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ORIGINAL PAPER

Ball convergence of some iterative methods for nonlinear equations in Banach space under weak conditions

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Abstract The aim of this paper is to expand the applicability of a fast iterative method in a Banach space setting. Moreover, we provide computable radius of convergence, error bounds on the distances involved and a uniqueness of the solution result based on Lipschitz-type functions not given before. Furthermore, we avoid hypotheses on high order derivatives which limit the applicability of the method. Instead, we only use hypotheses on the first derivative. The convegence order is determined using the computational order of convergence or the approximate order of convergence. Numerical examples where earlier results cannot be applied to solve equations but our results can be applied are also given in this study.

Keywords Newton-type method · Radius of convergence · Local convergence · Restricted convergence domains

Mathematics Subject Classification 65D10 · 65D99 · 65J20 · 49M15 · 74G20 · 41A25

1 Introduction

Let $F : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchet-differentiable operator between the Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ and Ω be a convex set. Let $U(a, \rho)$, $\bar{U}(a, \rho)$ stand, respectively for the open and closed balls in $\mathcal{B}_1$ with center $a \in \mathcal{B}_1$ and of radius $\rho > 0$.

In this study, we consider the problem of approximating a solution α^* of nonlinear equation

$$F(x) = 0, \tag{1.1}$$

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using the method considered in [8] for increasing the order of convergence of iterative methods defined for each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} \beta_n &= \alpha_n - F'(\alpha_n)^{-1} F(\alpha_n) \\ \gamma_n &= \varphi(\alpha_n, \beta_n) \\ \alpha_{n+1} &= \gamma_n - F'(\beta_n)^{-1} F(\gamma_n), \end{aligned} \tag{1.2}$$

where $\alpha_0 \in \Omega$ is an initial point, $\varphi : \Omega \times \Omega \longrightarrow \mathcal{B}_1$ is a continuous operator describing a method of convergence order p(p a natural number). In [8], the above method is considered for solving systems of equations, when $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^i$ (i a natural number). Using Taylor expansion and the assumptions on derivatives of order up to four of F, Cordero et al. [8] proved the convergence order of method (1.2) is $p + 2$. The assumptions on the higher order Fréchet derivatives of the operator F restricts the applicability of method (1.2). For example consider the following:

Example 1.1 Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0, 1], \Omega = \bar{U}(\alpha^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [1,2,6–9,12] defined by

$$x(s) = \int_0^1 G(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right) dt,$$

where the kernel G is the Green's function defined on the interval $[0, 1] \times [0, 1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \le s \\ s(1-t), & s \le t. \end{cases}$$

The solution $\alpha^*(s) = 0$ is the same as the solution of Eq. (1.1), where $F : C[0, 1] \longrightarrow C[0, 1]$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right) dt.$$

Notice that

$$\left\| \int_0^1 G(s,t) dt \right\| \le \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)\left(\frac{3}{2}x(t)^{1/2} + x(t)\right) dt,$$

so since $F'(\alpha^*(s)) = I$,

$$\|F'(\alpha^*)^{-1}(F'(x) - F'(y))\| \le \frac{1}{8}\left(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|\right).$$

One can see that, higher order derivatives of F do not exist in this example.

Our goal is to weaken the assumptions in [8] and apply the method for solving equation (1.1) in Banach spaces, so that the applicability of the method (1.2) can be extended using Lipschitz-type functions. The study of the local convergence is important because it provides the degree of difficulty for choosing initial points. Notice that in the studies using Taylor expansions and high order derivatives the choice of the initial point is a shot in the dark. The technique introduced in this paper can be used on other iterative methods [1–18].

The rest of the paper is organized as follows. In Sect. 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and a uniqueness result. Special cases and numerical examples are given in the last section.

2 Local convergence analysis

Let $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ be a continuous and non-decreasing function with $w_0(0) = 0$. Define the parameter ρ_0 by

$$\rho_0 = \sup\{t \geq 0 : w_0(t) < 1\}. \tag{2.1}$$

Let $w : [0, \rho_0) \longrightarrow [0, +\infty)$, $v : [0, \rho_0) \longrightarrow [0, +\infty)$ be also continuous and non-decreasing functions with $w(0) = 0$. Define functions g_1 and h_1 on the interval $[0, \rho_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)},$$
$$h_1(t) = g_1(t) - 1.$$

We have that $h_1(0) = -1 < 0$ and $h_1(t) \longrightarrow +\infty$ as $t \longrightarrow \rho_0^-$. As a consequence of intermediate value theorem, the function h_1 has zeros in the interval $(0, \rho_0)$. Denote by ρ_1 the smallest such zero.
$(\mathcal{C})$ Suppose that there exist $\alpha, \beta \in [0, \rho_0)$ with $\alpha < \beta$ and a function $g_2 : [0, \rho_0) \longrightarrow [0, +\infty)$ such that for $h_2(t) = g_2(t) - 1$, $h_2(\alpha) < 0$ and $h_2(\beta) \longrightarrow +\infty$ or a positive number as $\beta \longrightarrow \rho_0^-$.

Notice that under condition $(\mathcal{C})$ function h_2 has zeros in the interval $(\alpha, \beta) \subseteq (0, \rho_0)$. Denote by ρ_2 smallest such zero. Define parameter $\bar{\rho}_0$ by

$$\bar{\rho}_0 = \max\{t \in [0, \rho_0] : g_1(t)t < 1\}. \tag{2.2}$$

Define functions g_3 and h_3 on the interval $[0, \bar{\rho}_0)$ by

$$g_3(t) = \left(1 + \frac{\int_0^1 v(\theta g_2(t)t)d\theta}{1 - w_0(g_1(t)t)}\right) g_2(t)$$

and

$$h_3(t) = g_3(t) - 1.$$

$(\mathcal{H})$ Suppose that there exist $\bar{\alpha}, \bar{\beta} \in [0, \bar{\rho}_0)$ with $\bar{\alpha} < \bar{\beta}$ such that $h_3(\bar{\alpha}) < 0$ and $h_3(\bar{\beta}) \longrightarrow +\infty$ or a positive number as $\bar{\beta} \longrightarrow \bar{\rho}_0^-$. Denote by ρ_3 the smallest zero of function h_3 on the interval $(\bar{\alpha}, \bar{\beta})$. Notice that condition $(\mathcal{H})$ implies condition $(\mathcal{C})$. Define the radius of convergence ρ by

$$\rho = \min\{\rho_i\}, \quad i = 1, 2, 3. \tag{2.3}$$

Then, for each $t \in [0, \rho)$

$$0 \leq g_i(t) < 1, \quad i = 1, 2, 3 \tag{2.4}$$

and

$$0 \leq g_1(t)t < 1. \tag{2.5}$$

The local convergence analysis of method (1.2) that follows is based on the preceding notations.

Theorem 2.1 *Let $F : \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a continuously Fréchet-differentiable operator. Let $\varphi : \Omega \times \Omega \longrightarrow \mathcal{B}_1$ be a continuous operator. Suppose there exist $\alpha^* \in \Omega$ and function $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ continuous and non-decreasing with $w_0(0) = 0$ such that for each $\alpha \in \Omega$*

$$F(\alpha^*) = 0, \quad F'(\alpha^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \tag{2.6}$$

and

$$\|F'(\alpha^*)^{-1}(F'(\alpha) - F'(\alpha^*)\| \leq w_0(\|\alpha - \alpha^*\|); \tag{2.7}$$

Let $w, v : [0, \rho_0) \longrightarrow [0, +\infty)$ be continuous and non-decreasing with $w(0) = 0$. Let $\Omega_0 = \Omega \cap U(\alpha^*, \rho_0)$. For each $\alpha, \beta \in \Omega_0$

$$\|F'(\alpha^*)^{-1}(F'(\alpha) - F'(\beta)\| \leq w(\|\alpha - \beta\|), \tag{2.8}$$

$$\|F'(\alpha^*)^{-1}F'(\alpha)\| \leq v(\|\alpha - y\|), \tag{2.9}$$

For each $\alpha \in \Omega$ and $\beta = \alpha - F'(\alpha)^{-1}F(\alpha)$

$$\|\varphi(\alpha, \beta) - \alpha^*\| \leq g_2(\|\alpha - \alpha^*\|)\|\alpha - \alpha^*\| \tag{2.10}$$

$$\bar{U}(\alpha^*, \rho) \subseteq \Omega \tag{2.11}$$

and condition ($\mathcal{H}$) holds where radius $\rho_0, \bar{\rho}_0$ and ρ are given in (2.1), (2.2) and (2.3), respectively. Then, the sequence $\{\alpha_n\}$ generated for $\alpha_0 \in U(\alpha^*, \rho) - \{\alpha^*\}$ by method (1.2) is well defined in $U(\alpha^*, \rho)$, remains in $U(\alpha^*, \rho)$ for each $n = 0, 1, 2, \ldots$ and converges to α^*. Moreover, the following estimates hold

$$\|\beta_n - \alpha^*\| \leq g_1(\|\alpha_n - \alpha^*\|)\|\alpha_n - \alpha^*\| \leq \|\alpha_n - \alpha^*\| < \rho, \tag{2.12}$$

$$\|\gamma_n - \alpha^*\| \leq g_2(\|\alpha_n - \alpha^*\|)\|\alpha_n - \alpha^*\| \leq \|\alpha_n - \alpha^*\| \tag{2.13}$$

and

$$\|\alpha_{n+1} - \alpha^*\| \leq g_3(\|\alpha_n - \alpha^*\|)\|\alpha_n - \alpha^*\| \leq \|\alpha_n - \alpha^*\|, \tag{2.14}$$

where the functions g_i, $i = 1, 2, 3$ are defined previously. Furthermore, if there exists $\rho^* \geq \rho$ such that

$$\int_0^1 w_0(\theta\rho^*)d\theta < 1, \tag{2.15}$$

then the limit point α^* is the only solution of equation $F(x) = 0$ in $\Omega_1 = \Omega \cap \bar{U}(\alpha^*, \rho^*)$.

Proof We shall show that the estimates (2.12)–(2.14) hold using mathematical induction on the integer k. Let $\alpha \in U(\alpha^*, \rho) - \{\alpha^*\}$. Then using (2.6) and (2.7), we have that

$$\|F'(\alpha^*)^{-1}(F'(\alpha) - F'(\alpha^*))\| \leq w_0(\|\alpha - \alpha^*\|) \leq w_0(\rho) < 1. \tag{2.16}$$

It follows from (2.16) and the Banach Lemma on invertible operators [2,4,16] that $F'(\alpha)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(\alpha)^{-1}F'(\alpha^*)\| \leq \frac{1}{1 - w_0(\|\alpha - \alpha^*\|)}. \tag{2.17}$$

In particular iterate β_0 is well defined by the first substep of method (1.2) for $n = 0$. We can write by (2.6) and (1.2) that

$$\begin{aligned}\beta_0 - \alpha^* &= \alpha_0 - \alpha^* - F'(\alpha_0)^{-1}F(\alpha_0) \\ &= -F'(\alpha_0)^{-1}\int_0^1 [F'(\alpha^* + \theta(\alpha_0 - \alpha^*)) - F'(\alpha_0)](\alpha_0 - \alpha^*)d\theta.\end{aligned} \tag{2.18}$$

Using (2.3), (2.4) (for $i = 1$), (2.8), (2.17) (for $a = a_0$) and (2.18), we get in turn that

$$\begin{aligned}\|\beta_0 - \alpha^*\| &\le \|F'(\alpha_0)^{-1}F'(\alpha^*)\| \\ &\quad \times \left\| \int_0^1 F'(\alpha^*)^{-1}(F'(\alpha^* + \theta(\alpha_0 - \alpha^*)) - F'(\alpha_0))(\alpha_0 - \alpha^*)d\theta \right\| \\ &\le \frac{\int_0^1 w((1-\theta)\|\alpha_0 - \alpha^*\|)d\theta\|\alpha_0 - \alpha^*\|}{1 - w_0(\|\alpha_0 - \alpha^*\|)} \\ &= g_1(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\| \le \|\alpha_0 - \alpha^*\| < \rho, \end{aligned} \tag{2.19}$$

which shows (2.12) for $n = 0$ and $\beta_0 \in U(\alpha^*, \rho)$. It follows that γ_0 is well defined by second substep of method (1.2) for $n = 0$. Using (2.3), (2.4) (for $i = 2$) and (2.10), we get in turn that

$$\begin{aligned}\|\gamma_0 - \alpha^*\| &= \|\varphi(\alpha_0, \beta_0) - \alpha^*\| \\ &\le g_2(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\| \le \|\alpha_0 - \alpha^*\| < \rho, \end{aligned} \tag{2.20}$$

so (2.13) holds for $n = 0$ and $\gamma_0 \in U(\alpha^*, \rho)$. It follows from (2.17) for $\alpha_0 = \beta_0$ and (2.19) that $F'(\beta_0)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\begin{aligned}\|F'(\beta_0)^{-1}F'(\alpha^*)\| &\le \frac{1}{1 - w_0(\|\beta_0 - \alpha^*\|)} \\ &\le \frac{1}{1 - w_0(g_1(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\|)}. \end{aligned} \tag{2.21}$$

We also have α_1 is well defined by (2.21) and the third substep of method (1.2) for $n = 0$. Then by method (1.2) for $n = 0$, we have that

$$\|\alpha_1 - \alpha^*\| \le \|\gamma_0 - \alpha^*\| + \|F'(\beta_0)^{-1}F'(\alpha^*)\|\|F'(\alpha^*)^{-1}F(\gamma_0)\|. \tag{2.22}$$

In view of (2.6) we can write

$$F(\gamma_0) = F(\gamma_0) - F(\alpha^*) = \int_0^1 F'(\alpha^* + \theta(\gamma_0 - \alpha^*))d\theta(\gamma_0 - \alpha^*), \tag{2.23}$$

so since $\|\alpha^* + \theta(\alpha_0 - \alpha^*) - \alpha^*\| = \theta\|\alpha_0 - \alpha^*\| < \rho$, (i.e., $\alpha^* + \theta(\alpha_0 - \alpha^*) \in U(\alpha^*, \rho)$ for each $\theta \in [0, 1]$), (2.9) and (2.20) gives (2.23) and

$$\begin{aligned}\|F'(\alpha^*)^{-1}F'(\gamma_0)\| &\le \int_0^1 v(\theta\|\gamma_0 - \alpha^*\|)d\theta\|\gamma_0 - \alpha^*\| \\ &\le \int_0^1 v(g_2(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\|)g_2(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\|. \end{aligned} \tag{2.24}$$

Next, from (2.4) (for $n = 0$), (2.20)–(2.24), we get in turn that

$$\begin{aligned}\|\alpha_1 - \alpha^*\| &= \left(1 + \frac{\int_0^1 v(\theta g_2(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\|)d\theta}{1 - w_0(g_1(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\|)}\right) \\ &\quad \times g_2(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\| \qquad (2.25) \\ &= g_3(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\| \le \|\alpha_0 - \alpha^*\| < \rho, \qquad (2.26) \end{aligned}$$

which shows (2.14) for $n = 0$ and $\alpha_1 \in U(\alpha^*, \rho)$. The induction for estimates (2.12)–(2.14) is completed in an analogous way, if we simply replace $\alpha_0, \beta_0, \gamma_0, \alpha_1$ by $\alpha_k, \beta_k, \gamma_k, \alpha_{k+1}$ in the preceding estimates. Then, from the estimates

$$\|\alpha_{k+1} - \alpha^*\| \leq c\|\alpha_k - \alpha^*\| < r, \tag{2.27}$$

where $c = g_3(\|\alpha_0 - \alpha^*\|) \in [0, 1)$, we deduce that $\lim_{k\to\infty} \alpha_k = \alpha^*$ and $\alpha_{k+1} \in U(\alpha^*, \rho)$. Finally, to show the uniqueness part, let $\beta^* \in \Omega_2$ with $F(\beta^*) = 0$. Define $T = \int_0^1 F'(\alpha^* + \theta(\beta^* - \alpha^*))d\theta$. Then, using (2.7), we obtain that

$$\begin{aligned}\|F'(\alpha^*)^{-1}(T - F'(\alpha^*))\| &\leq \int_0^1 w_0(\theta\|\alpha^* - \beta^*\|)d\theta \\ &\leq \int_0^1 w_0(\theta\rho^*)d\theta < 1.\end{aligned} \tag{2.28}$$

Hence, we have that $T^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Then, from the identity $0 = F(\beta^*) - F(\alpha^*) = T(\beta^* - \alpha^*)$, we conclude that $\alpha^* = \beta^*$. □

Remark 2.2 (a) In the case when $w_0(t) = L_0 t$, $w(t) = Lt$ and $\Omega_0 = \Omega$, the radius $\rho_A = \frac{2}{2L_0+L}$ was obtained by Argyros in [2] as the convergence radius for Newton's method under condition (2.7)–(2.9). Notice that the convergence radius for Newton's method given independently by Rheinboldt [16] and Traub [18] is given by

$$\rho_{TR} = \frac{2}{3L} < \rho_A. \tag{2.29}$$

As an example, let us consider the function $F(x) = e^x - 1$. Then $\alpha^* = 0$. Set $\Omega = B(0, 1)$. Then, we have that $L_0 = e - 1 < L = e$, so $\rho_{TR} = 0.24252961 < \rho_A = 0.324947231$. Moreover, the new error bounds [2] are:

$$\|\alpha_{n+1} - \alpha^*\| \leq \frac{L}{1 - L_0\|\alpha_n - \alpha^*\|}\|\alpha_n - \alpha^*\|^2,$$

whereas the old ones [5,7]

$$\|\alpha_{n+1} - \alpha^*\| \leq \frac{L}{1 - L\|\alpha_n - \alpha^*\|}\|\alpha_n - \alpha^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, the radius of convergence of method (1.2) given by ρ cannot be be larger than ρ_A.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [1–5].

(c) The results can be also be used to solve equations where the operator F' satisfies the autonomous differential equation [2–4]:

$$F'(x) = P(F(x)),$$

where $P : \mathcal{B}_2 \longrightarrow \mathcal{B}_2$ is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) are not changing if we use the new instead of the old conditions [7]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{ln\frac{\|\alpha_{n+2}-\alpha^*\|}{\|\alpha_{n+1}-\alpha^*\|}}{ln\frac{\|\alpha_{n+1}-\alpha^*\|}{\|\alpha_n-\alpha^*\|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{ln \frac{\|\alpha_{n+2}-\alpha_{n+1}\|}{\|\alpha_{n+1}-\alpha_n\|}}{ln \frac{\|\alpha_{n+1}-\alpha_n\|}{\|\alpha_n-\alpha_{n-1}\|}}, \quad \text{for each } \; n = 1, 2, \ldots.$$

(e) In view of (2.4) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + w_0(\|x - x^*\|)\end{aligned}$$

condition (2.6) can be dropped and can be replaced by

$$v(t) = 1 + w_0(t)$$

or

$$v(t) = 1 + w_0(\rho_0),$$

since $t \in [0, \rho_0)$.

(f) Let us choose $\alpha = 1$ and $\varphi(x, y) = y - F'(y)^{-1}F(y)$. Then, we have in (2.22) with α_k replaced by β_k

$$\|\varphi(\alpha_k, \beta_k) - \alpha^*\| \leq \frac{\int_0^1 w((1-\theta)\|\beta_k - \alpha^*\|)d\theta\|\beta_k - \alpha^*\|}{1 - w_0(g_1(\|\alpha_k - \alpha^*\|)\|\alpha_k - \alpha^*\|)},$$

so we can choose $p = 1$ and

$$\psi(t) = \frac{\int_0^1 w((1-\theta)g_1(t)t)d\theta g_1(t)}{1 - w_0(t)}.$$

(g) Let us consider two special choices for φ leading to an extension of Frontini and Sormani method denoted by (M_5) as well as the Cordero method denoted by (M_6) [7]:
Case (M_5). Let

$$\begin{aligned}&\varphi(\alpha_n, \beta_n) = \beta_n - F'(\alpha_n)^{-1}[2I - F'(\beta_n)F'(\alpha_n)^{-1}]F(\beta_n) \\ &\|\varphi(\alpha_n, \beta_n) - \alpha^*\| \leq \|\beta_n - \alpha^*\| \\ &\quad + \frac{2\|w_0(\|\alpha_n - \alpha^*\|) + w_0(\|\beta_n - \alpha^*\|) + 1] \int_0^1 v(\theta\|\beta_n - \alpha^*\|)d\theta\|\beta_n - \alpha^*\|}{(1 - w_0(\|\alpha_n - \alpha^*\|))^2} \\ &\leq \left(1 + \frac{(2\|w_0(t) + w_0(g_1(t)t) + 1) \int_0^1 v(\theta g_1(t)t)d\theta}{(1 - w_0(t)^2)}\right) g_1(t)t\end{aligned}$$

so for $\|\alpha_n - \alpha^*\| \leq t$, we can choose

$$g_2(t) = \left(1 + \frac{(2\|w_0(t) + w_0(g_1(t)t) + 1) \int_0^1 v(\theta g_1(t)t)d\theta}{(1 - w_0(t))^2}\right) g_1(t).$$

Case (M_6). Let

$$\begin{aligned}\varphi(\alpha_n, \beta_n) &= \alpha_n - 2(F'(\beta_n) + F'(\alpha_n))^{-1}]F(\alpha_n) \\ \varphi(\alpha_n, \beta_n) - \alpha^* &= \alpha_n - \alpha^* - 2(F'(\beta_n) + F'(\alpha_n))^{-1}F(\alpha_n) \\ &= \alpha_n - \alpha^* - F'(\alpha_n)^{-1}F(\alpha_n) + F'(\alpha_n)^{-1}F(\alpha_n) - 2(F'(\beta_n) + F'(\alpha_n))^{-1} \\ &\quad -F'(\alpha_n) + F'(\alpha^*))F(\alpha_n) = \alpha_n - \alpha^* - F'(\alpha_n)^{-1}F(\alpha_n) \\ &\quad +F'(\alpha_n)^{-1}[F(\alpha_n) + F'(\beta_n) - 2F'(\beta_n))(F'(\beta_n) + F'(\alpha_n))^{-1}F(\alpha_n).\end{aligned}$$

So

$$\|\varphi(\alpha_n,\beta_n)-\alpha^*\| \leq g_1(\|\alpha_n-\alpha^*\|)\|\alpha_n-\alpha^*\| + \frac{(\|w_0(\|\beta_n-\alpha^*\|)+w_0(\|\alpha_n-\alpha^*\|))\int_0^1 v(\theta\|\alpha_n-\alpha^*\|)d\theta\|\alpha_n-\alpha^*\|}{2(1-w_0(\|\alpha_n-\alpha^*\|))(1-p(\|\alpha_n-\alpha^*\|))}$$

so for $\|\alpha_n-\alpha^*\|\leq t$, we can choose

$$g_2(t)=g_1(t)+\frac{(w_0(t)+w_0(g_1(t)t))\int_0^1 v(\theta t)d\theta}{2(1-w_0(t))(1-p(t))}.$$

3 Numerical examples

We present two examples in this section. We choose φ as in Remark 2.2 (g) in both examples.

Example 3.1 Let $\mathcal{B}_1=\mathcal{B}_2=C[0,1]$, $D=\bar{U}(0,1)$. Define function F on D by

$$F(\varphi)(x)=\varphi(x)-5\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

Then, the Fréchet-derivative is given by

$$F'(\varphi(\xi))(x)=\xi(x)-15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta,\quad \text{for each } \xi\in D.$$

Then, we have that $x^*=0$, $w_0(t)=L_0t$, $w(t)=Lt$, $v(t)=2$, $L_0=7.5<L=15$. Then, the radius of convergence ρ is given by

$$\rho_1=0.0667,\ \rho_2=0.0045=\rho,\ \rho_3=0.0050.$$

Example 3.2 Returning back to the motivational example given at the introduction of this study, we can choose (see also Remark 2.2 (5) for function v) $w_0(t)=w(t)=\frac{1}{8}(\frac{3}{2}\sqrt{t}+t)$ and $v(t)=1+w_0(\rho_0)$, $\rho_0\backsimeq 4.7354$. Then, the radius of convergence ρ is given by

$$\rho_1=3.5303,\ \rho_2=2.0048,\ \rho_3=1=\rho.$$

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