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ASYMPTOTIC SOLUTION FOR STEADY STREAMING MHD FLOW IN A CHANNEL OF VARIABLE CROSS SECTION

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Abstract—The oscillatory flow of a homogeneous, conducting, viscous fluid in a channel of varying cross section is considered. The solutions for large Womersley number α are obtained by the method of matched asymptotic expansions, supplementing the earlier work of Ramachandra Rao and Deshikachar. It is observed that the magnetic field provides an appropriate length scale necessary for matching the solutions and thus resolves the difficulty arising in the corresponding non-magnetic case.

1. INTRODUCTION

The importance of steady and oscillatory flows of viscous fluid through channels of variable cross section and their applications to physiological fluid dynamics has drawn the attention of many research workers. The purely oscillatory viscous flows over curved boundaries exhibit a steady streaming component due to nonlinearity of the governing equations. This phenomena of steady streaming is of great interest and has been discussed at length by Schlichting [1], Batchelor [2] and Telionis [3]. Following Benjamin [4], Duck [5] has studied the oscillatory viscous flow in a channel or pipe with slightly perturbed walls. In the above problems, the oscillatory pressure gradient was prescribed. Whereas Chow and Soda [6] have obtained the hydrodynamic solution for a laminar flow in a plane asymmetric channel with the assumption that the spread of surfaceness is large compared to the half mean width of the channel, by prescribing the volume flux. The corresponding magnetohydrodynamic (MHD) steady and oscillatory flows have been studied by Deshikachar and Ramachandra Rao [7] and Ramachandra Rao and Deshikachar [8]. Following Stuart [9], Grotberg [10] has obtained an asymptotic solution, for the oscillatory flow in a tapered channel, for large Womersley number α. The difficulties in matching the solutions have been removed by him [10], by introducing a steady drift layer that is thicker than the Stokes layer.

Ramachandra Rao and Deshikachar [8] have presented a solution for the oscillatory flow in a variable channel for small values of α, in the presence of a transverse magnetic field. In this paper, we obtain the solutions for large values of α, by the method of matched asymptotic expansions, supplementing the earlier work. It is interesting to note that there is no difficulty in matching the solutions, whereas in the corresponding non-magnetic case the asymptotic method breaks down. The matching here is possible essentially due to the magnetic field providing another necessary appropriate length scale.

2. MATHEMATICAL FORMULATION

Consider the two-dimensional flow of an incompressible, viscous, homogeneous, conducting fluid in a channel with the walls given by

$$y' = \eta_1(x')d = d + a_1(x'), \tag{1}$$

$$y' = \eta_2(x')d = d + a_2(x'), \tag{2}$$

where x'-axis is along the channel, y'-axis is perpendicular to it, the functions $a_1(x')$ and $a_2(x')$ give the slow variations of the walls and d is the half-mean width of the channel.

The equations of motion under the usual MHD approximations in terms of stream function ψ defined by

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{3}$$

are given by

$$\frac{\partial^2\psi'}{\partial t'\,\partial y'}+\frac{\partial\psi'}{\partial y'}\frac{\partial^2\psi'}{\partial x'\,\partial y'}-\frac{\partial\psi'}{\partial x'}\frac{\partial^2\psi'}{\partial y'^2}=-\frac{1}{\rho}\frac{\partial p'}{\partial x'}+\nu\left(\frac{\partial^3\psi'}{\partial x'^2\,\partial y'}+\frac{\partial^3\psi'}{\partial y'^2}\right)-\frac{\sigma B_0^2}{\rho}\frac{\partial\psi'}{\partial y'},\tag{4}$$

$$\frac{\partial^2\psi'}{\partial t'\,\partial x'}+\frac{\partial\psi'}{\partial y'}\frac{\partial^2\psi'}{\partial x'^2}-\frac{\partial\psi'}{\partial x'}\frac{\partial^2\psi'}{\partial x'\,\partial y'}=-\frac{1}{\rho}\frac{\partial p'}{\partial y'}+\nu\left(\frac{\partial^3\psi'}{\partial x'\,\partial y'^2}+\frac{\partial^3\psi}{\partial x'^3}\right),\tag{5}$$

where ρ is the density, ν is the coefficient of viscosity, σ is the electrical conductivity and B_0 is the uniform magnetic field applied in the y'-direction.

The boundary conditions on the walls are given by

$$\frac{\partial\psi'}{\partial y'}=0=\frac{\partial\psi'}{\partial x'}\quad\text{on}\quad y'=\eta_i d,\qquad i=1,2,\tag{6}$$

and further the volume flux oscillating with frequency ω is prescribed by

$$\int_{\eta_1 d}^{\eta_2 d}\frac{\partial\psi'}{\partial y'}\,\mathrm{d}y'=Q'\cos(\omega t'),\qquad\text{for all } x'.\tag{7}$$

Introducing the non-dimensional variables

$$\psi=\psi'/u_0 d,\qquad x=x'/\lambda,\qquad y=y'/d,\qquad t=\omega t',\qquad Q=Q'/u_0 d,\tag{8}$$

where λ is the characteristic length along the channel, eqns (4) and (5) after the elimination of pressure term, reduce to

$$\alpha^2\nabla^2\frac{\partial\psi}{\partial t}+\mathrm{Re}\,\delta\left(\frac{\partial\psi}{\partial y}\nabla^2\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial x}\nabla^2\frac{\partial\psi}{\partial y}\right)=\nabla^4\psi-H^2\frac{\partial^2\psi}{\partial y^2}.\tag{9}$$

The corresponding boundary conditions are

$$\frac{\partial\psi}{\partial y}=0=\frac{\partial\psi}{\partial x}\quad\text{on}\quad y=\eta_i,\qquad i=1,2,\tag{10}$$

$$\int_{\eta_1}^{\eta_2}\frac{\partial\psi}{\partial y}\,\mathrm{d}y=Q\quad\cos\quad t,\qquad\text{for all } x,\tag{11}$$

where $\alpha^2=\omega d^2/\nu$ is the Womersley number, $\mathrm{Re}=u_0 d/\nu$ is the Reynolds number, $H^2=B_0^2 d^2\sigma/\rho\nu$ is the Hartmann number, $\delta=d/\lambda$ $(d\ll\lambda)$ and $\nabla^2=\delta^2(\partial^2/\partial x^2)+(\partial^2/\partial y^2)$. The solution of (9) satisfying conditions (10) and (11) has been presented by Ramachandra Rao and Deshikachar [8], for small values of α.

3. SOLUTION FOR LARGE α

When α is large, by taking the limit of (H^2/α^2) as $\alpha\to\infty$ in eqn (9), we arrive at two different cases. In the inviscid core region, first, if we allow both H^2 and α^2 to vary as $\alpha\to\infty$, $(H^2/\alpha^2)\to q^2(=B_0^2\sigma/\rho\nu)$, a finite nonzero value that is independent of viscosity ν. In this case we have, in addition to the length scale of the Stokes layer, a second length scale due to the magnetic field affected drift layer. On the other hand, if H^2 is fixed then $(H^2/\alpha^2)\to 0$ as $\alpha\to\infty$. In this case, the magnetic field has no effect on the drift layer and its behaviour is same as that in the non-magnetic case, discussed by Grotberg [10].

Case (i): Let $(H^2/\alpha^2)\to q^2$ *as* $\alpha\to\infty$

The governing equation, (9), in this case, neglecting higher orders in δ, reduces to

$$\frac{\partial^3\psi}{\partial y^2\,\partial t}+q^2\frac{\partial^2\psi}{\partial y^2}+\beta\left(\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial y^2\,\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}\right)=0,\tag{12}$$

where $\beta=\mathrm{Re}\,\delta/\alpha^2$ and this equation corresponds to the inviscid equation. Now we expand ψ

in (12) in powers of $\beta(\ll 1)$ as

$$\psi = \frac{1}{2}\psi_0 e^{it} + \text{c.c.} + \beta\left(\frac{1}{2}\psi_1 e^{2it} + \text{c.c.} + \psi_1^s\right) + O(\beta^2), \tag{13}$$

where ψ_0 and ψ_1 correspond to the first and second harmonics and ψ_1^s correspond to the steady streaming (Telionis [3]), and the abbreviation c.c. denotes the complex conjugate of the preceding term. Using the expansion of ψ given by (13) in (12) and solving the corresponding equations in the transformed coordinates,

$$X = x, \qquad Y = y/\eta + \xi, \tag{14}$$

where $\eta = \eta_1 - \eta_2/2$, $\xi = (\eta_1 + \eta_2)/(\eta_2 - \eta_1)$, which transforms the boundary walls $y = \eta_i$ $(i = 1, 2)$ into $Y = \pm 1$, we obtain

$$\psi = \frac{1}{2}[\eta Y D_0(x)e^{it} + \beta(\eta Y D_1(x)e^{2it} + \eta Y D_2(X)] + \text{c.c.} + O(\beta^2), \tag{15}$$

where $D_0(x)$ and $D_2(x)$ will be determined by using the matching procedure. By expanding the pressure gradient as

$$\frac{\partial p}{\partial x} = \frac{1}{\beta}\left(\frac{\partial p_0}{\partial x} + \beta\frac{\partial p_1}{\partial x} + O(\beta^2)\right) + O(1/\alpha), \tag{16}$$

the momentum equations in (4) and (5) to the above order in transformed coordinates give the following equations:

$$\frac{\partial p_0}{\partial Y} = 0, \tag{17}$$

$$\frac{\partial p_0}{\partial X} = -\gamma^2 D_0 e^{it} + \text{c.c.}, \tag{18}$$

$$\frac{\partial p_1}{\partial X} = -D_0 D_0^{*\prime} - q^2 D_2 + \text{c.c.}, \tag{19}$$

where $\gamma^2 = i + q^2$, "*" denotes the complex conjugate and prime denotes derivative with respect to X. The unsteady component of p_1 is omitted here as we are interested only in the steady streaming. In order to examine the boundary layer or the inner region, we use the transformation of variables, given by

$$\xi_1 = X, \qquad \varphi_1 = \alpha\eta(1 - Y). \tag{20}$$

Now using $\phi(\xi_1, \varphi_1, t) = \alpha\psi(x, y, t)$ in (4), we obtain, for large α,

$$\frac{\partial^2\phi}{\partial\varphi_1\,\partial t} + q^2\frac{\partial\phi}{\partial\varphi_1} - \frac{\partial^3\phi}{\partial\varphi_1^3} = \left(\frac{\partial p_0}{\partial\xi_1} + \beta\frac{\partial p_1}{\partial\xi_1}\right) + \beta\left(\frac{\partial\phi}{\partial\zeta_1}\frac{\partial^2\phi}{\partial\xi_1\,\partial\varphi_1} - \frac{\partial\phi}{\partial\xi_1}\frac{\partial^2\phi}{\partial\varphi_1^2}\right) + O(\beta^2), \tag{21}$$

The boundary conditions at the wall reduce to

$$\frac{\partial\phi}{\partial\varphi_1} = 0, \qquad \phi = \alpha e^{it} + \text{c.c.} \quad \text{on } \varphi_1 = 0. \tag{22}$$

Expanding ϕ in powers of β by

$$\phi = \phi_0 + \beta\phi_1 + O(\beta^2) + O(1/\alpha), \tag{23}$$

and using (23) in (21), the leading term ϕ_0 satisfying the corresponding boundary conditions is obtained as

$$\phi_0 = [D_0(1 - e^{-\gamma\varphi_1} - \gamma\varphi_1)/\gamma + \alpha]e^{it} + \text{c.c.} \tag{24}$$

Using the matching conditions at zeroth order for both X and Y components of velocity field, one gets $D_0 = 1/\eta$. The solution of velocity field for the entire flow region to the leading order is obtained from (3) and (24) and is given by

$$u_0 = (1 - e^{-\gamma\varphi_1})e^{it}/\eta + \text{c.c.}, \tag{25}$$

$$v_0 = (Y - e^{-\gamma\varphi_1})\eta' e^{it}/\eta + \text{c.c.} \tag{26}$$

The steady streaming component of ϕ_1 at β order given by (21) and (23) is obtained as

$$\phi_1^s = [-D_2(\xi_1)\varphi_1 + A + Be^{-q\varphi_1} + (\eta'/\eta^3)F(\varphi_1)] + \text{c.c.}, \tag{27}$$

where

$$F(\varphi_1) = \frac{(\gamma - \gamma^*)e^{-(\gamma-\gamma^*)\varphi_1}}{\gamma^*(\gamma+\gamma^*)[(\gamma+\gamma^*)^2 - q^2]} + \frac{e^{-\gamma^*\varphi_1}}{\gamma^*(\gamma^{*2} - q^2)} + \frac{[2\gamma^*(2\gamma^2 - q^2) - \gamma(\gamma^2 - q^2)(1 - \gamma^*\varphi_1)]}{\gamma\gamma^*(\gamma^2 - q^2)^2} e^{-\gamma\varphi_1}, \tag{28}$$

A and B are determined by using the boundary conditions (22). Whereas $D_2(\xi_1)$ is determined by the corresponding matching condition and it is given by

$$D_2 = \eta'(iq - \gamma_2)/2q\eta^3, \tag{29}$$

where

$$\gamma_2 = \text{Im}(\gamma) = [\{q^4 + 1\}^{\frac{1}{2}} - q^2\}/2]^{\frac{1}{2}}. \tag{30}$$

Thus the steady state component of the axial velocity is given by

$$u_1^s = -\frac{\partial\phi_1^s}{\partial\varphi_1} = \frac{\eta'}{\eta^3}[(iq - \gamma_2)(1 - e^{-q\varphi_1})/2q - (G(\varphi_1) - G(0)e^{-q\varphi_1})] + \text{c.c.}, \tag{31}$$

where $G(\varphi_1) = \partial F/\partial\varphi_1$.

Here it is interesting to observe that there is no difficulty in using the asymptotic method to match the solutions and obtain the value of $D_2(\xi_1)$, whereas in the corresponding non-magnetic case this asymptotic method breaks down. The matching in this case is possible, essentially due to the magnetic field providing another necessary appropriate length scale.

Case (ii): Let $H^2/\alpha^2 \to 0$ as $\alpha \to \infty$

In this case eqn (9) reduces to

$$\frac{\partial^3\psi}{\partial y^2\,\partial t} + \beta\left(\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x\,\partial y^2} - \frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}\right) = 0, \tag{32}$$

and this corresponds to the case without magnetic field. Equation (32) and its solution for the oscillatory flow in a tapered channel has been discussed by Grotberg's, which is applicable to a channel of any variable cross section and hence a more general one.

By following the same procedure as in the case case (i), we obtain the zeroth order velocity components u_0 and v_0 given by (25) and (26), respectively, with γ replaced $\gamma_1 = 1 + i/\sqrt{2}$. Proceeding to the first order, we obtain the steady component of ϕ_1 as

$$\phi_1^s = \frac{\eta'}{\eta^3}\left[\frac{1}{2}(1 - i)e^{-2\varphi_1} + (5 + i(3 + 2\varphi_1)e^{-\gamma_1\varphi_1} + (1 - i)e^{-\gamma_1\varphi_1} + A_1(\xi_1) + B_1(\xi_1)\varphi_1 + C_1(\xi_1)\varphi_1^2\right] + \text{c.c.} \tag{33}$$

In (33), A_1 and B_1 can be determined by using the boundary conditions (22) but C_1 cannot be determined and the asymptotic method fails here for fixed amplitude oscillations. In the limit $\varphi_1 \to \infty$, we should be able to match the Stokes layer velocities to the outer region. Following Stuart [9], we choose the coefficient of φ_1^2(i.e. C_1) as zero, leaving only the singular term with φ_1. The linear term in φ_1 gives the following steady velocity components which must be matched to the steady drift $D_2(X)$ in the inviscid core:

$$\lim_{\varphi_1\to\infty} u_1^s = -\lim_{\varphi_1\to\infty}\left(\frac{\partial\phi_1^s}{\partial\varphi_1} = -B_1(\xi_1) = \frac{3\eta'}{\eta^3}\right), \tag{34}$$

$$\lim_{\varphi_1\to\infty} = -\lim_{\varphi_1\to\infty}\left(\frac{\partial\phi_1^3}{\partial\eta_1} = \frac{3\eta'^2}{\eta^3}\right). \tag{35}$$

Direct matching is not possible here because the length scale given by the dimensionless Stokes layer $\delta_s = (\nu/\omega d^2)^{\frac{1}{2}}$ is not sufficient to match the steady drift phenomena. Introduction of another length scale given by $\delta_D = \delta_s/\beta$ is necessary for matching.

Following Stuart [9], we express ϕ as

$$\phi = \phi(\xi_1, t) - \varphi_1 u_0(\xi_1, t) + \phi_a(\xi_1, \varphi_1, t). \tag{36}$$

The potential flow given by first two terms on the right-hand side of (36) balances the pressure gradient in (21). As we are interested only in the steady part, writing ϕ_a as

$$\phi_a = \phi_a^s + \phi_a^t \tag{37}$$

and averaging with respect to time, we obtain, by neglecting interacting terms of ϕ_a^s and ϕ_a^t, the equation for ϕ_a^s as

$$\frac{\partial^3\phi_a^s}{\partial\varphi_1^3} + \beta\left(\frac{\partial^2\phi_a^s}{\partial\xi_1\,\partial\varphi_1}\frac{\partial\phi_a^s}{\partial\varphi_1} - \frac{\partial^2\phi_a^s}{\partial\varphi_1^2}\frac{\partial\phi_a^s}{\partial\xi_1}\right) = 0. \tag{38}$$

Using the new boundary layer coordinate and stream function defined by

$$\varphi = \beta\varphi_1,\ \Phi(\xi_1, \varphi) = -\phi_a^s(\xi_1, \varphi_1) \tag{39}$$

in (38), we obtain

$$\frac{\partial^3\Phi}{\partial\varphi^3} + \left(\frac{\partial^2\Phi}{\partial\xi_1\,\partial\varphi} - \frac{\partial^2\Phi}{\partial\varphi^2}\frac{\partial\Phi}{\partial\xi_1}\right) = 0. \tag{40}$$

The following boundary conditions must be satisfied by the function Φ:

$$\Phi = 0 \quad \text{on} \quad \varphi = 0, \tag{41}$$

$$\lim_{\varphi\to\infty}\frac{\partial\Phi}{\partial\varphi} = -\lim_{\varphi\to\infty} u_1^s = \frac{3\eta'}{\eta^3}, \tag{42}$$

$$\lim_{\varphi\to\infty}\frac{\partial\Phi}{\partial\varphi} = D_2(x) + \text{c.c.} \tag{43}$$

The conditions (41)–(43), respectively, represent steady mass flow condition, the matching of the Stokes layer solution to the drift layer and the matching of the drift layer to the inviscid core. An approximate solution of (40) satisfying (41)–(43) is obtained by the method of Fettis, extended to partial differential equations by Stuart [9]. Since we are considering a variable wall, the η dependent function in (34) and (35) can be written as

$$B_1(\xi_1) + \text{c.c.} = -b^2\chi(\xi_1) \tag{44}$$

and a similar form is expected for $D_2(\xi_1)$ given by

$$D_2(\xi_1) + \text{c.c.} = -C^2\chi(\xi_1). \tag{45}$$

We rewrite the boundary conditions on Φ as

$$\frac{\partial\Phi}{\partial\varphi} = \varepsilon b^2\chi(\xi_1) \quad \text{at} \quad \varphi = 0, \tag{46}$$

$$\frac{\partial\Phi}{\partial\varphi} = -\varepsilon c^2\chi(\xi_1) \quad \text{as} \quad \varphi\to\infty \tag{47}$$

and expand the solution of (40) as

$$\Phi = \gamma(\xi_1) + \varepsilon\Phi_1(\xi_1, \varphi) + \varepsilon^2\Phi_2(\xi_1, \varphi) + \varepsilon^3\Phi_3(\xi_1, \varphi) + O(\varepsilon^4), \tag{48}$$

where ε is a parameter that will be set to unity later, $\gamma(\xi_1)$ is a function of ξ_1 to be determined and gives the form of Φ as $\varphi \to \infty$. The equations and the solutions for Φ_1, Φ_2 and Φ_3 satisfying the corresponding boundary conditions are presented in the Appendix. Using (A14), (A15) and (A16) in (48), setting $\varepsilon = 1$ and making use of the boundary condition $\Phi = 0$ on $\varphi = 0$, we obtain the following ordinary differential equation for γ:

$$\gamma\gamma' - a^2\chi + (\gamma'' a^4\chi^2)/4\gamma'^3 + 4a^2c^2\chi^2\gamma''/\gamma'^3 + l_1 a\gamma'^2 + (2l_2\gamma'^2 + l_3\gamma' + 3l_4)/8\gamma'^4 + (12l_5\gamma'^3 + 24l_6\gamma'^2 + 72l_7\gamma' + 288l_8)/12\gamma'^5 = 0. \tag{49}$$

In general it is very difficult to obtain a closed form solution of (49). So we assume that $\chi(\xi_1)$ can be expanded, near $\xi_1 = 0$, in the form

$$\chi(\xi_1) = \xi_1(1 + \beta_2\xi_1^2 + \ldots), \tag{50}$$

where β_2 is a constant. In the analysis of Grotberg [10], $\chi(\xi_1)$ is taken in the linear form as $\chi(\xi_1) = \xi_1$. Using (50) in (49), we obtain the solution for γ subject to $\gamma(0) = 0$ as

$$\gamma(\xi_1) = a\xi_1(1 + k\beta_2\xi_1^2 + \ldots), \tag{51}$$

where

$$k = \frac{3}{17}\left[1 + \frac{45}{4}\left(\frac{c}{a}\right)^2 + \left(\frac{1080}{17}\right)\left(\frac{c}{a}\right)^4\right]^{-1}. \tag{52}$$

By taking $X(\xi_1) = \xi_1$, i.e. $\beta_2 = 0$ in the above results, we get

$$\Phi(\xi_1, \varphi) = [(b^2 + c^2)/a]\xi_1(1 - e^{-a\varphi}) - c^2\xi_1, \tag{53}$$

which coincides with the corresponding result presented by Grotberg [10]. Further, by putting $c = 0$ and $b = a$ in the above results, we recover the result given by Stuart [9].

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APPENDIX

Substituting eqn (48) in (40) and collecting like powers of ε, we obtain the following differential equations:

$$\dddot{\Phi}_1 + \gamma'\ddot{\Phi}_1 = 0, \tag{A1}$$

$$\dddot{\Phi}_2 + \gamma'\ddot{\Phi}_2 = (a^4\gamma''\chi^2/\gamma')e^{-2\gamma'\varphi} - 2a^2c^2\chi\chi' e^{-\gamma'\varphi} + a^2c^2(\chi^2\gamma'' - \chi\chi'\gamma')\varphi e^{-\gamma'\varphi} + c^4\chi\chi', \tag{A2}$$

$$\dddot{\Phi}_3 + \gamma'\ddot{\Phi}_3 = l_1(\xi_1)e^{-3\gamma'\varphi} + (l_2 + l_3\varphi + l_4\varphi^2)e^{-2\gamma'\varphi} + (l_5 + l_6\varphi + l_7\varphi^2 + l_8\varphi^3)e^{-\gamma'\varphi}, \tag{A3}$$

where

$$l_1 = a^6\chi^3(\gamma'\gamma'' + 2\gamma'^2)/4\gamma'^4, \tag{A4}$$

$$l_2 = a^6\chi^3[(2c^2/a^2)(9\chi'\gamma\gamma'/\chi + 7\gamma'\gamma'' - 25\gamma''^2) - 4\gamma''^2]/4\gamma'^4, \tag{A5}$$

$$l_3 = a^4c^2\chi^3[2\chi'\gamma'^2\gamma''/\chi + 4\gamma'^2\gamma'' - 18\gamma'\gamma'']/2\gamma'^4, \quad (A6)$$

$$l_4 = -2a^4c^2\chi^3\gamma''^2/\gamma'^2, \quad (A7)$$

$$l_5 = a^4c^2\chi^3[\chi'\gamma'^2\gamma'' - 3(\gamma''^2 - \chi'\gamma'\gamma'''/\chi)]2\gamma'^4, \quad (A8)$$

$$l_6 = a^4c^2\chi^3[\chi'\gamma'^2\gamma''/\chi - \gamma'\gamma''^2 + c^2(8\chi'\gamma''/\chi + 4\gamma'^2\gamma''' - 8\gamma'^2\gamma''^2)/a^2]2\gamma'^4, \quad (A9)$$

$$l_7 = a^2c^4\chi^3[3\chi'\gamma'^3\gamma''/\chi + \gamma'^3\gamma''' - 3\gamma'^2\gamma''^2]/\gamma'^4 \quad (A10)$$

$$l_8 = -a^2c^4\chi^3[\gamma'^3\gamma''^2 - \chi'\gamma'\gamma''/\chi']/\gamma'^4; \qquad a^2 = b^2 + c^2. \quad (A11)$$

The corresponding boundary conditions for Φ_1, Φ_2, Φ_3 etc. are

$$\dot{\Phi}_1 = b^2\chi,\ \Phi_i = 0,\ i = 2, 3, 4, \dots, \quad \text{at} \quad \varphi = 0, \quad (A12)$$

$$\dot{\Phi}_1 = -c^2\chi,\ \Phi_i = 0,\ i = 2, 3, 4, \dots, \quad \text{as} \quad \varphi \to \infty. \quad (A13)$$

The solutions of (A1)–(A3) satisfying the boundary conditions (A12) and (A13) are, respectively,

$$\Phi_1 = -a^2\chi e^{-\gamma'\varphi}/\gamma' - c^2\chi\varphi, \quad (A14)$$

$$\Phi_2 = a^4\chi^2\gamma''(e^{-\gamma'\varphi} - e^{-2\gamma'\varphi}/2)2\gamma'^4 + a^2c^2\chi^2\gamma''(4 + 4\gamma'\varphi + \gamma''\varphi^2)e^{-\gamma'\varphi}/2\gamma'^4, \quad (A15)$$

$$\begin{aligned}\Phi_3 = {} & l_1(3e^{-\gamma'\varphi} - e^{-3\gamma'\varphi})/18\gamma'^3 + l_2(2e^{-\gamma'\varphi} - e^{-2\gamma'\varphi})/4\gamma'^3 \\ & + l_3[6e^{-\gamma'\varphi} - 2(2 + \gamma'\varphi)e^{-2\gamma'\varphi}]/18\gamma'^4 + l_4[14e^{-\gamma'\varphi} - (11 + 8\gamma'\varphi + 2\gamma'^2\varphi^2)e^{-2\gamma'\varphi}]/8\gamma'^5 \\ & + [12l_5\gamma'^3(\gamma'\varphi + 1)]/12\gamma'^6 + 6l_6\gamma'^2(\gamma'^2\varphi^2 + 4\gamma'\varphi + 4) + 4l_7\gamma'(\gamma'^3\varphi^3 + 6\gamma'^2\varphi^2 + 18\gamma'\varphi + 18) \\ & + 3l_8(\gamma'^4\varphi^4 + 8\gamma'\varphi^3 + 36\gamma'^2\varphi^2 + 96\gamma'\varphi + 96).\end{aligned} \quad (A16)$$