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Ioannis K. Argyros & Santhosh George

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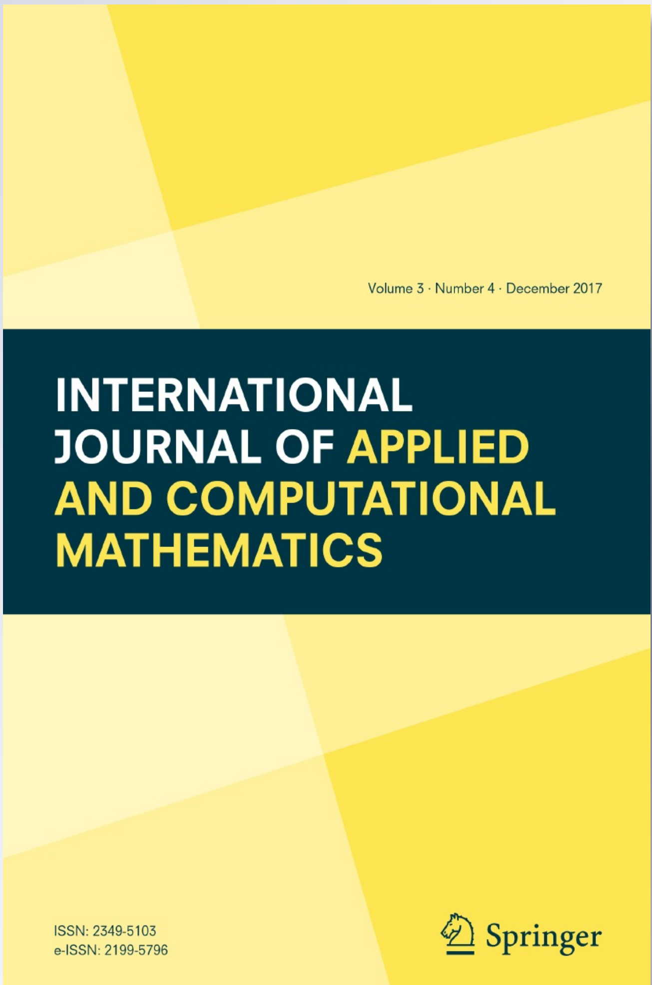

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ORIGINAL PAPER

Expanding the Applicability of the Kantorovich's Theorem for Solving Generalized Equations Using Newton's Method

Ioannis K. Argyros[1] · Santhosh George[2]

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Abstract In this paper we consider the Kantorovich's theorem for solving generalized equations $F(x) + Q(x) \ni 0$ using Newton's method, where F is a Fréchet differentiable function and Q is a set-valued and maximal monotone function acting between Hilbert spaces. We used our new idea of restricted convergence domains to obtain better location about where the iterates are located leading to a tighter convergence analysis than in the earlier studies and under the same or less computational cost of the majorant functions involved.

Keywords Generalized equation · Kantorovich's theorem · Newton's method · Restricted convergence domains · Maximal monotone operator

Mathematics Subject Classification 65G99 · 90C30 · 49J53

Introduction

In [18], G. S. Silva considered the problem of approximating the solution of the generalized equation

$$F(x) + Q(x) \ni 0, \tag{1}$$

where $F : D \longrightarrow H$ is a Fréchet differentiable function, H is a Hilbert space with inner product $\langle ., . \rangle$ and corresponding norm $\|.\|$, $D \subseteq H$ an open set and $T : H \rightrightarrows H$ is set-valued and maximal monotone. It is well known that the system of nonlinear equations and abstract

✉ Santhosh George
sgeorge@nitk.ac.in

Ioannis K. Argyros
iargyros@cameron.edu

1 Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA

2 Department of Mathematical and Computational Sciences, NIT Karnataka, Mangalore 575 025, India

inequality system can be modeled as equation of the form (1) [17]. If $\psi : H \longrightarrow (-\infty, +\infty]$ is a proper lower semi continuous convex function and

$$Q(x) = \partial\psi(x) = \{u \in H : \psi(y) \geq \psi(x) + \langle u, y - x\rangle\}, \quad \text{for all } y \in H \tag{2}$$

then (1) becomes the variational inequality problem

$$F(x) + \partial\psi(x) \ni 0,$$

including linear and nonlinear complementary problems. Newton's method for solving (1) for an initial guess x_0 is defined by

$$F(x_k) + F'(x_k)(x_{k+1} - x_k) + Q(x_{k+1}) \ni 0, \quad k = 0, 1, 2 \ldots \tag{3}$$

has been studied by several authors [1–24]. In [13], Kantorovich obtained a convergence result for Newton's method for solving the equation $F(x) = 0$ under some assumptions on the derivative $F'(x_0)$ and $\|F'(x_0)^{-1}F(x_0)\|$. Kantorovich, used the majorization principle to prove his results. Later in [16], Robinson considered generalization of the Kantorovich theorem of the type $F(x) \in K$, where K is a nonempty closed and convex cone, and obtained convergence results and error bounds for this method. Josephy [12], considered a semilocal Newton's method of the kind (3) in order to solving (1) with $F = N_C$ the normal cone mapping of a convex set $C \subset \mathbb{R}^2$.

The main concern in this paper is the enlargement of the convergence domain which is small in general, the improvement of the error bounds and a better information on the location of the solution. The novelty of this paper is in the fact that the above are obtained under the same computational cost on the majorizing functions involved. In particular, the sufficient convergence conditions are weaker as demonstrated in the theoretical part of the paper as well as through the numerical example. So it is clear that readers in this area will go and use the new sufficient criteria and abandon the old ones, since those will never be better than the new ones (unless if all majorizing sequences are the same, which is not true in general and thats the whole point of this paper). These are main concerns in computational mathematics.

The rest of this paper is organized as follows. Preliminaries are given in "Preliminaries" section and the main results are presented in the concluding "Semilocal Convergence" section.

Preliminaries

Let $U(x, \rho)$ and $\bar{U}(x, \rho)$ stand respectively for open and closed balls in H with center $x \in H$ and radius $\rho > 0$. The following Definitions and Lemmas are used for proving our results. These items are stated briefly here in order to make the paper as self contains as possible. More details can be found in [18].

Definition 2.1 Let $D \subseteq H$ be an open nonempty subset of H, $h : D \longrightarrow H$ be a Fréchet differentiable function with Fréchet derivative h' and $Q : H \rightrightarrows H$ be a set mapping. The partial linearization of the mapping $h + Q$ at $x \in H$ is the set-valued mapping $L_h(x, .) : H \rightrightarrows H$ given by

$$L_h(x, y) := h(x) + h'(x)(y - x) + Q(y). \tag{4}$$

For each $x \in H$, the inverse $L_h(x, .)^{-1} : H \rightrightarrows H$ of the mapping $L_h(x, .)$ at $z \in H$ is defined by

$$L_h(x, .)^{-1} := \{y \in H : z \in h(x) + h'(x)(y - x) + Q(y)\}. \tag{5}$$

Definition 2.2 Let $Q : H \rightrightarrows H$ be a set-valued operator. Q is said to be monotone if for any $x, y \in domQ$ and $u \in Q(y)$, $v \in Q(x)$ implies that the following inequality holds:

$$\langle u - v, y - x\rangle \geq 0.$$

A subset of $H \times H$ is monotone if it is the graph of a monotone operator. If $\varphi : H \longrightarrow (-\infty, +\infty]$ is a proper function then the subgradient of φ is monotone.

Definition 2.3 Let $Q : H \rightrightarrows H$ be monotone. Then Q is maximal monotone if the following implication holds for all $x, u \in H$:

$$\langle u - v, y - x\rangle \geq 0 \text{ for each } y \in domQ \text{ and } v \in Q(y) \Rightarrow x \in domQ \text{ and } v \in Q(y). \tag{6}$$

Lemma 2.4 ([22]) *Let G be a positive operator (i.e.,$\langle G(x), x\rangle \geq 0$). The following statements about G hold:*

- $\|G^2\| = \|G\|^2$;
- *If G^{-1} exists, then G^{-1} is a positive operator.*

Lemma 2.5 ([21, Lemma 2.2]) *Let G be a positive operator. Suppose G^{-1} exists, then for each $x \in H$ we have*

$$\langle G(x), x\rangle \geq \frac{\|x\|^2}{\|G^{-1}\|}.$$

Semilocal Convergence

We present the semilocal convergence analysis of generalized Newton's method using some more flexible scalar majorant functions than in [18].

Theorem 3.1 *Let $F : D \subseteq H \longrightarrow H$ be continuous with Fréchet derivative F' continuous on D. Let also $Q : H \rightrightarrows H$ be a set-valued operator. Suppose that there exists $x_0 \in D$ such that $F'(x_0)$ is a positive operator and $\hat{F}'(x_0)^{-1}$ exists. Let $R > 0$ and $\rho := \sup\{t \in [0, R) : U(x_0, t) \subseteq D\}$. Suppose that there exists $f_0 : [0, R) \longrightarrow \mathbb{R}$ twice continuously differentiable such that for each $x \in U(x_0, \rho)$*

$$\|\hat{F}'(x_0)^{-1}\|\|F'(x) - F'(x_0)\| \leq f_0'(\|x - x_0\|) - f_0'(0). \tag{7}$$

Moreover, suppose that

(h_1^0) $f_0(0) > 0$ *and* $f_0'(0) = -1$.

(h_2^0) f_0' *is convex and strictly increasing.*

(h_3^0) $f_0(t) = 0$ *for some* $t \in (0, R)$.

Then, sequence $\{t_n^0\}$ generated by $t_0^0 = 0$,

$$t_{n+1}^0 = t_n^0 - \frac{f_0(t_n^0)}{f_0'(t_n^0)}, \quad n = 0, 1, 2, \ldots$$

is strictly increasing, remains in $(0, t_0^*)$ and converges to t_0^*, where t_0^* is the smallest zero of function f_0 in $(0, R)$. Furthermore, suppose that for each $x, y \in D_1 := \bar{U}(x_0, \rho) \cap U(x_0, t_0^*)$ there exists $f_1 : [0, \rho_1) \longrightarrow \mathbb{R}$, $\rho_1 = \min\{\rho, t_0^*\}$ such that

$$\begin{aligned}&\|\hat{F}'(x_0)^{-1}\|\|F'(y)-F'(x)\| \leq f_1'(\|x-y\|+\|x-x_0\|)-f_1'(\|x-x_0\|)\\ &\textit{and}\ \ \|x_1-x_0\| \leq f_1'(0)\end{aligned} \tag{8}$$

Moreover, suppose that

(h_1^1) $f_1(0) > 0$ *and* $f_1'(0) = -1$.
(h_2^1) f_1' *is convex and strictly increasing.*
(h_3^1) $f_1(t) = 0$ *for some* $t \in (0, \rho_1)$.
(h_4^1) $f_0(t) \leq f_1(t)$ *and* $f_0'(t) \leq f_1'(t)$ *for each* $t \in [0, \rho_1)$.

Then, f_1 *has a smallest zero* $t_1^* \in (0, \rho_1)$, *the sequences generated by generalized Newton's method for solving the generalized equation* $F(x) + Q(x) \ni 0$ *and the scalar equation* $f_1(0) = 0$, *with initial point* x_0 *and* t_0^1 *(or* $s_0 = 0$*), respectively,*

$$\begin{aligned}&0 \in F(x_n) + F'(x_n)(x_{n+1}-x_n) + Q(x_{n+1}),\\ &t_{n+1}^1 = t_n^1 - \frac{f_1(t_n^1)}{f_1'(t_n^1)}\ \ (or\ s_{n+1} = s_n - \frac{f_1(s_n)}{f_0'(s_n)})\end{aligned}$$

are well defined, $\{t_n^1\}$ *(or* s_n*) is strictly increasing, remains in* $(0, t_1^*)$ *and converges to* t_1^*. *Moreover, sequence* $\{x_n\}$ *generated by generalized Newton's method (3) is well defined, remains in* $U(x_0, t_1^*)$ *and converges to a point* $x^* \in \bar{U}(x_0, t_1^*)$, *which is the unique solution of generalized equation* $F(x) + Q(x) \ni 0$ *in* $\bar{U}(x_0, t_1^*)$. *Furthermore, the following estimates hold:*

$$\begin{aligned}\|x_n - x^*\| &\leq t_1^* - t_n^1,\ \ \|x_n - x^*\| \leq t_1^* - s_n,\\ \|x_{n+1} - x^*\| &\leq \frac{t_1^* - t_{n+1}^1}{(t_1^* - t_n^1)^2}\|x_n - x^*\|^2\\ \|x_{n+1} - x^*\| &\leq \frac{t_1^* - s_{n+1}^1}{(t_1^* - s_n^1)^2}\|x_n - x^*\|^2\\ s_{n+1} - s_n &\leq t_{n+1}^1 - t_n^1,\end{aligned}$$

and sequences $\{t_n^1\}$, $\{s_n\}$ *and* $\{x_n\}$ *converge* $Q-$*linearly as follows:*

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \frac{1}{2}\|x_n - x^*\|,\\ t_1^* - t_{n+1}^1 &\leq \frac{1}{2}(t_1^* - t_n),\end{aligned}$$

and

$$t_1^* - s_{n+1}^1 \leq \frac{1}{2}(t_1^* - s_n).$$

Finally, if
(h_5^1) $f_1'(t_1^*) < 0$,
then the sequences $\{t_n^1\}$, $\{s_n\}$ *and* $\{x_n\}$ *converge* $Q-$*quadratically as follows:*

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \frac{D^- f_1'(t_1^*)}{-2f_1'(t_1^*)}\|x_n - x^*\|^2,\\ \|x_{n+1} - x^*\| &\leq \frac{D^- f_1'(t_1^*)}{-2f_0'(t_1^*)}\|x_n - x^*\|^2,\end{aligned}$$

$$t_1^* - t_{n+1} \leq \frac{D^- f_1'(t_1^*)}{-2 f_1'(t_1^*)}(t_1^* - t_n)^2,$$

and

$$t_1^* - s_{n+1} \leq \frac{D^- f_1'(t_1^*)}{-2 f_0'(t_1^*)}(t_1^* - s_n)^2,$$

where D^- *stands for the left directional derivative of function* f_1.

Remark 3.2 (a) Suppose that there exists $f : [0, R) \longrightarrow \mathbb{R}$ twice continuously differentiable such that for each $x, y \in U(x_0, \rho)$

$$\|\hat{F}'(x_0)^{-1}\| \|F'(y) - F'(x)\| \leq f'(\|x - y\| + \|x - x_0\|) - f'(\|x - x_0\|). \tag{9}$$

If $f_1(t) = f_0(t) = f(t)$ for each $t \in [0, R)$, then Theorem 3.1 specializes to Theorem 4 in [18]. Otherwise, i.e., if

$$f_0(t) \leq f_1(t) \leq f(t) \text{ for each } t \in [0, \rho_1), \tag{10}$$

then, our Theorem is an improvement of Theorem 4 under the same computational cost, since in practice the computation of function f requires the computation of functions f_0 or f_1 as special cases. Moreover, we have that for each $t \in [0, \rho_1)$

$$f_0(t) \leq f(t) \tag{11}$$

and

$$f_1(t) \leq f(t) \tag{12}$$

leading to $t_n^1 \leq t_n, \ s_n \leq t_n,$

$$t_{n+1}^1 - t_n^1 \leq t_{n+1} - t_n \tag{13}$$
$$s_{n+1}^1 - s_n^1 \leq s_{n+1} - s_n \tag{14}$$

and

$$t_1^* \leq t^*, \tag{15}$$

where $\{t_n\}$ is defined by

$$t_0 = 0, \ t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)},$$

$t^* = \lim_{n \longrightarrow \infty} t_n$ and t^* is the smallest zero of function f in $(0, R)$ (provided that the "h" conditions hold for function f replacing f_1 and f_0). If

$$-\frac{f_1(t)}{f_1'(t)} \leq -\frac{f(s)}{f'(s)} \tag{16}$$

or

$$-\frac{f_1(t)}{f_0'(t)} \leq -\frac{f(s)}{f'(s)}, \tag{17}$$

respectively for each $t \leq s$. Estimates (13) and (14) can be strict if (16) and (17) hold as strict inequalities.

(b) So far we have improved the error bounds and the location of the solution x^* but not necessarily the convergence domain of the generalized Newton's method (3). We can also show that convergence domain can be improved in some interesting special cases. Let $F \equiv \{0\}$,

$$f(t) = \frac{L}{2}t^2 - t + \eta,$$
$$f_0(t) = \frac{L_0}{2}t^2 - t + \eta$$

and

$$f_1(t) = \frac{L_1}{2}t^2 - t + \eta,$$

where $\|x_1 - x_0\| \leq \eta$ and L, L_0 and L_1 are Lipschitz constants satisfying:

$$\|\hat{F}'(x_0)^{-1}\|\|F'(y) - F'(x)\| \leq L\|y - x\|$$
$$\|\hat{F}'(x_0)^{-1}\|\|F'(x) - F'(x_0)\| \leq L_0\|x - x_0\|$$

and

$$\|\hat{F}'(x_0)^{-1}\|\|F'(y) - F'(x)\| \leq L_1\|y - x\|,$$

on the corresponding balls. Then, we have that

$$L_0 \leq L$$

and

$$L_1 \leq L.$$

The corresponding majorizing sequences are

$$t_0 = 0,\ t_1 = \eta,\ t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)} = t_n + \frac{L(t_n - t_{n-1})^2}{2(1 - Lt_n)},\ \ n = 1, 2, \ldots$$
$$t_0^1 = 0,\ t_1^1 = \eta,\ t_{n+1}^1 = t_n^1 - \frac{f_1(t_n^1)}{f_1'(t_n^1)} = t_n^1 + \frac{L_1(t_n^1 - t_{n-1}^1)^2}{2(1 - Lt_n^1)},\ \ n = 1, 2, \ldots$$
$$s_0 = 0,\ s_1 = \eta,$$
$$s_{n+1} = s_n - \frac{f_1(s_n) - f_1(s_{n-1}) - f_1'(s_{n-1})(s_n - s_{n-1})}{f_0'(s_n)}$$
$$= s_n + \frac{L_1(s_n - s_{n-1})^2}{2(1 - L_0 s_n)},\ \ n = 1, 2, \ldots.$$

Then, sequences converge provided, respectively that

$$q = L\eta \leq \frac{1}{2} \tag{18}$$

and for the last two

$$q_1 = L_1\eta \leq \frac{1}{2},$$

so

$$q \leq \frac{1}{2} \Longrightarrow q_1 \leq \frac{1}{2}.$$

It turns out from the proof of Theorem 3.1 that sequence $\{r_n\}$ defined by [6]

$$r_0 = 0,\ r_1 = \eta,\ r_2 = r_1 + \frac{L_0(r_1 - r_0)^2}{2(1 - L_0 r_1)},$$
$$r_{n+2} = r_{n+1} + \frac{L_1(r_{n+1} - r_n)^2}{2(1 - L_0 r_{n+1})}$$

is also a tighter majorizing sequence than the preceding ones for $\{x_n\}$. The sufficient convergence condition for $\{r_n\}$ is given by [6]:

$$q_2 = K\eta \le \frac{1}{2},$$

where

$$K = \frac{1}{8}(4L_0 + \sqrt{L_1 L_0 + 8L_0^2} + \sqrt{L_0 L_1}).$$

Then, we have that

$$q_1 \le \frac{1}{2} \Longrightarrow q_2 \le \frac{1}{2}.$$

Hence, the old results in [18] have been improved. Similar improvements can follow for the Smale's alpha theory [2,6] or Wang's theory [18,22,24]. Examples where $L_0 < L$ or $L_1 < L$ or $L_0 < L_1$ can be found in [6]. It is worth noticing that (18) is the famous for its simplicity and clarity Newton-Kantorovich hypothesis for solving nonlinear equations using Newton's method [13] employed as a sufficient convergence condition in all earlier studies other than ours.

(c) The introduction of (8) depends on (7) (i.e., f_1 depends on f_0). Such an introduction was not possible before (i.e., when f was used instead of f_1).

Proof of Theorem 3.1 Simply notice that the iterates $\{x_n\} \in D_1$ which is a more precise location than $\bar{U}(x_0, \rho)$ used in [18], since $D_1 \subseteq \bar{U}(x_0, \rho)$. Then, the definition of function f_1 becomes possible and replaces f in the proof [18], whereas for the computation on the upper bounds $\|\hat{F}'(x)^{-1}\|$ we use the more precise f_0 than f as it is shown in the next perturbation Banach lemma [13]. □

Lemma 3.3 *Let $x_0 \in D$ be such that $\bar{F}'(x_0)$ is a positive operator and $\bar{F}'(x_0)^{-1}$ exists. If $\|x - x_0\| \le t < t^*$, then $\bar{F}'(x)$ is a positive operator and $\bar{F}'(x)^{-1}$ exists. Moreover,*

$$\|\bar{F}'(x)^{-1}\| \le \frac{\bar{F}'(x_0)^{-1}\|}{f_0'(t)}. \tag{19}$$

Proof Observe that

$$\|\bar{F}'(x) - \bar{F}'(x_0)\| \le \frac{1}{2}\|\bar{F}'(x) - \bar{F}'(x_0)\| + \frac{1}{2}\|(\bar{F}'(x) - \bar{F}'(x_0))^*\| = \|\bar{F}'(x) - \bar{F}'(x_0)\|. \tag{20}$$

Let $x \in \bar{U}(x_0, t),\ 0 \le t < t^*$. Thus $f'(t) < 0$. Using, (h_1^1) and (h_2^1), we obtain that

$$\begin{aligned} \|\bar{F}'(x_0)^{-1}\|\|\bar{F}'(x) - \bar{F}'(x_0)\| &\le \|\bar{F}'(x_0)^{-1}\|\|\bar{F}'(x) - \bar{F}'(x_0)\| \\ &\le f_0'(\|x - x_0\|) - f_0(0) \\ &< f_0'(t) + 1 < 1. \end{aligned} \tag{21}$$

So by Banach's Lemma on invertible operators, we have $\bar{F}'(x)^{-1}$ exists. Moreover by above inequality,

$$\|\bar{F}'(x)^{-1}\| \leq \frac{\|\bar{F}'(x_0)^{-1}\|}{1 - \|\bar{F}'(x_0)^{-1}\|\|F'(x) - F'(x_0)\|} \leq \frac{\|\bar{F}'(x_0)^{-1}\|}{1 - (f_0'(t) + 1)} = -\frac{\|F'(x_0)^{-1}\|}{f_0'(t)}.$$

Using (21) we have

$$\|\bar{F}'(x) - \bar{F}'(x_0)\| \leq \frac{1}{\|\bar{F}'(x_0)^{-1}\|}. \tag{22}$$

Thus, we have

$$\langle(\bar{F}'(x_0) - \bar{F}'(x))y, y\rangle \leq \|\bar{F}'(x_0) - \bar{F}'(x)\|\|y\|^2 \leq \frac{\|y\|^2}{\|\bar{F}'(x_0)^{-1}\|},$$

which implies,

$$\langle\bar{F}'(x_0)y, y\rangle - \frac{\|y\|^2}{\|\bar{F}'(x_0)^{-1}\|} \leq \langle\bar{F}'(x)y, y\rangle. \tag{23}$$

Now since $\bar{F}'(x_0)$ is a positive operator and $\bar{F}'(x_0)^{-1}$ exists by assumption, we obtain that

$$\langle\bar{F}'(x_0)y, y\rangle \geq \frac{\|y\|^2}{\|\bar{F}'(x_0)^{-1}\|}. \tag{24}$$

The result now follows from (23) and (24). □

Remark 3.4 This result improves the corresponding one in [18, Lemma 8](using function f instead of f_0 or f_1) leading to more precise estimates on the distances $\|x_{n+1} - x^*\|$ which together with idea of restricted convergence domains lead to the aforementioned advantages stated in Remark 3.2.

Next, we present an academic example to show that: (11), (12) hold as strict inequalities; the old convergence criteria (18) does not hold but the new involving q_1 and q_2 do hold and finally the new majorizing sequences $\{s_n\}$ and $\{r_n\}$ are tighter than the old majorizing sequence $\{t_n\}$. Hence, the advantages of our approach (as already stated in the abstract of this study) over the corresponding ones such as the ones in [18] follow.

Example 3.5 Let $F \equiv \{0\}$, $H = \mathbb{R}$, $D = U(x_0, 1-p)$, $x_0 = 1$, $p \in (0, \frac{1}{2})$ and define function F on D by

$$F(x) = x^3 - p. \tag{25}$$

Then, we get using (25) that $L = 2(2-p)$, $L_0 = 3-p$, $L_1 = 2(1 + \frac{1}{L_0})$ and $\eta = \frac{1}{3}(1-p)$, so we have

$$L_0 < L_1 < L$$

and

$$f_0(t) < f_1(t) < f(t) \quad \text{for each} \quad p \in \left(0, \frac{1}{2}\right).$$

In particular the sufficient Kantorovich convergence criterion (18) used in [18] is not satisfied, since

$$q = L\eta > \frac{1}{2} \quad \text{for each} \quad p \in \left(0, \frac{1}{2}\right). \tag{26}$$

Table 1 Error bound comparison

n	$t_{n+1} - t_n$	$s_{n+1} - s_n$	$r_{n+1} - r_n$
2	0.0397	0.0370	0.0314
3	0.0043	0.0033	0.0023
4	5.1039e−05	2.6253e−05	1.2571e−05
5	7.2457e−09	1.6743e−09	3.7365e−10

Hence, there is no guarantee under the results in [18] that Newton's method converges to the solution $x^* = \sqrt[3]{p}$. However, our convergence criteria are satisfied:

$$q_1 = L_1 \eta \leq \frac{1}{2} \quad \text{for each} \quad p \in \left(0.4619831630, \frac{1}{2}\right). \tag{27}$$

and

$$q_2 = K \eta \leq \frac{1}{2} \quad \text{for each} \quad p \in \left(0.2756943786, \frac{1}{2}\right). \tag{28}$$

In order for us to compare the majorizing sequences $\{t_n\}$, $\{s_n\}$, $\{r_n\}$, let $p = 0.6$. Then, old condition (26) as well as new conditions (27) and (28) are satisfied. Then, we present the following table showing that the new error bounds are tighter. It is worth noticing that these advantages are obtained under the same computational cost since the evaluation of the old Lipschitz constants L requires the computation of the new constants L_0, L_1 and K as special cases (Table 1).

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