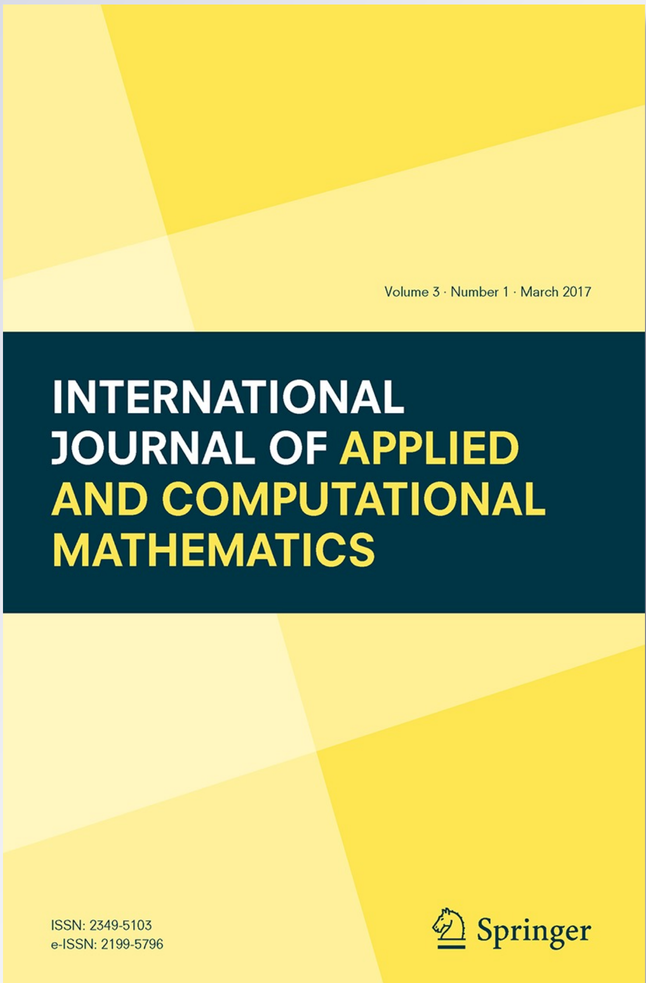

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ORIGINAL PAPER

Ball Convergence for an Inverse Free Jarratt-Type Method Under Hölder Conditions

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Abstract We present a local convergence analysis of an inverse free Jarratt-type method in order to approximate a locally unique solution of an equation in a Banach space setting. Earlier studies have used hypotheses up to the third Fréchet-derivative of the operator involved to show convergence although only the first derivative is used in the method. We show convergence using only the first Fréchet derivative under Hölder conditions. This way we expand the applicability of the method. Numerical examples are also provided in this study.

Keywords Jarratt method · Banach space · Fréchet derivative · Local convergence · Hölder condition

Mathematics Subject Classification 65D10 · 65D99 · 65G99 · 47J25 · 45J05

Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Denote by $L(X, Y)$ the space of bounded linear operators from X into Y.

A lot of problems from Computational Sciences and other disciplines can be brought in the form of equation (1) using Mathematical Modeling [1,2]. The solution of these equations

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can rarely be found in closed form. That is why the solution methods for these equations are iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method.

The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [3,4]. In order to obtain a higher order of convergence, Newton-like methods have been studied such as Potra–Ptak [5], Chebyshev and Cauchy Halley method [6,7].

We study the local convergence analysis of the inverse-free Jarratt-type method defined for each $n = 0, 1, 2...$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ H_n &= \frac{3}{2}F'(x_n)^{-1}\left(F'\left(x_n + \frac{2}{3}(y_n - x_n)\right) - F'(x_n)\right) \\ x_{n+1} &= y_n - \frac{1}{2}H_n(I - H_n)(y_n - x_n), \end{aligned} \tag{2}$$

where x_0 is an initial point. Method (2) has been studied in [8,9] under Lipschitz or Hölder-type hypotheses reaching up to the fourth Fréchet-derivative of F although only the first Fréchet-derivative appears in method (2). These hypotheses limit the applicability of method (2). As a motivational example, let us define function F on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \;\; F'(1) = 3, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously function F does not have bounded third derivative in X. The results in earlier studies (see for example [8,9]) show that if the initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^* the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (2) in section "Local Convergence Analysis". The same technique can be used to other methods.

The paper is organized as follows. In section "Local Convergence Analysis" we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in the earlier studies using Taylor expansions. Special cases and numerical examples are presented in the concluding section "Numerical Examples".

Local Convergence Analysis

We present the local convergence analysis of the method (2) in this section. Let $L_0 > 0$, $L > 0$, $M \geq 1$ and $p \in (0, 1]$ be given parameters. It is convenient for the local convergence analysis of method (2) that follows to define some scalar functions and parameters. Define functions g_1, h_1, g_2 and h_2 on the interval $[0, (\frac{1}{L_0})^{\frac{1}{p}})$ by

$$g_1(t) = \frac{Lt^p}{(1+p)(1-L_0t^p)},$$
$$h_1(t) = g_1(t) - 1,$$
$$g_2(t) = \frac{1}{(1+p)(1-L_0t^p)}\left[L + \left(\frac{1+p}{2}\right)\left(1 + \frac{LM^pt}{(1-L_0t^p)^{1+p}}\right)\frac{LM^{1+p}}{(1-L_0t^p)^{1+p}}\right]t^p,$$
$$h_2(t) = g_2(t) - 1$$

and parameters r_1 by

$$r_1 = \left(\frac{1+p}{(1+p)L_0 + L}\right)^{\frac{1}{p}}.$$

We have that $h_2(0) = -1 < 0$ and $h_2(r_1) = \frac{1}{2}[1 + (\frac{LM^pr_1}{(1-L_0r_1^p)^{1+p}})\frac{LM^{1+p}r_1^p}{(1-L_0r_1^p)^{2+p}}] > 0$. It then, follows from the intermediate value theorem that function h_2 has zeros in the interval $(0, r_1)$. Denote by r the smallest such zero. Then, we have that

$$0 < r \leq r_1 \tag{3}$$

and for each $t \in [0, r)$

$$0 \leq g_1(t) < 1 \tag{4}$$

and

$$0 \leq g_2(t) < 1. \tag{5}$$

Let $U(\gamma, \rho)$, $\bar{U}(\gamma, \rho)$, stand respectively for the open and closed balls in X with center $\gamma \in X$ and radius $\rho > 0$. Next, we present the local convergence analysis of the method (2), using the preceding notation.

Theorem 2.1 *Let $F : D \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$, $M \geq 1$ and $p \in (0, 1]$ such that for each $x, y \in D$*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X) \tag{6}$$
$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|^p, \tag{7}$$
$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - x^*\|^p, \tag{8}$$
$$\|F'(x^*)^{-1}F'(x)\| \leq M \tag{9}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{10}$$

hold, where the radius r is defined previously. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (2) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2.....$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{11}$$

and

$$\|x_{n+1} - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| \tag{12}$$

where the "g" functions are defined previously. Furthermore, if there exist $T \in [r, (\frac{1+p}{L_0})^{\frac{1}{p}})$, the limit point x^* is the only solution of the equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap D$.

Proof We shall show estimates (11) and (12) using mathematical induction. By hypothesis $x_0 \in U(x^*, r) - \{x^*\}$ and (7), we get that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\|^p < L_0 r^p < 1. \tag{13}$$

It follows from (13) and Banach Lemma on invertible operators [1,10] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|^p}. \tag{14}$$

Hence, y_0 is well defined by the first sub-step of method (2) for $n = 0$. Using (2), (4), (8) and (14), we obtain in turn that

$$\begin{aligned}\|y_0 - x^*\| &\leq \|x_0 - x^* - F'(x_0)^{-1}F'(x^*)\| \\ &\leq \|F'(x_0)^{-1}F'(x^*)\| \left\| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) \right. \\ &\quad \left. - F'(x_0))(x_0 - x^*)d\theta \right\| \\ &\leq \frac{L\|x_0 - x^*\|^{1+p}}{(1 + p)(1 - L_0\|x_0 - x^*\|^p)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,\end{aligned} \tag{15}$$

which shows (11) for $n = 0$ and $y_0 \in U(x^*, r)$. Then, we have that

$$\begin{aligned}\|x_0 - x^* + \frac{2}{3}(y_0 - x_0)\| &\leq \frac{2}{3}\|y_0 - x^*\| + \frac{1}{3}\|x_0 - x^*\| \\ &< \frac{2}{3}r + \frac{1}{3}r = r\end{aligned}$$

and

$$\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r,$$

which shows $x_0 + \frac{2}{3}(y_0 - x_0), x^* + \theta(x_0 - x^*) \in U(x^*, r)$. We can write by (6) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{16}$$

Then, using (9) and (16), we have that

$$\|F'(x^*)^{-1}F(x_0)\| \leq M\|x_0 - x^*\|. \tag{17}$$

Next, we need an estimate on $\|H_0\|$. Using (8), (14), (15) and (17), we get in turn that

$$\begin{aligned}\|H_0\| &\leq \frac{3}{2}\|F'(x_0)^{-1}F'(x^*)\| \left\| F'(x^*)^{-1}\left(F'\left(x_0 + \frac{2}{3}(y_0 - x_0)\right) - F'(x_0)\right)\right\| \\ &\leq \frac{L\|y_0 - x_0\|^p}{1 - L_0\|x_0 - x^*\|^p} \\ &\leq \frac{LM^p\|x_0 - x^*\|^p}{(1 - L_0\|x_0 - x^*\|^p)^{1+p}}.\end{aligned} \tag{18}$$

Then, in view of (2), (5), (14), (15), (17) and (18), we get that

$$\begin{aligned}\|x_1 - x^*\| &\le \|y_0 - x^*\| + \frac{1}{2}\|H_0\|(1+\|H_0\|)\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\ &\le \frac{L\|x_0 - x^*\|^{1+p}}{(1+p)(1-L_0\|x_0-x^*\|^p)} + \frac{LM^p\|x_0-x^*\|^p}{2(1-L_0\|x_0-x^*\|^p)^{1+p}} \\ &\quad\times\left(1+\frac{LM^p\|x_0-x^*\|^p}{2(1-L_0\|x_0-x^*\|^p)^{1+p}}\right)\frac{M\|x_0-x^*\|}{1-L_0\|x_0-x^*\|^p} \\ &= g_2(\|x_0-x^*\|)\|x_0-x^*\| < \|x_0-x^*\| < r, \end{aligned} \tag{19}$$

which shows (12) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding estimates we arrive at estimates (11) and (12). Then, from the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\| < r$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Finally, to show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$. Then, using (7), we have that

$$\begin{aligned}\|F'(x^*)^{-1}(Q - F'(x^*))\| &\le \frac{L_0}{1+p}\|x^* - y^*\|^p \\ &\le \frac{L_0}{1+p}T^p < 1. \end{aligned} \tag{20}$$

It follows from (20) that Q is invertible. Then, using the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we conclude that $x^* = y^*$. □

Remark 2.2 1. In view of (7) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\le 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le 1 + L_0\|x - x^*\|^p \end{aligned}$$

condition (9) can be dropped and be replaced by

$$M(t) = 1 + L_0 t^p,$$

or

$$M = M(t) = 2,$$

since $t \in [0, (\frac{1}{L_0})^{1/p})$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [1] of the form

$$F'(x) = G(F(x))$$

where T is a continuous operator. Then, since $F'(x^*) = G(F(x^*)) = G(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $G(x) = x + 1$.

3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [1,2].

4. The parameter r_1 was shown by us to be the convergence radius of Newton's method [1,2]
$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{21}$$
under the conditions (6)–(8). It follows from the definitions of radii r that the convergence radius r of the preceding method cannot be larger than the convergence radius r_1 of the second order Newton's method (21). As already noted in [1,2] r_1 is at least as large as the convergence ball given by Rheinboldt [10] (for $p = 1$)
$$r_R = \frac{2}{3L}.$$
In particular, for $L_0 < L$ we have that
$$r_R < r_1$$
and
$$\frac{r_R}{r_1} \to \frac{1}{3} \quad as \quad \frac{L_0}{L} \to 0.$$
That is our convergence ball r_1 is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [11].
5. It is worth noticing that the studied method is not changing when we use the conditions of the preceding Theorem instead of the stronger conditions used in [8,9,11]. Moreover, the preceding Theorem we can compute the computational order of convergence (COC) defined by
$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) \Big/ \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$
or the approximate computational order of convergence
$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) \Big/ \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$
This way we obtain in practice the order of convergence.

Numerical Examples

The numerical examples are presented in this section. In the first four examples, we take $p = 1$.

Example 3.1 Let $D = (-\infty, +\infty)$. Define function f of D by
$$f(x) = \sin(x). \tag{22}$$
Then we have for $x^* = 0$ that $L_0 = L = M = 1$. Then for $p = 1$ the parameters are
$$r_1 = 0.6667, \quad r_2 = 0.2006 = r.$$

Example 3.2 Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0, 1)$, $x^* = (0, 0, 0)^T$. Define function F on D for $w = (x, y, z)^T$ by
$$F(w) = \left(e^x - 1, \quad \frac{e-1}{2}y^2 + y, \quad z\right)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using (7)– (9) conditions, we get $L_0 = e - 1$, $L = e$, $M = 2$. Then for $p = 1$ the parameters are

$$r_1 = 0.3249, \quad r_2 = 0.0672 = r.$$

Example 3.3 Let $X = Y = C[0, 1]$, the space of continuous functions defined on $[0, 1]$ and be equipped with the max norm. Let $D = \overline{U}(0, 1)$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{23}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \quad \text{for each } \xi \in D.$$

Then, we get that $x^* = 0,\ L_0 = 7.5,\ L = 15,\ M = 2$. Then for $p = 1$ the parameters are

$$r_1 = 0.0667, \quad r_2 = 0.0132 = r.$$

Example 3.4 Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073,\ M = 2$. Then for $p = 1$ the parameters are

$$r_1 = 0.0045, \quad r_2 = 0.0010 = r.$$

Example 3.5 Let $X = Y = \mathbb{R}$ and $D = [1, 3]$. Define function F on D by

$$F(x) = \frac{2}{3}x^{3/2} - x.$$

Then, we have $x^* = \frac{9}{4}$, $L_0 = 1$, $L = M = 2$. Then for $p = 1/2$ the parameters are

$$r_1 = 0.5, \quad r_2 = 0.0229 = r.$$

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