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EXPANDING THE APPLICABILITY OF AN ITERATIVE REGULARIZATION METHOD FOR ILL-POSED PROBLEMS

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Abstract. An iteratively regularized projection method, which converges quadratically, is considered for stable approximate solutions to a nonlinear ill-posed operator equation $F(x) = y$, where $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator defined on the real Hilbert space X. We assume that only a noisy data y^δ with $\|y - y^\delta\| \leq \delta$ are available. Under the assumption that the Fréchet derivative F' of F is Lipschitz continuous, a choice of the regularization parameter using an adaptive selection of the parameter and a stopping rule for the iteration index using a majorizing sequence are presented. We prove that, under a general source condition on $x_0 - \hat{x}$, the error $\|x^{h,\delta}_{n,\alpha} - \hat{x}\|$ between the regularized approximation $x^{h,\delta}_{n,\alpha}$, ($x^{h,\delta}_{0,\alpha} := P_h x_0$, where P_h is an orthogonal projection on to a finite dimensional subspace X_h of X) and the solution $\hat{x}$ is of optimal order.

Keywords. Majorizing sequence; Monotone operator; Nonlinear ill-posed operator; Quadratic convergence; Regularized Projection method.

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1. INTRODUCTION

Let X be a real Hilbert space. Let $F : D(F) \to X$ with domain $D(F) \subseteq X$ be a monotone operator. We consider the problem of solving the nonlinear ill-posed operator equation

$$F(x) = y \tag{1.1}$$

approximately when the data y is not known exactly. Assume that $y^\delta \in X$ are the available noisy data with

$$\|y - y^\delta\| \leq \delta, \tag{1.2}$$

and that (1.1) has a solution $\hat{x}$. Equation (1.1) is ill-posed in the sense that the Fréchet derivative $F'(.)$ is not boundedly invertible (see, [19, page 26]). Since (1.1) is ill-posed, one has to replace equation (1.1) by a nearby equation whose solution is less sensitive to perturbation in the right side y. This replacement is known as regularization. A well known method for regularizing (1.1), when F is monotone, is the method of the Lavrentiev regularization (see, [20]). In this method, approximation x^δ_α is obtained by solving the singularly perturbed operator equation

$$F(x) + \alpha(x - x_0) = y^\delta. \tag{1.3}$$

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In practice, one has to deal with some sequence $(x_{n,\alpha}^{\delta})$ converging to $\hat{x}$, the solution of (1.1). Recently, many authors considered such sequences; see [6, 7, 8, 12, 17, 18] and the references therein.

In [6], Bakushinsky and Smirnova considered an iteratively regularized Lavrentiev method:

$$x_{k+1}^{\delta} = x_k^{\delta} - (A_k^{\delta} + \alpha_k I)^{-1}(F(x_k^{\delta}) - y^{\delta} + \alpha_k(x_k^{\delta} - x_0)), \tag{1.4}$$

for $k = 0, 1, 2, \cdots$, where $A_k^{\delta} := F'(x_k^{\delta})$ and (α_k) is a sequence of positive real numbers such that $\lim_{k\to\infty} \alpha_k = 0$ as an approximate solution for (1.1). A general discrepancy principle was considered in [6] for choosing the stopping index k_δ and showed that $x_{k_\delta}^{\delta} \to \hat{x}$ as $\delta \to 0$. However, no error estimate for $\|x_{k_\delta}^{\delta} - \hat{x}\|$ was given in [6]. Later, Mahale and Nair [13] considered method (1.4) and obtained an error estimate for $\|x_{k_\delta}^{\delta} - \hat{x}\|$ under weaker assumptions than the assumptions in [6].

In [9], George and Elmahdy considered the iterative regularization method

$$x_{n+1,\alpha}^{\delta} = x_{n,\alpha}^{\delta} - (F'(x_0) + \alpha I)^{-1}(F(x_{n,\alpha}^{\delta}) - y^{\delta} + \alpha(x_{n,\alpha}^{\delta} - x_0)), \tag{1.5}$$

where $x_{0,\alpha}^{\delta} := x_0$ and proved that $(x_{n,\alpha}^{\delta})$ converges to the unique solution x_{α}^{δ} of (1.3) under the following Assumptions.

Assumpion 1.1. There exists $r_0 > 0$ such that $B_{r_0}(\hat{x}) \subseteq D(F)$ and F is Fréchet differentiable at all $x \in B_{r_0}(\hat{x})$.

Assumpion 1.2. There exists a constant $L > 0$ such that, for every $x, u \in B_{r_0}(\hat{x})$ and $v \in X$, there exists an element $\Phi(x,u,v) \in X$ satisfying

$$[F'(x) - F'(u)]v = F'(u)\Phi(x,u,v), \qquad \|\Phi(x,u,v)\| \le L\|v\|,$$

for all $x, u \in B_{r_0}(\hat{x})$.

Assumpion 1.3. There exists a continuous, strictly monotonically increasing function $\varphi : (0,a] \to (0,\infty)$ with $a \ge \|F'(\hat{x})\|$ satisfying $\lim_{\lambda\to 0} \varphi(\lambda) = 0$ and a vector $v \in X$ with $\|v\| \le 1$ such that

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v$$

and

$$\sup_{\lambda \ge 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \le \varphi(\alpha), \forall \alpha \in (0,a].$$

The main drawback of the method considered in [9] is that the initial guess x_0 of the iterative sequence $(x_{n,\alpha}^{\delta})$ is highly dependent on l_0 (see Lemma 2.4 and Theorem 2.6 in [9]), so it is hard to obtain such an initial guess x_0 when l_0 is not small enough. One of the purposes of this paper is to overcome this drawback.

In this paper we use the following modified form of Assumption 1.2.

Assumpion 1.4. Let $x_0 \in B_r(\hat{x})$ be fixed. There exists a constant $l_0 > 0$ such that, for every $x, x_0 \in B_{r_0}(\hat{x})$ and $v \in X$, there exists an element $\Phi(x, x_0, v) \in X$ satisfying

$$[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x,x_0,v), \qquad \|\Phi(x,x_0,v)\| \le l_0\|v\|\|x - x_0\|,$$

for all $x \in B_{r_0}(\hat{x})$ and $v \in X$.

From Assumption 1.4, one sees that the first hypotheses in Assumption 1.4 is weaker but the second hypotheses is stronger (but more practical) than the corresponding ones in Assumption 1.2. Hence Assumption 1.4 is stronger than Assumption 1.2. The autoconvolution problem discussed in [10] is an example of the nonlinear ill-posed problem satisfying Assumption 1.2 but not Assumption 1.4.

Further note that $l_0 \leq L$ holds in general and $\frac{L}{l_0}$ can be arbitrarily large [1, 2, 3, 4, 5]. The results in [9] really require Assumption 1.4 not Assumption 1.2. If $l_0 = L$, the results of this paper coincide with the results in [9]. Otherwise, i.e., if $l_0 < L$, then our convergence results are better under weaker majorizing sequences. The error estimates are tighter and the information on the location of the solution as well at least as precise and the stopping rule at least as tight. Hence, the applicability of method (1.5) has been extended under less computational cost since, in practice, computing L is more expensive (if at all possible) than computing l_0.

The main advantage of using the stronger Assumption 1.4 is that the majorizing sequence we are going to use in this paper is independent of the regularization parameter α. Further the majorizing sequence gives an a priori error estimate which can be used to determine the number of iterations needed to achieve a prescribed solution accuracy before actual computation takes place.

Remark 1.1. It can be seen that functions

$$\varphi(\lambda) = \lambda^{\nu}, \lambda > 0,$$

for $0 < \nu \leq 1$ and

$$\varphi(\lambda) = \begin{cases} (\ln \frac{1}{\lambda})^{-p}, & 0 < \lambda \leq e^{-(p+1)} \\ 0, & otherwise \end{cases}$$

for $p \geq 0$ satisfy the above assumption (see [15]).

2. Convergence Analysis

To prove the main results in this paper, we consider the sequence $(t_n), n \geq 0$ defined iteratively by $t_0 = 0, t_1 = \eta$,

$$t_{n+1} = t_n + \frac{l_0 \eta}{(1-r)}(t_n - t_{n-1}), \tag{2.1}$$

where $r \in [0,1)$ as a majorizing sequence of the sequence $(x^{\delta}_{n,\alpha})$.

The following lemma is a essential reformulation of a Lemma in [9]. For the sake of completeness, we give its proof as well.

Lemma 2.1. *Assume there exist nonnegative numbers l_0, η and $r \in [0,1)$ such that*

$$\frac{l_0}{(1-r)}\eta \leq r. \tag{2.2}$$

Then the sequence (t_n) defined in (2.1) is increasing, bounded above by $t^{**} := \frac{\eta}{1-r}$, and converges to some t^*, such that $0 < t^* \leq \frac{\eta}{1-r}$. Moreover, for $n \geq 0$,

$$0 \leq t_{n+1} - t_n \leq r(t_n - t_{n-1}) \leq r^n \eta \tag{2.3}$$

and

$$t^* - t_n \leq \frac{r^n}{1-r}\eta. \tag{2.4}$$

Proof. Since the result holds for $\eta = 0, l_0 = 0$ or $r = 0$, we assume that $l_0 \neq 0, \eta \neq 0$ and $r \neq 0$. Observe that $t_1 - t_0 = \eta \geq 0$. We assume that $t_{i+1} - t_i \geq 0$, for all $i \leq k$ for some k. Hence,

$$t_{k+2} - t_{k+1} = \frac{l_0 \eta}{(1-r)}(t_{k+1} - t_k) \geq 0.$$

and $t_{n+1} - t_n \geq 0$ for all $n \geq 0$. From $\frac{l_0\eta}{(1-r)} \leq r$, estimate (2.3) follows from (2.1). Further observe that

$$\begin{aligned} t_{k+1} &\leq t_k + r(t_k - t_{k-1}) \\ &\leq \cdots \\ &\leq \eta + r\eta + \cdots + r^k \eta \\ &< \frac{\eta}{1-r}. \end{aligned}$$

Hence (t_n) is bounded above by $\frac{\eta}{1-r}$ and nondecreasing. So, it converges to some $t^* \leq \frac{\eta}{1-r}$, and

$$t^* - t_n = \lim_{i\to\infty} t_{n+i} - t_n \leq \lim_{i\to\infty} \sum_{j=0}^{i-1} (t_{n+1+j} - t_{n+j}) \leq \frac{r^n}{1-r}\eta.$$

This completes the proof of the Lemma. □

To prove the convergence of the sequence $(x_{n,\alpha}^\delta)$ defined in (1.5), we introduce the following notations. Let $R_\alpha(x_0) := F'(x_0) + \alpha I$ and

$$G(x) := x - R_\alpha(x_0)^{-1}[F(x) - y^\delta + \alpha(x - x_0)]. \tag{2.5}$$

Note that $G(x_{n,\alpha}^\delta) = x_{n+1,\alpha}^\delta$ and

$$\|R_\alpha(x_0)^{-1} F'(x_0)\| \leq 1. \tag{2.6}$$

The following Lemma based on the Assumption 1.4 will be used later.

Lemma 2.2. *For $u, v, x_0 \in B_{r_0}(\hat{x})$,*

$$F(v) - F(u) - F'(x_0)(v-u) = F'(x_0)\int_0^1 \Phi(u + t(v-u), x_0, v-u)dt.$$

Proof. Using the Fundamental Theorem of Integration, for $u, v, x_0 \in B_{r_0}(\hat{x})$ we have

$$F(v) - F(u) = \int_0^1 F'(u + t(v-u))(v-u)dt.$$

From Assumption 1.4, we have

$$F(v) - F(u) - F'(x_0)(v-u) = F'(x_0)\int_0^1 \Phi(u + t(v-u), x_0, v-u)dt.$$

This completes the proof of the Lemma. □

Hereafter we assume that $\|x_0 - \hat{x}\| \leq \rho$ and

$$\frac{l_0}{2}\rho^2 + \rho + \frac{\delta}{\alpha} \leq \eta \leq \min\{\frac{r(1-r)}{l_0}, r_0(1-r)\}. \tag{2.7}$$

Theorem 2.1. *Suppose that (2.1) holds. Let the assumptions in Lemma 2.1 with η be as in (2.7) and Assumption 1.4 be satisfied. Then the sequence $(x_{n,\alpha}^{\delta})$ defined in (1.5) is well defined and $x_{n,\alpha}^{\delta} \in B_{t^*}(x_0)$ for all $n \geq 0$. Further $(x_{n,\alpha}^{\delta})$ is a Cauchy sequence in $B_{t^*}(x_0)$ (converges to $x_{\alpha}^{\delta} \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$) and $F(x_{\alpha}^{\delta}) + \alpha(x_{\alpha}^{\delta} - x_0) = y^{\delta}$. Moreover, the following estimate hold, for all $n \geq 0$,*

$$\|x_{n+1,\alpha}^{\delta} - x_{n,\alpha}^{\delta}\| \leq t_{n+1} - t_n \tag{2.8}$$

and

$$\|x_{n,\alpha}^{\delta} - x_{\alpha}^{\delta}\| \leq t^* - t_n \leq \frac{r^n \eta}{(1-r)}. \tag{2.9}$$

Proof. Let G be as in (2.5). Then, for $u, v \in B_{t^*}(x_0)$,

$$\begin{aligned}
G(u) - G(v) &= u - v - R_{\alpha}(x_0)^{-1}[F(u) - y^{\delta} + \alpha(u - x_0)] \\
&\quad + R_{\alpha}(x_0)^{-1}[F(v) - y^{\delta} + \alpha(v - x_0)] \\
&= R_{\alpha}(x_0)^{-1}[R_{\alpha}(x_0)(u-v) - (F(u) - F(v))] \\
&\quad + \alpha R_{\alpha}(x_0)^{-1}(v-u) \\
&= R_{\alpha}(x_0)^{-1}[F'(x_0)(u-v) - (F(u) - F(v)) + \alpha(u-v)] \\
&\quad + \alpha R_{\alpha}(x_0)^{-1}(v-u) \\
&= R_{\alpha}(x_0)^{-1}[F'(x_0)(u-v) - (F(u) - F(v))].
\end{aligned}$$

Using Lemma 2.2, Assumption 1.4, (2.6) and (2.7) we have

$$\|G(u) - G(v)\| \leq l_0 t^* \|u - v\|. \tag{2.10}$$

Now we prove that the sequence (t_n) defined in Lemma 2.1 is a majorizing sequence of $(x_{n,\alpha}^{\delta})$ and $x_{n,\alpha}^{\delta} \in B_{t^*}(x_0)$, for all $n \geq 0$. Since $F(\hat{x}) = y$, one has

$$\begin{aligned}
\|x_{1,\alpha}^{\delta} - x_0\| &= \|R_{\alpha}(x_0)^{-1}(F(x_0) - y^{\delta})\| \\
&= \|R_{\alpha}(x_0)^{-1}(F(x_0) - y + y - y^{\delta})\| \\
&= \|R_{\alpha}(x_0)^{-1}(F(x_0) - F(\hat{x}) - F'(x_0)(x_0 - \hat{x}) \\
&\quad + F'(x_0)(x_0 - \hat{x}) + y - y^{\delta})\| \\
&\leq \|R_{\alpha}(x_0)^{-1}(F(x_0) - F(\hat{x}) - F'(x_0)(x_0 - \hat{x}))\| \\
&\quad + \|R_{\alpha}(x_0)^{-1}F'(x_0)(x_0 - \hat{x})\| + \|R_{\alpha}(x_0)^{-1}(y - y^{\delta})\| \\
&\leq \|R_{\alpha}(x_0)^{-1}F'(x_0)\int_0^1 \Phi(\hat{x} + t(x_0 - \hat{x}), x_0, (x_0 - \hat{x}))dt\| \\
&\quad + \|R_{\alpha}(x_0)^{-1}F'(x_0))(x_0 - \hat{x})\| + \frac{\delta}{\alpha} \\
&\leq \frac{l_0}{2}\|x_0 - \hat{x}\|^2 + \|x_0 - \hat{x}\| + \frac{\delta}{\alpha} \\
&\leq \frac{l_0}{2}\rho^2 + \rho + \frac{\delta}{\alpha} \leq \eta = t_1 - t_0.
\end{aligned}$$

The last but one step follows from Assumption 1.4. Assume that

$$\|x_{i+1,\alpha}^{\delta} - x_{i,\alpha}^{\delta}\| \leq t_{i+1} - t_i, \qquad \forall i \leq k \tag{2.11}$$

for some k. Then

$$\begin{aligned}\|x^{\delta}_{k+1,\alpha}-x_0\| &= \|x^{\delta}_{k+1,\alpha}-x^{\delta}_{k,\alpha}+x^{\delta}_{k,\alpha}-x^{\delta}_{k-1,\alpha}+\cdots+x^{\delta}_{1,\alpha}-x_0\|\\ &\leq \|x^{\delta}_{k+1,\alpha}-x^{\delta}_{k,\alpha}\|+\|x^{\delta}_{k,\alpha}-x^{\delta}_{k-1,\alpha}\|+\cdots+\|x^{\delta}_{1,\alpha}-x_0\|\\ &\leq t_{k+1}-t_k+t_k-t_{k-1}+\cdots+t_1-t_0\\ &= t_{k+1}\leq t^*.\end{aligned}$$

So $x^{\delta}_{i+1,\alpha}\in B_{t^*}(x_0)$ for all $i\leq k$. Hence, by (2.10) and (2.11), one has

$$\|x^{\delta}_{k+2,\alpha}-x^{\delta}_{k+1,\alpha}\|\leq l_0t^*\|x^{\delta}_{k+1,\alpha}-x^{\delta}_{k,\alpha}\|\leq \frac{l_0\eta}{(1-r)}(t_{k+1}-t_k)=t_{k+2}-t_{k+1}.$$

Thus by induction, $\|x^{\delta}_{n+1,\alpha}-x^{\delta}_{n,\alpha}\|\leq t_{n+1}-t_n$ for all $n\geq 0$. Hence $(t_n), n\geq 0$ is a majorizing sequence of $(x^{\delta}_{n,\alpha})$. In particular, $\|x^{\delta}_{n,\alpha}-x_0\|\leq t_n\leq t^*$, i.e., $x^{\delta}_{n,\alpha}\in B_{t^*}(x_0)$, for all $n\geq 0$. Hence, $(x^{\delta}_{n,\alpha})$ is a Cauchy sequence and converges to some $x^{\delta}_{\alpha}\in\overline{B_{t^*}(x_0)}\subset B_{t^{**}}(x_0)$ and

$$\|x^{\delta}_{\alpha}-x^{\delta}_{n,\alpha}\|\leq t^*-t_n\leq\frac{r^n\eta}{(1-r)}.$$

Letting $n\to\infty$ in (1.5), we obtain $F(x^{\delta}_{\alpha})+\alpha(x^{\delta}_{\alpha}-x_0)=y^{\delta}$. This completes the proof of the Theorem. □

3. Error Bounds Under Source Conditions

We will use the error estimates in the following Proposition, which can be found in [20] for our error analysis.

Proposition 3.1. [20, Proposition 3.1] *Let $\hat{x}\in D(F)$ be a solution of (1.1) and let $F: D(F)\subseteq X\mapsto X$ be a monotone operator in X. Let x_α be the unique solution of*

$$F(x)+\alpha(x-x_0)=y \tag{3.1}$$

and let x^{δ}_{α} be the unique solution of (1.3). Then

$$\|x^{\delta}_{\alpha}-x_\alpha\|\leq\frac{\delta}{\alpha} \tag{3.2}$$

and

$$\|x_\alpha-\hat{x}\|\leq\|x_0-\hat{x}\|.$$

To obtain an error estimate for $\|x^{\delta}_{\alpha}-\hat{x}\|$, it is enough to obtain an error estimate for $\|x^{\delta}_{\alpha}-x_\alpha\|$ and $\|x_\alpha-\hat{x}\|$.

Let us introduce the following operators:

$$A:=F'(\hat{x}) \tag{3.3}$$

and

$$M_\alpha:=\int_0^1 F'(\hat{x}+t(x_\alpha-\hat{x}))dt. \tag{3.4}$$

Using the Mean Value Theorem in Integral form, we have

$$F(x_\alpha)-F(\hat{x})=M_\alpha(x_\alpha-\hat{x}). \tag{3.5}$$

The following Theorem gives an estimate for $\|x_\alpha-\hat{x}\|$.

Theorem 3.1. *Let x_α be the unique solution of (3.1) and let the Assumptions 1.1, 1.3 and 1.4 be satisfied. Then*

$$\|x_\alpha - \hat{x}\| \leq (1 + l_0 r_0) c_\varphi \varphi(\alpha). \tag{3.6}$$

Proof. Since $F(x_\alpha) + \alpha(x_\alpha - x_0) = y$, for any $\alpha > 0$, we obtain from (3.5) that

$$M_\alpha(x_\alpha - \hat{x}) + \alpha(x_\alpha - \hat{x}) = \alpha(x_0 - \hat{x}).$$

Hence

$$\begin{aligned}
x_\alpha - \hat{x} &= (M_\alpha + \alpha I)^{-1} \alpha(x_0 - \hat{x}) \\
&= [(M_\alpha + \alpha I)^{-1} - (A + \alpha I)^{-1}] \alpha(x_0 - \hat{x}) + \alpha(A + \alpha I)^{-1}(x_0 - \hat{x}) \\
&= (M_\alpha + \alpha I)^{-1}(A - M_\alpha) \alpha(A + \alpha I)^{-1}(x_0 - \hat{x}) + \alpha(A + \alpha I)^{-1}(x_0 - \hat{x}) \\
&= (M_\alpha + \alpha I)^{-1} M_\alpha \Phi(\hat{x}, \hat{x} + t(x_\alpha - \hat{x}), \alpha(A + \alpha I)^{-1}(x_0 - \hat{x})) \\
&\quad + \alpha(A + \alpha I)^{-1}(x_0 - \hat{x}),
\end{aligned}$$

which follows from Assumption 1.4. From Proposition 3.1, Assumptions 1.1 , 1.3 and 1.4 we have

$$\begin{aligned}
\|x_\alpha - \hat{x}\| &= \|(M_\alpha + \alpha I)^{-1} M_\alpha \Phi(\hat{x}, \hat{x} + t(x_\alpha - \hat{x}), \alpha(A + \alpha I)^{-1}(x_0 - \hat{x})) \\
&\quad + \alpha(A + \alpha I)^{-1}(x_0 - \hat{x})\| \\
&\leq \|(M_\alpha + \alpha I)^{-1} M_\alpha \Phi(\hat{x}, \hat{x} + t(x_\alpha - \hat{x}), \alpha(A + \alpha I)^{-1}(x_0 - \hat{x}))\| \\
&\quad + \|\alpha(A + \alpha I)^{-1}(x_0 - \hat{x})\| \\
&\leq (l_0 r_0 + 1) c_\varphi \varphi(\alpha).
\end{aligned}$$

This completes the proof. □

Combining the estimates in Theorem 2.1, (3.2) and (3.6), we obtain the following Theorem.

Theorem 3.2. *Let $x_{n,\alpha}^\delta$ be as in (1.5) and let the assumptions in Theorem 2.1, (3.2) and (3.6) be satisfied. Then*

$$\|x_{n,\alpha}^\delta - \hat{x}\| \leq \frac{r^n \eta}{1 - r} + \frac{\delta}{\alpha} + (l_0 r_0 + 1) c_\varphi \varphi(\alpha). \tag{3.7}$$

Let

$$n_\delta := \min\{n : r^n \leq \delta\} \tag{3.8}$$

and

$$C := \max\{\frac{\eta}{1 - r} + 1, (l_0 r_0 + 1) c_\varphi\}. \tag{3.9}$$

Theorem 3.3. *Let $x_{n,\alpha}^\delta$ be as in (1.5) and let the assumptions in Theorem 3.2 be satisfied. Let n_δ be as in (3.8) and let C be as in (3.9). Then, for all $0 < \alpha \leq 1$,*

$$\|x_{n_\delta,\alpha}^\delta - \hat{x}\| \leq C(\varphi(\alpha) + \frac{\delta}{\alpha}). \tag{3.10}$$

3.1. **A priori choice of the parameter.** Note that the error $\varphi(\alpha)+\frac{\delta}{\alpha}$ in (3.10) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies $\alpha_\delta\varphi(\alpha_\delta)=\delta$. Using $\psi(\lambda):=\lambda\varphi^{-1}(\lambda)$, $0<\lambda\leq a$, we have

$$\delta=\alpha_\delta\varphi(\alpha_\delta)=\psi(\varphi(\alpha_\delta)).$$

Hence, $\alpha_\delta=\varphi^{-1}(\psi^{-1}(\delta))$. Using (3.10), we have the following.

Theorem 3.4. *Let $\psi(\lambda):=\lambda\varphi^{-1}(\lambda)$ for $0<\lambda\leq a$, and let the assumptions in Theorem 3.3 holds. For $\delta>0$, let $\alpha:=\alpha_\delta=\varphi^{-1}(\psi^{-1}(\delta))$. Let n_δ be as in (3.8). Then*

$$\|x^\delta_{n_\delta,\alpha}-\hat{x}\|=\bigcirc(\psi^{-1}(\delta)).$$

3.2. **An adaptive choice of the parameter.** In this subsection, we present a parameter choice rule based on the adaptive method studied in [14, 16].

In practice, the regularization parameter α is often selected from some finite set

$$D_M(\alpha):=\{\alpha_i=\mu^i\alpha_0, i=0,1,\cdots,M\}, \tag{3.11}$$

where $\mu>1$ and M is such that $\alpha_M<1\leq\alpha_{M+1}$. We choose $\alpha_0:=\sqrt{\delta}$ because in general $\varphi(\lambda)=\lambda^\nu, 0<\nu\leq 1$. In this case, the best possible error estimate is of order $\bigcirc(\sqrt{\delta})$. From Theorem 3.4, it follows that such an accuracy cannot be guaranteed for $\alpha<\sqrt{\delta}$. Let

$$n_M:=\min\{n: r^n\leq\delta\}. \tag{3.12}$$

Then, for $i=0,1,\cdots,M$,

$$\|x^\delta_{n_M,\alpha_i}-x^\delta_{\alpha_i}\|\leq\frac{\delta}{\alpha_i}, \quad \forall i=0,1,\cdots M. \tag{3.13}$$

Let $x_i:=x^\delta_{n_M,\alpha_i}$. We selects $\alpha=\alpha_i$ from $D_M(\alpha)$ and operates only with corresponding x_i, $i=0,1,\cdots,M$.

Theorem 3.5. *Assume that there exists $i\in\{0,1,2,\cdots,M\}$ such that $\varphi(\alpha_i)\leq\frac{\delta}{\alpha_i}$. Let assumptions of Theorem 3.3 and Theorem 3.4 hold and let*

$$l:=\max\{i:\varphi(\alpha_i)\leq\frac{\delta}{\alpha_i}\}<M,$$

$$k:=\max\{i:\|x_i-x_j\|\leq 4C\frac{\delta}{\alpha_j}, j=0,1,2,\cdots .i\}. \tag{3.14}$$

Then, $l\leq k$, $\|\hat{x}-x_k\|\leq c\psi^{-1}(\delta)$, where $c=6C\mu$.

Proof. To see that $l\leq k$, it is enough to show that, for each $i\in\{1,2,\cdots,M\}$,

$$\varphi(\alpha_i)\leq\frac{\delta}{\alpha_i}\Longrightarrow\|x_i-x_j\|\leq 4C\frac{\delta}{\alpha_j}, \quad \forall j=0,1,\cdots,i.$$

For $j\leq i$, we conclude from (3.10)that

$$\begin{aligned}
\|x_i-x_j\| &\leq \|x_i-\hat{x}\|+\|\hat{x}-x_j\| \\
&\leq C(\varphi(\alpha_i)+\frac{\delta}{\alpha_i})+C(\varphi(\alpha_j)+\frac{\delta}{\alpha_j}) \\
&\leq 2C\frac{\delta}{\alpha_i}+2C\frac{\delta}{\alpha_j}. \\
&\leq 4C\frac{\delta}{\alpha_j}.
\end{aligned}$$

Thus the relation $l \leq k$ is proved. Next we observe that

$$\begin{aligned}\|\hat{x}-x_k\| &\leq \|\hat{x}-x_l\|+\|x_l-x_k\| \\ &\leq C(\varphi(\alpha_l)+\frac{\delta}{\alpha_l})+4C\frac{\delta}{\alpha_l} \\ &\leq 6C\frac{\delta}{\alpha_l}.\end{aligned}$$

Since $\alpha_\delta \leq \alpha_{l+1} \leq \mu\alpha_l$, one has

$$\frac{\delta}{\alpha_l} \leq \mu\frac{\delta}{\alpha_\delta} = \mu\varphi(\alpha_\delta) = \mu\psi^{-1}(\delta).$$

This completes the proof. □

4. Implementation of the adaptive choice rule

In this section, we provide an algorithm for the determination of a parameter fulfilling the balancing principle (3.14) and provide a starting point for the iteration (1.5) to approximate the unique solution x_α^δ of (1.3). The choice of the starting point involves the following steps:

- Choose $\alpha_0 = \sqrt{\delta}$ and $\mu > 1$.
- Choose $x_0 \in D(F)$ such that $\|x_0-\hat{x}\| \leq \rho$.
- Choose η satisfying (2.7).

The choice of the stopping index n_M involves the following step:

- Choose n_M such that $n_M = min\{n : r^n \leq \delta\}$.

Finally, the adaptive algorithm associated with the choice of the parameter specified in Theorem 3.5 involves the following steps.

4.1. The algorithm.

- Set $i \leftarrow 0$.
- Solve $x_i := x^\delta_{n_M,\alpha_i}$ via iteration (1.5).
- If $\|x_i - x_j\| > 4C\frac{\sqrt{\delta}}{\mu^j}, \quad j \leq i$, then we take $k = i-1$.
- Set $i = i+1$ and return to step 2.

5. The Iteratively Regularized Projection Method

Let H be a bounded subset of positive real numbers such that zero is a limit point of H. Let $\{P_h\}_{h\in H}$ be a family of orthogonal projections from X into itself. Let

$$\Gamma_h := \|(I-P_h)F'(x_0)\| \tag{5.1}$$

and

$$\gamma_h := \|F'(P_hx_0)(I-P_h)\|. \tag{5.2}$$

We assume that

$$b_h := \|(I-P_h)x_0\| \to 0, \quad \text{as } \to 0. \tag{5.3}$$

The above assumption is satisfied if $P_h \to I$ pointwise. Let $(t_{n,h}), n \geq 0$ be defined iteratively by $t_{0,h} = 0, t_{1,h} = \eta_h$,

$$t_{n+1,h} = t_{n,h} + (1+\frac{\gamma_h}{\alpha})\frac{l_0\eta_h}{(1-r_h)}(t_{n,h}-t_{n-1,h}), \tag{5.4}$$

where l_0, α and $r_h \in [0,1)$ are nonnegative numbers with $(1+\frac{\gamma_h}{\alpha})\frac{l_0}{(1-r_h)}\eta_h \le r_h$. We need the following Lemma. Its proof is analogous to the proof of Lemma 2.1. So, we omit the proof.

Lemma 5.1. *Assume there exist nonnegative numbers l_0, α and $r_h \in [0,1)$ such that*

$$(1+\frac{\gamma_h}{\alpha})\frac{l_0}{(1-r_h)}\eta_h \le r_h. \tag{5.5}$$

Then the sequence $(t_{n,h})$ defined in (5.4) is increasing, bounded above by $t_h^{**} := \frac{\eta_h}{1-r_h}$, and converges to some t_h^*, such that $0 < \frac{\eta_h}{1-r_h}$. Moreover, for $n \ge 0$,

$$0 \le t_{n+1,h} - t_{n,h} \le r_h(t_{n,h} - t_{n-1,h}) \le r_h^n \eta_h \tag{5.6}$$

and

$$t_h^* - t_{n,h} \le \frac{r_h^n}{1-r_h}\eta_h. \tag{5.7}$$

We considered the following iteratively regularized projection method

$$x_{n+1,\alpha}^{h,\delta} := x_{n,\alpha}^{h,\delta} - (P_hF'(P_hx_0)+\alpha I)^{-1}P_h(F(x_{n,\alpha}^{h,\delta}) - y^\delta + \alpha(x_{n,\alpha}^{h,\delta} - x_0)), \tag{5.8}$$

where $x_{0,\alpha}^{h,\delta} := P_hx_0$ for $(x_{n,\alpha}^{h,\delta})$ in a finite dimensional subspace X_h of X.

Next we prove that sequence $(t_{n,h})$ is a majorizing sequence of $(x_{n,\alpha}^{h,\delta})$. Let

$$(1+\frac{\gamma_h}{\alpha})(\frac{l_0}{2}(b_h+\rho)^2 + b_h + \rho) + \frac{\delta}{\alpha} \quad \le \eta_h \le \min\{\tfrac{r_h(1-r_h)}{l_0(1+\gamma_h/\alpha)}, r_0(1-r_h)\}. \tag{5.9}$$

Theorem 5.1. *Let the assumptions in Lemma 5.1 with η_h be as in (5.9) and let Assumption 1.4 be satisfied. Then the sequence $(t_{n,h})$ defined in (5.4) is a majorizing sequence of the sequence $(x_{n,\alpha}^{h,\delta})$ defined in (5.8) and $x_{n,\alpha}^{h,\delta} \in B_{t_h^*}(P_hx_0)$ for all $n \ge 0$.*

Proof. Let

$$G(x) = x - R_\alpha(P_hx_0)^{-1}[F(x) - y^\delta + \alpha(x - x_0)],$$

where $R_\alpha(P_hx_0)^{-1} = (P_hF'(P_hx_0)P_h + \alpha P_h)^{-1}$. Since

$$R_\alpha(P_hx_0)^{-1} = R_\alpha(P_hx_0)^{-1}P_h = P_hR_\alpha(P_hx_0)^{-1},$$

for $u, v \in B_{t_h^*}(P_hx_0)$, one has

$$\begin{aligned} G(u) - G(v) &= u - v - R_\alpha(P_hx_0)^{-1}[F(u) - y^\delta + \alpha(u - x_0)] + R_\alpha(P_hx_0)^{-1}[F(v) - y^\delta + \alpha(v - x_0)] \\ &= R_\alpha(P_hx_0)^{-1}[R_\alpha(P_hx_0)(u - v) - (F(u) - F(v))] + \alpha R_\alpha(P_hx_0)^{-1}(v - u) \\ &= R_\alpha(P_hx_0)^{-1}[F'(P_hx_0)P_h(u - v) - (F(u) - F(v)) + \alpha(u - v)] + \alpha R_\alpha(P_hx_0)^{-1}(v - u) \\ &= R_\alpha(P_hx_0)^{-1}[F'(P_hx_0)P_h(u - v) - (F(u) - F(v))]. \end{aligned}$$

Since $G(x_{n,\alpha}^{h,\delta}) = x_{n+1,\alpha}^{h,\delta}$ and $P_h(x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) = (x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta})$, we have from Lemma 2.2 that

$$\begin{aligned}(x_{n+1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}) &= G(x_{n,\alpha}^{h,\delta}) - G(x_{n-1,\alpha}^{h,\delta})\\ &= R_\alpha(P_hx_0)^{-1}[F'(P_hx_0)(x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) - (F(x_{n,\alpha}^{h,\delta}) - F(x_{n-1,\alpha}^{h,\delta}))]\\ &= R_\alpha(P_hx_0)^{-1}F'(P_hx_0)\int_0^1 \Phi(x_{n,\alpha}^{h,\delta} + t(x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}), P_hx_0, x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta})dt\\ &= R_\alpha(P_hx_0)^{-1}[F'(P_hx_0)P_h + F'(P_hx_0)(I-P_h)]\\ &\times \int_0^1 \Phi(x_{n,\alpha}^{h,\delta} + t(x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}), P_hx_0, x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta})dt.\end{aligned}$$

Using Assumption 1.4 and

$$\|R_\alpha(P_hx_0)^{-1}[F'(P_hx_0)P_h + F'(P_hx_0)(I-P_h)]\| \le 1 + \frac{\gamma_h}{\alpha}, \tag{5.10}$$

we have

$$\|x_{n+1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}\| \le (1+\frac{\gamma_h}{\alpha})l_0\|x_{n,\alpha}^{h,\delta} + t(x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}) - P_hx_0\|\|x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\|.$$

Now we prove that the sequence $(t_{n,h})$ defined in (5.4) is a majorizing sequence of $(x_{n,\alpha}^{h,\delta})$ and $x_{n,\alpha}^{h,\delta} \in B_{t_h^*}(P_hx_0)$, for all $n \ge 0$. In view of $F(\hat{x}) = y$, Assumption 1.4, (5.10) and inequality $\|P_hx_0 - \hat{x}\| \le b_h + \rho$, one has

$$\begin{aligned}\|x_{1,\alpha}^{h,\delta} - P_hx_0\| &= \|(P_hF'(P_hx_0) + \alpha I)^{-1}P_h(F(P_hx_0) - y^\delta)\|\\ &= \|(P_hF'(P_hx_0) + \alpha I)^{-1}P_h(F(P_hx_0) - F(\hat{x}) + y - y^\delta)\|\\ &\le \|(P_hF'(P_hx_0) + \alpha I)^{-1}P_h(F(P_hx_0) - F(\hat{x}) - F'(P_hx_0)(P_hx_0 - \hat{x}))\|\\ &\quad + \|(P_hF'(P_hx_0) + \alpha I)^{-1}P_hF'(P_hx_0)(P_hx_0 - \hat{x})\|\\ &\quad + \|(P_hF'(P_hx_0) + \alpha I)^{-1}P_h(y - y^\delta)\|\\ &\le (1+\frac{\gamma_h}{\alpha})(\frac{l_0}{2}\|P_hx_0 - \hat{x}\|^2 + \|P_hx_0 - \hat{x}\|) + \frac{\delta}{\alpha}\\ &\le (1+\frac{\gamma_h}{\alpha})(\frac{l_0}{2}(b_h+\rho)^2 + b_h + \rho) + \frac{\delta}{\alpha}\\ &\le \eta_h.\end{aligned}$$

So, $\|x_{1,\alpha}^{h,\delta} - P_hx_0\| \le t_{1,h} - t_{0,h}$. Assume that

$$\|x_{i+1,\alpha}^{h,\delta} - x_{i,\alpha}^{h,\delta}\| \le t_{i+1,h} - t_{i,h}, \quad \forall i \le k, \tag{5.11}$$

for some k. Then

$$\begin{aligned}\|x_{k+1,\alpha}^{h,\delta} - P_hx_0\| &= \|x_{k+1,\alpha}^{h,\delta} - x_{k,\alpha}^{h,\delta} + x_{k,\alpha}^{h,\delta} - x_{k-1,\alpha}^{h,\delta} + \cdots + x_{1,\alpha}^{h,\delta} - P_hx_0\|\\ &\le \|x_{k+1,\alpha}^{h,\delta} - x_{k,\alpha}^{h,\delta}\| + \|x_{k,\alpha}^{h,\delta} - x_{k-1,\alpha}^{h,\delta}\| + \cdots + \|x_{1,\alpha}^{h,\delta} - P_hx_0\|\\ &\le t_{k+1,h} - t_{k,h} + t_{k,h} - t_{k-1,h} + \cdots + t_{1,h} - t_{0,h}\\ &= t_{k+1,h} \le t_h^*.\end{aligned}$$

So $x_{i+1,\alpha}^{h,\delta} \in B_{t_h^*}(P_hx_0)$ for all $i \le k$. Hence,

$$x_{k+1,\alpha}^{h,\delta} + t(x_{k,\alpha}^{h,\delta} - x_{k+1,\alpha}^{h,\delta}) \in B_{t_h^*}(P_hx_0).$$

By (5.11) and (5.11), we have

$$\begin{aligned}\|x_{k+2,\alpha}^{h,\delta}-x_{k+1,\alpha}^{h,\delta}\| &\leq l_0(1+\frac{\gamma_h}{\alpha})t_h^*\|x_{k+1,\alpha}^{h,\delta}-x_{k,\alpha}^{h,\delta}\|\\ &= t_{k+2,h}-t_{k+1,h}.\end{aligned}$$

Thus, $\|x_{n+1,\alpha}^{h,\delta}-x_{n,\alpha}^{h,\delta}\|\leq t_{n+1,h}-t_{n,h}$, $\forall n\geq 0$. Hence $(t_{n,h}), n\geq 0$ is a majorizing sequence of $(x_{n,\alpha}^{h,\delta})$. In particular $\|x_{n,\alpha}^{h,\delta}-P_hx_0\|\leq t_{n,h}\leq t_h^*$, i.e., $x_{n,\alpha}^{h,\delta}\in B_{t_h^*}(P_hx_0)$, for all $n\geq 0$. Hence

$$\|x_{n,\alpha}^{h,\delta}-P_hx_0\|\leq t_h^*\leq \frac{\eta_h}{1-r_h}. \tag{5.12}$$

This completes the proof. □

Let

$$\tilde{\tilde{r}}:=\max\{r,r_h\} \tag{5.13}$$

and

$$q:=\frac{1}{2}[2\tilde{\tilde{r}}+l_0b_h]. \tag{5.14}$$

For $0<b_h<\frac{2(1-\tilde{\tilde{r}})}{l_0}$, $q<1$, one has the following.

Theorem 5.2. *Let $x_{n,\alpha}^{h,\delta}$ be as in (5.8) and let $x_{n,\alpha}^{\delta}$ be as in (1.5). Let assumptions in Theorem 2.1 and Theorem 5.1 hold. Then*

$$\|x_{n,\alpha}^{h,\delta}-x_{n,\alpha}^{\delta}\|\leq q^nb_h+\frac{\Gamma_h+l_0\|F'(x_0)\|b_h}{\alpha}\frac{q^n}{q-r_h}\eta_h.$$

Proof. Note that

$$\begin{aligned}x_{n,\alpha}^{h,\delta}-x_{n,\alpha}^{\delta} &= x_{n-1,\alpha}^{h,\delta}-x_{n-1,\alpha}^{\delta}-[(P_hF'(P_hx_0)+\alpha I)^{-1}P_h-(F'(x_0)+\alpha I)^{-1}]\\ &\quad\times(F(x_{n-1,\alpha}^{h,\delta})-y^\delta+\alpha(x_{n-1,\alpha}^{h,\delta}-x_0))\\ &\quad-(F'(x_0)+\alpha I)^{-1}[F(x_{n-1,\alpha}^{h,\delta})-F(x_{n-1,\alpha}^{\delta})+\alpha(x_{n-1,\alpha}^{h,\delta}-x_{n-1,\alpha}^{\delta})]\\ &= (F'(x_0)+\alpha I)^{-1}[F'(x_0)(x_{n-1,\alpha}^{h,\delta}-x_{n-1,\alpha}^{\delta})-(F(x_{n-1,\alpha}^{h,\delta})-F(x_{n-1,\alpha}^{\delta}))]\\ &\quad-(F'(x_0)+\alpha I)^{-1}[F'(x_0)P_h-P_hF'(P_hx_0)P_h](P_hF'(P_hx_0)+\alpha I)^{-1}\\ &\quad\times P_h[(F(x_{n-1,\alpha}^{h,\delta})-y^\delta+\alpha(x_{n-1,\alpha}^{h,\delta}-x_0))]\\ &= (F'(x_0)+\alpha I)^{-1}[F'(x_0)(x_{n-1,\alpha}^{h,\delta}-x_{n-1,\alpha}^{\delta})-(F(x_{n-1,\alpha}^{h,\delta})-F(x_{n-1,\alpha}^{\delta}))]\\ &\quad-(F'(x_0)+\alpha I)^{-1}[F'(x_0)-P_hF'(x_0)+P_hF'(x_0)-P_hF'(P_hx_0)](x_{n,\alpha}^{h,\delta}-x_{n-1,\alpha}^{h,\delta})\\ &:= \Gamma_1-\Gamma_2,\end{aligned} \tag{5.15}$$

where

$$\Gamma_1=(F'(x_0)+\alpha I)^{-1}[F'(x_0)(x_{n-1,\alpha}^{h,\delta}-x_{n-1,\alpha}^{\delta})-(F(x_{n-1,\alpha}^{h,\delta})-F(x_{n-1,\alpha}^{\delta}))]$$

and

$$\Gamma_2 = (F'(x_0)+\alpha I)^{-1}[F'(x_0)-P_hF'(x_0)+P_hF'(x_0)-P_hF'(P_hx_0)](x_{n,\alpha}^{h,\delta}-x_{n-1,\alpha}^{h,\delta}).$$

Using Lemma 2.2, one has

$$\begin{aligned}
\|\Gamma_1\| &\leq l_0\int_0^1 \|x_0-(x_{n-1,\alpha}^{h,\delta}+t(x_{n-1,\alpha}^{\delta}-x_{n-1,\alpha}^{h,\delta}))\|\|x_{n-1,\alpha}^{\delta}-x_{n-1,\alpha}^{h,\delta}\|dt\\
&\leq l_0\int_0^1 [t\|x_0-x_{n-1,\alpha}^{\delta}\|+(1-t)\|P_hx_0-x_{n-1,\alpha}^{h,\delta}\|\\
&\quad +(1-t)\|P_hx_0-x_0\|]\|x_{n-1,\alpha}^{\delta}-x_{n-1,\alpha}^{h,\delta}\|dt\\
&\leq \frac{l_0}{2}[\frac{\eta}{1-r}+\frac{\eta_h}{1-r_h}+b_h]\|x_{n-1,\alpha}^{h,\delta}-x_{n-1,\alpha}^{\delta}\|\\
&\leq \frac{1}{2}[2\tilde{\tilde{r}}+l_0b_h]\|x_{n-1,\alpha}^{h,\delta}-x_{n-1,\alpha}^{\delta}\|\\
&\leq q\|x_{n-1,\alpha}^{h,\delta}-x_{n-1,\alpha}^{\delta}\|. \qquad (5.16)
\end{aligned}$$

From Assumption 1.4, one has

$$\begin{aligned}
\|\Gamma_2\| &= \|(F'(x_0)+\alpha I)^{-1}[(I-P_h)F'(x_0)-P_h(F'(P_hx_0)-F'(x_0))](x_{n,\alpha}^{h,\delta}-x_{n-1,\alpha}^{h,\delta})\|\\
&\leq \|(F'(x_0)+\alpha I)^{-1}(I-P_h)F'(x_0)\|\\
&\quad +\|(F'(x_0)+\alpha I)^{-1}P_hF'(x_0)\Phi(P_hx_0,x_0,x_{n,\alpha}^{h,\delta}-x_{n-1,\alpha}^{h,\delta})\\
&\leq \frac{\Gamma_h+l_0\|F'(x_0)\|b_h}{\alpha}\|x_{n,\alpha}^{h,\delta}-x_{n-1,\alpha}^{h,\delta}\|. \qquad (5.17)
\end{aligned}$$

It follows from (5.15), (5.16) and (5.17) that

$$\begin{aligned}
\|x_{n,\alpha}^{h,\delta}-x_{n,\alpha}^{\delta}\| &\leq q\|x_{n-1,\alpha}^{h,\delta}-x_{n-1,\alpha}^{\delta}\|+\frac{\Gamma_h+l_0\|F'(x_0)\|b_h}{\alpha}\|x_{n,\alpha}^{h,\delta}-x_{n-1,\alpha}^{h,\delta}\|\\
&\leq q^nb_h+\frac{\Gamma_h+l_0\|F'(x_0)\|b_h}{\alpha}\eta_h(r_h^{n-1}+qr_h^{n-2}+\cdots+q^{n-1})\\
&\leq q^nb_h+\frac{\Gamma_h+l_0\|F'(x_0)\|b_h}{\alpha}\frac{q^n}{q-r_h}\eta_h.
\end{aligned}$$

This completes the proof. □

6. Error Bounds Under Source Conditions

It follows from Proposition 3.1 and Theorem 3.1 that

$$\|x_\alpha^\delta-x_\alpha\|\leq\frac{\delta}{\alpha} \qquad (6.1)$$

and

$$\|x_\alpha-\hat{x}\|\leq(l_0r_0+1)\varphi(\alpha), \qquad (6.2)$$

where x_α is the unique solution of $F(x)+\alpha(x-x_0)=y$.

Combining the estimates in Theorem 2.1, Theorem 5.2, equation (6.1) and equation (6.2), we obtain the following Theorem.

Theorem 6.1. *Let $x_{n,\alpha}^{h,\delta}$ be as in (5.8) and let the assumptions in Theorem 2.1 and Theorem 5.2 be satisfied. Then*

$$\|x_{n,\alpha}^{h,\delta}-\hat{x}\|\leq q^nb_h+\frac{\Gamma_h+l_0\|F'(x_0)\|b_h}{\alpha}\frac{q^n}{q-r_h}\eta_h+\frac{r^n\eta}{1-r}+\frac{\delta}{\alpha}+(l_0r_0+1)\varphi(\alpha). \qquad (6.3)$$

Let

$$n_\delta := \min\{n : \max\{q^n, r^n\} \le \delta\} \tag{6.4}$$

and

$$C_m := \max\{b_h + \frac{\Gamma_h + l_0\|F'(x_0)\|b_h}{q - r_h}\eta_h + \frac{\eta}{1-r} + 1, (l_0 r_0 + 1)\}. \tag{6.5}$$

Theorem 6.2. *Let $x_{n,\alpha}^{h,\delta}$ be as in (5.8) and let the assumptions in Theorem 2.1 and Theorem 5.2 be satisfied. Let n_δ be as in (6.4) and let C_m be as in (6.5). Then, for all $0 < \alpha \le 1$,*

$$\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| \le C_m(\varphi(\alpha) + \frac{\delta}{\alpha}). \tag{6.6}$$

6.1. **A priori choice of the parameter.** We observe that the error $\varphi(\alpha) + \frac{\delta}{\alpha}$ in (6.6) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies $\alpha_\delta \varphi(\alpha_\delta) = \delta$. Using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \le a$, we have $\delta = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$. Hence, $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Using (6.6), we have the following result.

Theorem 6.3. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \le a$, and assumptions in Theorem 6.2 holds. For $\delta > 0$, let $\alpha =: \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Let n_δ be as in (6.4). Then $\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| = O(\psi^{-1}(\delta))$.*

6.2. **An adaptive choice of the parameter.** We will present a parameter choice rule based on the adaptive method studied in [14, 16]. The regularization parameter α is selected from the finite set

$$D_M(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0, 1, \cdots, M\}, \tag{6.7}$$

where $\mu > 1$ and M is such that $\alpha_M < 1 \le \alpha_{M+1}$. We choose $\alpha_0 := \sqrt{\delta}$ because in general $\varphi(\lambda) = \lambda^\nu, 0 < \nu \le 1$ and in this case the best possible error estimate is order $\bigcirc(\sqrt{\delta})$. From Theorem 6.3, it follows that such an accuracy cannot be guaranteed for $\alpha < \sqrt{\delta}$. Let

$$n_M := \min\{n : \max\{q^n, r^n\} \le \delta\} \tag{6.8}$$

and $x_i := x_{n_M,\alpha_i}^{h,\delta}$. We select $\alpha = \alpha_i$ from $D_M(\alpha)$ and operates only with corresponding x_i, $i = 0, 1, \cdots, M$.

Theorem 6.4. *(cf. Theorem 3.5) Assume that there exists $i \in \{0, 1, 2, \cdots, M\}$ such that $\varphi(\alpha_i) \le \frac{\delta}{\alpha_i}$. Let assumptions of Theorem 6.2 and Theorem 6.3 hold and let $l := \max\{i : \varphi(\alpha_i) \le \frac{\delta}{\alpha_i}\} < M$,*

$$k := \max\{i : \|x_i - x_j\| \le 4C_m\frac{\delta}{\alpha_j}, j = 0, 1, 2, \cdots, i\}. \tag{6.9}$$

Then $l \le k$ and $\|\hat{x} - x_k\| \le c\psi^{-1}(\delta)$, where $c = 6C_m\mu$.

7. Implementation of the adaptive choice rule

In this section, we provide an algorithm for the determination of a parameter fulfilling the balancing principle (6.9) and provide a starting point for iteration (5.8) to approximate the unique solution x_α^δ of (1.3). The choice of the starting point involves the following steps:

- Choose $\alpha_0 = \sqrt{\delta}$, $\mu > 1$ and $q < 1$.
- Choose $x_0 \in D(F)$ such that $\|x_0 - \hat{x}\| \le \rho$ and η_h satisfying (5.9).

The choice of the stopping index n_M involves the following step:

- Choose n_M such that $n_M = \min\{n : \max\{q^n, r^n\} \le \delta\}$.

Finally the adaptive algorithm associated with the choice of the parameter specified in Theorem 6.4 involves the following steps.

7.1. **Algorithm.**

- Set $i \leftarrow 0$.
- Solve $x_i := x^{h,\delta}_{n_M,\alpha_i}$ via iteration (5.8).
- If $\|x_i - x_j\| > 4C_m \frac{\sqrt{\delta}}{\mu^j}$, $\quad j \leq i$, then we take $k = i-1$.
- Set $i = i+1$ and return to step 2.

8. Examples

In this section, we consider some simple examples satisfying the assumptions made in this paper and presents a few of examples.

We consider the operator $F : L^2[0,1] \to L^2[0,1]$ defined by [15, Example 6.1]

$$F(x)(s) = K^*K(x)(s) + f(s), \quad x, f \in L^2[0,1], s \in [0,1], \tag{8.1}$$

where $K : L^2[0,1] \to L^2[0,1]$ is a compact linear operator such that the range of K denoted by $R(K)$ is not closed in $L^2[0,1]$. Then the equation $F(x) = y$ is ill-posed as K is compact with non-closed range. The Fréchet derivative $F'(.)$ of F is given by

$$F'(x)z = K^*Kz, \qquad \forall x, z \in L^2[0,1]. \tag{8.2}$$

So, F is monotone on $L^2[0,1]$. Further, for $x, y, z \in L^2[0,1]$, one has

$$[F'(x) - F'(y)]z = 0. \tag{8.3}$$

It is obvious that Assumption 1.4 holds. Since $\Phi(x,y,z) = 0 \leq l_0\|z\|\|x-y\|$, $\forall l_0 \geq 0$ we can choose η_h large enough in step 2 of the algorithm. Further, due to (8.2), $x^{h,\delta}_{m+1,\alpha}$ only needs one step to compute. This can be seen as follows:

$$x^{h,\delta}_{m+1,\alpha} = x^{h,\delta}_{m,\alpha} - (P_hF'(P_hx_0) + \alpha I)^{-1} P_h[F(x^{h,\delta}_{m,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{m,\alpha} - x_0)],$$

i.e.,

$$\begin{aligned}
(P_hF'(P_hx_0) + \alpha I)P_h x^{h,\delta}_{m+1,\alpha} &= (P_hF'(P_hx_0) + \alpha I)P_h x^{h,\delta}_{m,\alpha} - P_h[F(x^{h,\delta}_{m,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{m,\alpha} - x_0)] \\
&= (P_hK^*K + \alpha I)P_h x^{h,\delta}_{m,\alpha} - P_h[K^*Kx^{h,\delta}_{m,\alpha} + f - y^\delta + \alpha(x^{h,\delta}_{m,\alpha} - x_0)] \\
&= -P_h(f - y^\delta - \alpha x_0).
\end{aligned} \tag{8.4}$$

Next, we give the details for implementing the algorithm given in the above section. Let (V_n) be a sequence of finite dimensional subspaces of X and let $P_h, h = 1/n$ denote the orthogonal projection on X with range $R(P_h) = V_n$. We assume that $dim V_n = n+1$, and $\|P_hx - x\| \to 0$ as $h \to 0$ for all $x \in X$. Let$\{v_1, v_2, \cdots, v_{n+1}\}$ be a basis of $V_n, n = 1, 2, \cdots$. Note that $x^{h,\delta}_{m+1,\alpha} \in V_n$. Thus, $x^{h,\delta}_{m+1,\alpha}$ is of the form $\sum_{i=1}^{n+1} \lambda_i v_i$ for some scalars $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$. It can be seen that $x^{h,\delta}_{m+1,\alpha}$ is a solution of (8.4) if and only if $\bar{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{n+1})^T$ is the unique solution of

$$(M_n + \alpha B_n)\bar{\lambda} = \bar{a}, \tag{8.5}$$

where $M_n = (\langle Kv_i, Kv_j\rangle)$, $i, j = 1, 2, \cdots, n+1$ $B_n = (\langle v_i, v_j\rangle)$, $i, j = 1, 2, \cdots, n+1$ and

$$\bar{a} = (\langle P_h(y^\delta + \alpha x_0 - f), v_i\rangle)^T, i = 1, 2, \cdots, n+1.$$

Note that (8.5) is uniquely solvable because M_n is a positive definite matrix (i.e., $xM_nx^T > 0$ for all non-zero vector x) and B_n is an invertible matrix.

8.1. **Numerical Examples.** In order to illustrate the method considered in the above section, we consider the space $X = Y = L^2[0,1]$ and $K : L^2[0,1] \to L^2[0,1]$ as the Fredholm integral operator

$$K(x)(s) = \int_0^1 k(s,t)x(t)dt, \tag{8.6}$$

with

$$k(t,s) = \begin{cases} 0, & t \le s \\ t-s, & t > s. \end{cases} \tag{8.7}$$

We apply the Algorithm in Section 7 by choosing V_n as the space of linear splines in a uniform grid of $n+1$ points in $[0,1]$. Specifically, for fixed n, we consider $t_i = \frac{i-1}{n}, i = 1,2,\cdots,n+1$ as the grid points. We take the basis function $v_i, i = 1,2,\cdots,n+1$ of V_n as follows:

$$v_1(t) = \begin{cases} \frac{t_2-t}{t_2}, & 0 = t_1 \le t \le t_2 \\ 0, & t_2 \le t \le t_{n+1} = 1 \end{cases} \tag{8.8}$$

for $j = 2,3,\cdots,n$,

$$v_j(t) = \begin{cases} 0, & 0 = t_1 \le t \le t_{j-1}, \\ \frac{t-t_{j-1}}{t_j-t_{j-1}}, & t_{j-1} \le t \le t_j, \\ \frac{t_{j+1}-t}{t_{j+1}-t_j}, & t_j \le t \le t_{j+1}, \\ 0, & t_{j+1} \le t \le t_{n+1} = 1 \end{cases} \tag{8.9}$$

and

$$v_{n+1}(t) = \begin{cases} 0, & 0 \le t \le t_n, \\ \frac{t-t_n}{t_{n+1}-t_n}, & t_n \le t \le t_{n+1}. \end{cases} \tag{8.10}$$

Let P_h be the orthogonal projection onto V_n. We note that, for $x \in C[0,1]$,

$$\begin{aligned} \|P_h x - x\|_2 &= dist(x, R(P_h)) \\ &\le \|\pi_n x - x\|_2 \\ &\le \|\pi_n x - x\|_\infty, \end{aligned}$$

where π_n is the (piecewise linear) interpolatory projection onto V_n. It is known that $\|\pi_n x - x\|_\infty \to 0$ as $n \to \infty$. Therefore using the fact that $C[0,1]$ is dense in $L^2[0,1]$, it follows that $\|P_h x - x\|_2 \to 0$ for all $x \in L^2[0,1]$. The elements $Kv_i, i = 1,2,\cdots,n+1$, the entries of the matrix B_n, M_n and $\bar{a}$ are computed explicitly. For the operator K defined by (8.6) and (8.7), $\Gamma_h = \gamma_h = \|(I-P_h)F'(x_0)\| = \|(I-P_h)K^*K\| = O(n^{-2})$ (see, [11]).

Example 8.1. Take $y = \frac{1}{720}(26 + s^6 - 6s^5 + 15s^4 - 36s) + f(s)$, where $f(s) = s^2$ and $x_0 = 0$. Then the exact solution is $\hat{x} = \frac{1}{2}(s-1)^2$. Since $\hat{x} - x_0 = \hat{x} = K^*1 \in R(K^*) = R(F'(\hat{x})^{1/2})$, $\varphi(\lambda) = \lambda^{1/2}$. Hence $\psi^{-1}(\delta) = \varphi(\alpha_\delta) = (\delta)^{1/3}$. This implies that $\|\hat{x} - x_k\| \le c\psi^{-1}(\delta)$, where $c = 6C_m\mu$. The result are given in the following Table and Figures.

n	k	e_k	$\frac{e_k}{\psi^{-1}(\delta)}$
4	100	0.0377	0.1385
8	99	0.0385	0.1400
16	99	0.0385	0.1399
32	99	0.0385	0.1400
64	99	0.0385	0.1400
128	99	0.0385	0.1400
256	99	0.0385	0.1400
512	99	0.0385	0.1400
1024	99	0.0385	0.1400

Table 3.1: $\delta = 0.001$; $\mu = 1.002$.

Here $e_k := \|x_k - \hat{x}\|$ and $y^\delta = y + \delta$.

Remark 8.1. The last column of the Table shows that $e_k = O(\psi^{-1}(\delta))$. From computation, we observe that due to the round off error k and e_k remains as a constant for large values of n.

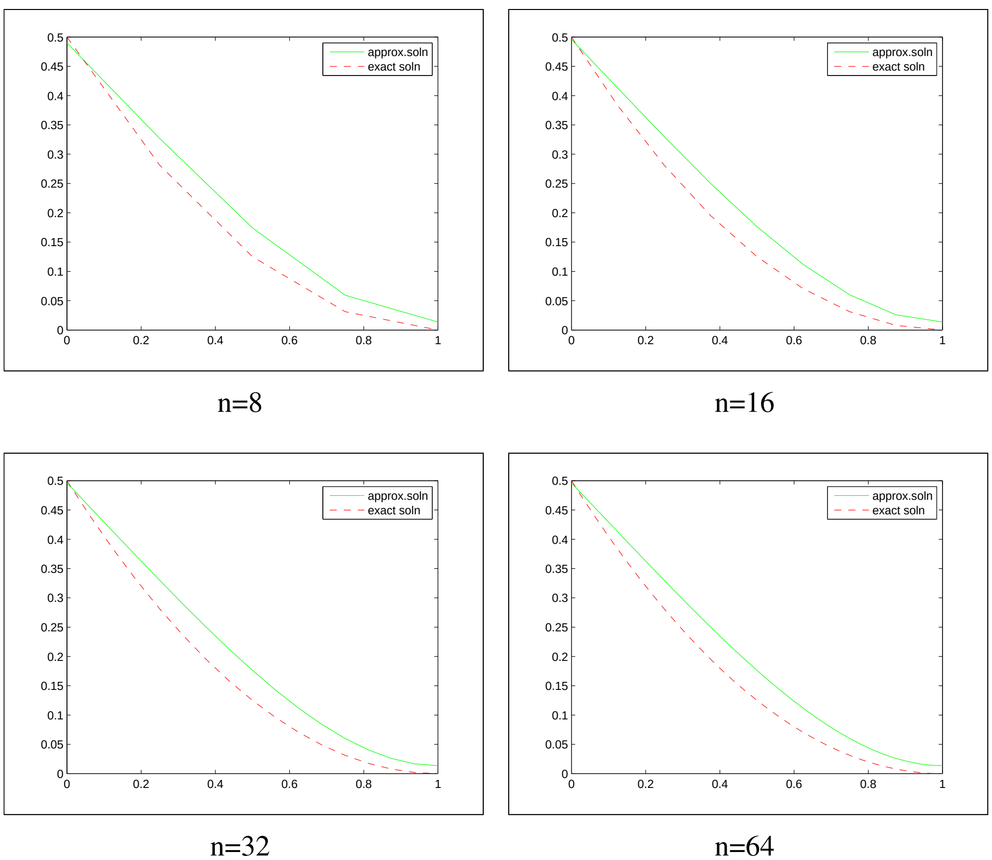

FIGURE 1. Curve of the exact and approximate solutions

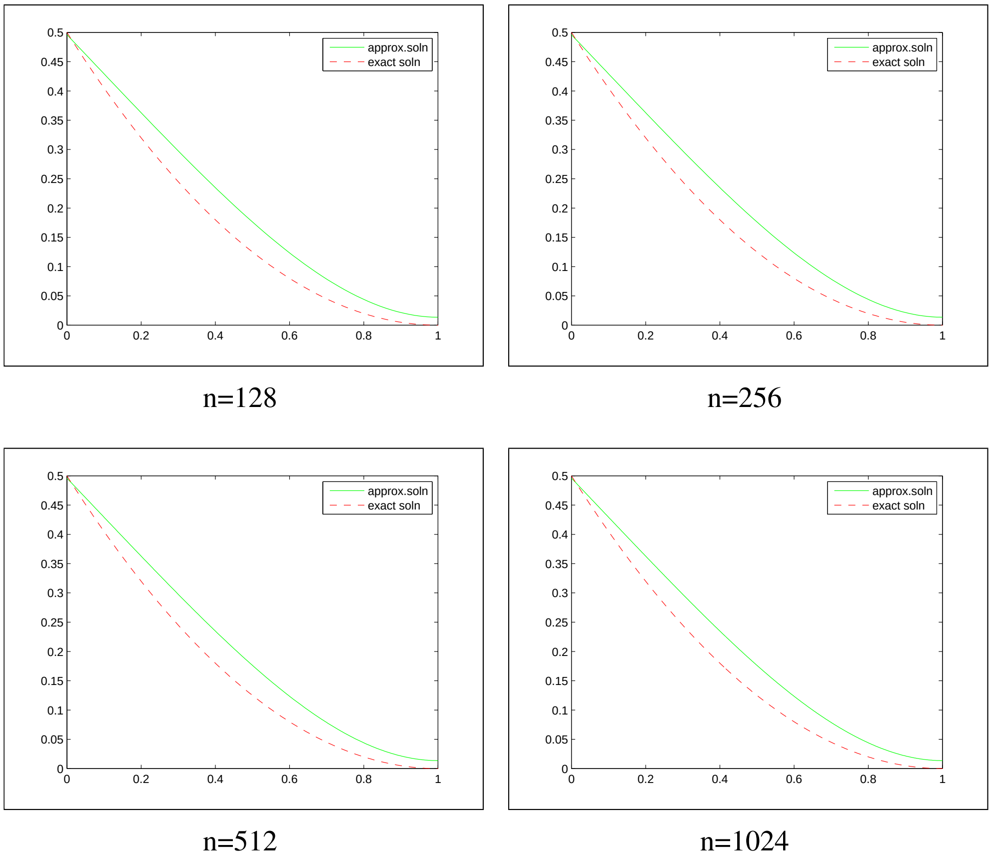

FIGURE 2. Curve of the exact and approximate solutions

9. CONCLUDING REMARK

In this paper, we considered an iteratively regularized projection method for solving the nonlinear ill-posed operator equation $F(x) = y$, when the available data is y^δ in place of the exact data y with $\|y - y^\delta\| \leq \delta$. It is assumed that F is Fréchet differentiable in a neighborhood of some initial guess x_0 of the actual solution $\hat{x}$. The procedure involves finding the fixed point of the function $G_h(x) := x - (P_h F'(P_h x_0) + \alpha I)^{-1} P_h(F(x) - y^\delta + \alpha(x - x_0))$, in a finite dimensional subspace X_h of X iteratively, where P_h is the orthogonal projection on to X_h. For choosing the regularization parameter α, we employed the adaptive method suggested by Pereversev and Schock in [16] and the stopping rule is based on a majorizing sequence. Our numerical experiments show that if α is chosen according to the balancing principle (6.9), then $\|x_k - \hat{x}\| \leq c\psi^{-1}(\delta)$.

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