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# Extended Semi-local Convergence of Newton's Method on Lie Groups Using Restricted Regions

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**Abstract**

We extend the applicability of Newton's method used to approximate a solution of a mapping involving Lie valued operators. Using our idea of the restricted convergence region, we locate a more precise set containing the Newton iterates leading to tighter majorizing sequences than before. This way and under the same computational cost as before, we show the semi-local convergence of Newton's method with the following advantages over earlier works: weaker sufficient convergence criteria, tighter error bounds on the distances involved and at least as precise information on the location of the solution.

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**Key words:** Newton's method, Lie group, Lie algebra, Riemannian manifold, semi-local convergence, majorizing sequence, Kantorovich hypothesis.

# 1 Introduction

In this study, we are concerned with the problem of approximating a zero $x_\star$ of $\mathcal{C}^1$-mapping $F : G \longrightarrow Q$, where $G$ is a Lie group and $Q$ the Lie algebra of G that is the tangent space $T_e G$ of $G$ at $e$, equipped with the Lie bracket $[.,.] : Q \times Q \longrightarrow Q$ [5, 6, 7, 17, 20, 22].

The study of numerical algorithms on manifolds for solving eigenvalue or optimization problems on Lie groups [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] is very important in Computational Mathematics. Newton-type methods are the most popular iterative procedures used to solve equations, when these equations contain differentiable operators. A local as well as a semilocal convergence of Newton-type methods has been given by several authors under various conditions [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. There are two types of convergence results: the first uses information from the domain of an operator (see the well-known Kantorovich theorem [23]); where as the second uses information only at a point (see Smale's paper [25]). In particular, a convergence analysis of Newton's method on manifolds can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 25].

Newton's method (NM) with initial point $x_0 \in G$ was first introduced by Owren and Welfert [24] in the form

$$x_{n+1} = x_n \cdot \exp\left(-dF_{x_n}^{-1} F(x_n)\right) \quad (n \geq 0). \tag{1.1}$$

NM is undoubtedly the most popular method for generating a sequence $\{x_n\}$ approximating $x_\star$.

In this article, we are motivated by the work in [21] and optimization considerations. The following advantages are obtained in the semi-local case $(\mathcal{A}_1)$:

(i) Weaker sufficient convergence criteria and a larger convergence region.

(ii) Tighter error estimates on the distances involved;

and

(iii) An at least as precise information on location of the solution $x_\star$.

That is, the applicability of NM is extended.

The rest of the paper is structured as follows. Section 2 contains the necessary background on Lie groups. In Section 3, we present the semi-local convergence of NM method.

# 2 Background

In this section, we re-introduce standard concepts and notations from [5, 6, 17, 20, 21, 22], to make the paper as self contained as possible,.

"A Lie group (G,·) is a Hausdorff topological group with countable bases which also has the structure of a smooth manifold such that the group product and the inversion are smooth operations in the differentiable structure given on the manifold. The dimension of a Lie group is that of the underlying manifold and we shall always assume that it is finite. The symbol $e$ designates the identity element of $G$. Let $Q$ be the Lie algebra of the Lie group $G$ which is the tangent space $T_eG$ of $G$ at $e$, equipped with Lie bracket $[\cdot,\cdot] : Q \times Q \to Q$. In the sequel we will make use of the left translation of the Lie group $G$. We define for each $y \in G$

$$\begin{array}{rcl} L_y : G & \longrightarrow & G \\ z & \longrightarrow & y \cdot z \end{array} \tag{2.1}$$

the left multiplication in the group. The differential of $L_y$ at $e$ denoted by $(dL_y)_e$ determines an isomorphism of $Q = T_eG$ with the tangent space $T_yG$ via the relation

$$(dL_y)_e(Q) = T_yG,$$

or, equivalently,

$$Q = (dL_y)_e^{-1}(T_yG) = (dL_{y-1})_y(T_yG).$$

The exponential map is noted by *exp* and defined by

$$\begin{array}{rcl} \exp : Q & \longrightarrow & G \\ u & \longrightarrow & \exp(u), \end{array}$$

which is certainly the most important construct associated to $G$ and $Q$. Given $u \in Q$, the left invariant vector field $X_u : y \longrightarrow (dL_y)_e(u)$ determines an one-parameter subgroup of $G$: $\sigma_u : \mathbb{R} \longrightarrow G$ such that

$$\sigma_u(0) = e \quad and \quad \sigma_u'(t) = X_u(\sigma_u(t)) = (dL_{\sigma_u(t)})_e(u) \quad \forall\, t \in \mathbb{R}. \tag{2.2}$$

Consequently, the exponential map is defined by the relation

$$\exp(u) = \sigma_u(1).$$

Consider $F : G \longrightarrow Q = T_eG$ be a $\mathcal{C}^1$-mapping. The differential of $F$ at a point $x \in G$ is a linear map $F_x^{'} : T_xG \longrightarrow Q$ defined by

$$F_x'(\triangle_x) = \frac{d}{dt} F(x \cdot \exp(t((dL_{x-1})_x)(\triangle_x)))\,|_{t=0} \;\text{ for any }\; \triangle_x \in T_xG. \tag{2.3}$$

The differential $F_x'$ can be expressed via a function $dF_x : Q \longrightarrow Q$ given by

$$dF_x = (F \circ L_x)_e' = F_x' \circ (dL_x)_e.$$

Thus, by (2.3), it follows that

$$dF_x(u) = F_x'((dL_x)_e(u)) = \frac{d}{dt} F(x \cdot \exp(tu))\,|_{t=0} \quad \text{for any} \quad u \in Q.$$

Therefore the following lemma is clear.

**Lemma 2.1** *Let $x \in G$, $u \in Q$ and $t \in \mathbb{R}$. Then*

$$\frac{d}{dt}F(x \cdot \exp(-tu)) = -dF_{x \cdot \exp(-tu)}(u) \tag{2.4}$$

*and*

$$F(x \cdot \exp(tu)) - F(x) = \int_0^t dF_{x \cdot \exp(su)}(u)\, ds.'' \tag{2.5}$$

## 3 Semi-local convergence

We shall study the semi-local convergence of NM. In the rest of the paper we assume $\langle \cdot, \cdot \rangle$ the inner product and $\| \cdot \|$ on $Q$. As in [5, 6, 21, 22] we define a distance on $G$ for $x, y \in G$ as follows:

$$m(x, y) = \inf \quad \{ \quad \sum_{i=1}^{k} \| z_i \| : \text{there exist } k \geq 1 \text{ and } z_1, \cdots, z_k \in Q \quad \text{such that } y = x \cdot exp\, z_1 \cdots exp\, z_k \}. \tag{3.1}$$

By convention $\inf \emptyset = +\infty$. It is easy to see that $m(\cdot, \cdot)$ is a distance on $G$ and the topology induced is equivalent to the original one on $G$. Let $w \in G$ and $r > 0$, we denote by $B(w, r) = \{y \in G : m(w, y) < r\}$ the open ball centered at $w$ and of radius $r$. Moreover, we denote the closure of $B(w, r)$ by $\overline{B}(w, r)$. Let also $L(Q)$ denotes the set of all linear operators on $Q$.

Let $L_0, L, L_1$ be nondecreasing integrable functions on $[0, \rho)$, where $\rho > 0$ is such that $\int_0^\rho (\rho - t)L_0(t)dt \geq \rho$, $\int_0^\rho (\rho - t)L(t)dt \geq \rho$ and $\int_0^\rho (\rho - t)L_1(t)dt \geq \rho$. Define parameter $r$ by $r = \sup\{t \geq 0 : B(x_0, r) \subseteq G\}$. We need the following definitions related to functions $L_0, L$ and $L_1$.

**Definition 3.1** *Let $x_0 \in G$ and $M : G \longrightarrow L(Q)$. Operator $M$ satisfies the center-$L_0$-Lipschitz condition on $B(x_0, r)$, if*

$$\|M(x.exp\, v) - M(x_0)\| \leq \int_0^{d(x_0,x)} L_0(t)dt \tag{3.2}$$

*holds for each $v, v_i \in Q, i = 0, 1, \ldots m$, $x \in B(x_0, r)$ such that $x = x_0 exp\, v_1\, exp\, v_2 \ldots exp\, v_m$ and $d(x_0, x) < r$, where $d(x_0, x) = \sum_{i=0}^m \|v_i\|$.*

Suppose that equation

$$\int_0^{\bar{\rho}} L_0(t)dt = 1 \tag{3.3}$$

for some $\bar{\rho} > 0$ has at least one positive solution. Denoted by $\rho_0$ the smallest such solution. Define the set $B_0$ by

$$B_0 = B(x_0, r_0),\ r_0 = \min\{r, \rho_0\}. \tag{3.4}$$

**Definition 3.2** *Let $x_0 \in G$ and $M : G \longrightarrow L(Q)$. Operator $M$ satisfies the restricted $L-$Lipschitz condition on $B_0$, if*

$$\|M(x.exp\, v) - M(x)\| \leq \int_{d(x_0,x)}^{d(x_0,x)+\|v\|} L(t)dt \tag{3.5}$$

*holds for each $x \in B_0$.*

**Definition 3.3** *[21] Let $x_0 \in G$ and $M : G \longrightarrow L(Q)$. Operator $M$ satisfies the $L_1-$Lipschitz condition on $B(x_0, r)$, if*

$$\|M(x.exp\, v) - M(x)\| \leq \int_{d(x_0,x)}^{d(x_0,x)+\|v\|} L_1(t)dt \tag{3.6}$$

*holds for each $x \in B(x_0, r)$.*

It follows from (3.2)-(3.6) that

$$L_0(t) \leq L_1(t) \tag{3.7}$$

and

$$L(t) \leq L_1(t). \tag{3.8}$$

It is worth noticing that function $L_1$ was used in the semi-local convergence analysis in [21]. In the present article tighter function $L$ shall replace $L_1$, since the iterates $x_n$ lie in the more precise ball $B_0$ than $B(x_0, r)$ used in [21]. This way we obtain the advantages as already stated in the introduction. These advantages are obtained under the same cost, since the computation of function $L_1$ requires the computation of functions $L_0$ and $L$ as special cases. Notice that function $L_0$ is used to define function $L$, i.e., $L = L(L_0)$.

From now on, we suppose that

$$L_0(t) \leq L(t). \tag{3.9}$$

If

$$L(t) \leq L_0(t), \tag{3.10}$$

then function $L_0$ can replace $L$ in the results that follow after Lemma 3.4.

We need the auxiliary Banach-type perturbation result.

**LEMMA 3.4** *Let $\rho \in (0, \rho_0)$ and $x_0 \in G$ be such that $d_{F_{x_0}}$ is invertible. Suppose $d_{F_{x_0}}^{-1} d_F$ satisfies the center-$L_0$ Lipschitz condition on $B(x_0, \rho)$. Then, $d_{F_x}$ is invertible and*

$$\|d_{F_x}^{-1} d_{F_{x_0}} d_{F_{x_0}}^{-1} d_{F_{x_0}}\| \leq \frac{1}{1 - \int_0^{d(x_0,x)} L_0(t)dt}. \tag{3.11}$$

**Proof.** Set $z_0 = x_0$ and $z_{i+1} = z_i.exp\, v_i$ for each $i = 0, 1, 2, \dots k$. Using (3.2) for $M = d_{F_{x_0}} d_F$, one has that

$$\|d_{F_x}^{-1} d_{F_{x_0}} d_{F_{x_0}}^{-1} (d_{F_{z_i.exp\, v_i}} - d_{F_{z_i}})\| \leq \int_{d(z_i,x_0)}^{d(z_{i+1},x_0)} L(t)dt \text{ for each } 0 \leq i \leq k. \quad (3.12)$$

Since $z_{k+1} = x$, we have

$$\begin{aligned} \|d_{F_x}^{-1} d_{F_{x_0}} d_{F_{x_0}} d_{F_x} - I_G\| &= \|d_{F_{x_0}}^{-1} (d_{F_{y_k.exp\, v_k}} - d_{F_{x_0}}\| \\ &\leq \sum_{i=0}^{k} \|d_{F_{x_0}}^{-1} (d_{F_{y_i.exp\, v_i}} - d_{F_{y_i}}\| \\ &= \int_0^{d(x,x_0)} L_0(t)dt < \int_0^{\rho_0} L_0(t)dt = 1. \quad (3.13) \end{aligned}$$

The results follows from (3.13) and the Banach Lemma on invertible operators [23]. □

**REMARK 3.5** *The corresponding Lemma 2.1 in [21] arrived at the estimate*

$$\|d_{F_x}^{-1} d_{F_{x_0}} d_{F_{x_0}}^{-1} d_{F_{x_0}}\| \leq \frac{1}{1 - \int_0^{d(x_0,x)} L_1(t)dt} \quad (3.14)$$

*which is less tight than (3.11) (by (3.7)).*

Let

$$\eta_0 = \int_0^{\rho} L_0(t)tdt. \quad (3.15)$$

The majorizing function $\varphi$ shall be used. Define the majorizing function $\varphi$ by

$$\varphi(t) = \eta - t + \int_0^t L(s)(t-s)ds \text{ for each } t \in [0, \rho]. \quad (3.16)$$

Some useful results are needed:

**PROPOSITION 3.6** *[21] The function $\varphi$ is monotonically decreasing on $[0, \rho_0]$ and monotonically increasing on $[\rho_0, \rho]$. Moreover, if $\eta \leq \eta_0, \varphi(t) = 0$ has a unique solution respectively in $[0, \rho_0]$ and $[\rho_0, \rho]$, which are denoted by $r_1$ and $r_2$.*

Let $\{t_n\}$ denote the sequence generated by Newton's method with initial data $t_0 = 0$ for $\varphi$ defined for each $n = 0, 1, 2, \dots$ by

$$t_{n=1} = t_n - \varphi'(t_n)^{-1}\varphi(t_n) \text{ for each } n = 0, 1, 2, \dots. \quad (3.17)$$

**PROPOSITION 3.7** *[21] Suppose that $\eta \leq \eta_0$. Then the sequence $\{t_n\}$ generated by (3.17) is monotonically increasing and converges to $r_1$.*

Suppose that $x_0 \in G$ is such that $d_{F_{x_0}}^{-1}$ exists and set $\eta := \|d_{F_{x_0}}^{-1}\|$. Let $\eta_0$ given by (3.15) and $r_1$ be given by Proposition 3.6.

**THEOREM 3.8** *Suppose that $d_{F_{x_0}}^{-1} d_F$ satisfies the center $L_0-$Lipschitz condition on $B_0$, $L-$Lipschitz condition on $B(x_0, r_1)$ and that*

$$\eta = \|d_{F_{x_0}}^{-1}\| \leq \eta_0. \tag{3.18}$$

*Then, the sequence $\{x_n\}$ generated by NM with initial point $x_0$ is well defined and converges to a zero $x_*$ of $F$. Moreover, the following items hold for each $n = 0, 1, 2, \ldots$;*

$$d(x_{n+1}, x_n) \leq \|d_{F_{x_n}}^{-1} F(x_n)\| \leq t_{n+1} - t_n; \tag{3.19}$$

$$d(x_n, x_*) \leq r_1 - t_n. \tag{3.20}$$

**Proof.** Set $v_n = -d_{F_{x_n}}^{-1} F(x_n)$ for each $n = 0, 1, 2, \ldots$. Using induction, we shall show that each $v_n$ is well defined and

$$\rho(x_{k+1}, x_k) \leq \|v_n\| \leq t_{k+1} - t_k \tag{3.21}$$

holds for each $n = 0, 1, 2, \ldots$. Then, the sequence $\{x_k\}$ generated by NM starting at $x_0$ is well defined and converges to a zero $x_*$ of $F$, since, by NM

$$x_{k+1} = x_k.exp\, v_k \text{ for each } n = 0, 1, 2, \ldots.$$

Moreover, items (3.19) and (3.20) hold for each $n$ and the proof of the theorem is completed.

Note that $v_0$ is well defined by assumption and $x_1 = x_0.exp\, v_0$. Hence, $\rho(x_1, x_0) \leq \|v_0\|$. Since $\|v_0\| = \| - d_{F_{x_0}}^{-1} F(x_0)\| = \eta = t_1 - t_0$, it follows that (3.17) is true for $k = 0$. Suppose that $v_k$ is well defined and (3.21) holds for each $n \leq k - 1$. Then

$$\sum_{i=0}^{n-1} \|v_i\| \leq t_n - t_0 = t_n < r_1 \text{ and } x_n = x_0.exp\, v_0. \ldots .exp\, v_{n-1}. \tag{3.22}$$

By Lemma 3.4 $d_{F_{x_k}}^{-1}$ exists and

$$\|d_{F_{x_k}}^{-1} d_{F_{x_0}}\| \leq \frac{1}{1 - \int_0^1 t_k L_0(s) ds} = -\varphi'(t_k)^{-1}. \tag{3.23}$$

Hence, $v_k$ is well defined. Using NM and (2.5), we can write

$$\begin{aligned} F(x_k) &= F(x_k) - F(x_{k-1}) - d_{F_{x_k}} v_{k-1} \\ &= \int_0^1 d_{F_{x_k}}.exp\,(tv_{k-1})v_{k-1}dt - d_{F_{x_{k-1}}} v_{k-1} \\ &= \int_0^1 [d_{F_{x_{k-1}}}.exp\,(tv_{k-1}) - d_{F_{x_{k-1}}}]v_{k-1}dt. \end{aligned} \tag{3.24}$$

Then, using NM, (3.5), (3.23) and (3.24), we obtain in turn that

$$\begin{aligned} \|d_{F_{x_0}}^{-1} F(x_k)\| &\leq \int_0^1 \|d_{F_{x_0}}^{-1}[d_{F_{x_{k-1}}}.exp\,(tv_{k-1}) - d_{F_{x_{k-1}}}]\|\|v_{k-1}\|dt \\ &\leq \int_0^1 \int_{\rho(x_{k-1},x_0)}^{\rho(x_{k-1},x_0)+t\|v_{k-1}\|} L(s)ds\|v_{k-1}\|dt \\ &\leq \int_0^1 \int_{t_{k-1}}^{t_{k-1}+t(t_k-t_{k-1})} L(s)ds(t_k - t_{k-1})dt \\ &= \int_{t_{k-1}}^{t_k} L(s)(t_k - s)ds \\ &= h(t_k) - h(t_{k-1}) - \varphi'(t_{k-1})(t_k - t_{k-1}) \\ &= \varphi(t_k). \end{aligned} \tag{3.25}$$

By (3.23) and (3.25), we have in turn that

$$\begin{aligned} \|v_k\| &= \| - d_{F_{x_k}}^{-1} F(x_k)\| \\ &\leq \|d_{F_{x_k}}^{-1} d_{F_{x_0}}\|\|d_{F_{x_0}}^{-1} F(x_k)\| \\ &\leq -\varphi'(t_k)^{-1}\varphi(t_k) \\ &= t_{k+1} - t_k. \end{aligned} \tag{3.26}$$

Then, by $x_{k+1} = x_k.exp\,v_k$, we get $d(x_{k+1}, x_k) \leq \|v_k\|$ which together with (3.21) and (3.26) complete the induction.

□

**REMARK 3.9** *The majorizing sequence used in [21] is defined for each* $n = 0, 1, 2, \ldots$ *by*

$$u_{n+1} = u_n - \psi'(u_n)^{-1}\psi(u_n),\ u_0 = 0, \tag{3.27}$$

*where*

$$\psi(t) = \eta - t + \int_0^t L_1(s)(t - s)ds \text{ for each } t \in [0, \rho] \tag{3.28}$$

*and the convergence criterion corresponding to (3.20) is*

$$\eta \leq \bar{\eta}_0, \tag{3.29}$$

*where*

$$\int_0^{\bar{\rho}_0} L_1(t)dt = 1 \text{ and } \bar{\eta}_0 = \int_0^{\bar{\rho}_0} L_1(t)tdt. \tag{3.30}$$

*Denote also by $\bar{r}_1$ and $\bar{r}_2$ the solutions of equation $\psi(t) = 0$ corresponding to $r_1$ and $r_2$, respectively. Notice that*

$$\varphi(t) \leq \psi(t), \tag{3.31}$$

$$\bar{\rho}_0 \leq \rho_0 \tag{3.32}$$

*and*

$$\bar{r}_1 \leq r_1. \tag{3.33}$$

*It turns out that (3.18) is weaker than (3.29). As an example, suppose functions $L_0, L$ and $L_1$ are constants. Then, we have*

$$\bar{\rho}_0 = \frac{1}{L_1}, \rho_0 = \frac{1}{L}, \bar{\eta}_0 = \frac{1}{2L_1} \text{ and } \eta_0 = \frac{1}{2L}. \tag{3.34}$$

*Therefore (3.18) and (3.29) reduce respectively to the Kantorovich criteria for NM [23]*

$$2L\eta \leq 1 \tag{3.35}$$

*and*

$$2L_1\eta \leq 1. \tag{3.36}$$

*Then, in this case since $L \leq L_1$, we have*

$$2L_1\eta \leq 1 \Longrightarrow 2L\eta \leq 1 \tag{3.37}$$

*but not necessarily vice versa unless, if $L = L_1$. It follows from (3.25) that sequence $\{s_n\}$ defined for each $n = 0, 1, 2, \ldots$ by*

$$\begin{aligned} s_0 &= 0,\ s_1 = \eta, \\ s_{n+2} &= s_{n=1} + \bar{\varphi}_0'(s_n)^{-1} \int_{s_{n-1}}^{s_n} L(t)(s_n - t)dt \end{aligned} \tag{3.38}$$

*is also a majorizing sequence which converges to $s_* = r_1$ (under (3.18)), where*

$$\bar{\varphi}_0(t) = \eta - t + \int_0^t L_0(s)(t-s)ds. \tag{3.39}$$

*Moreover, we have*

$$0 \leq s_n \leq t_n \text{ for each } n = 0, 1, 2, \ldots, \tag{3.40}$$

$$0 \leq s_{n+2} - s_{n+1} \leq t_{n+2} - t_{n+1} \text{ for each } n = 0, 1, 2, \ldots, \tag{3.41}$$

*and*

$$s_* = \lim_{n \longrightarrow \infty} s_n \leq r_1 = \lim_{n \longrightarrow \infty} . \tag{3.42}$$

*Hence, tighter sequence* $\{s_n\}$ *can replace* $\{t_n\}$ *in Theorem 3.8 (under (3.18)). It turns out that sequence* $\{s_n\}$ *can converge under weaker than (3.35) criteria. We refer the reader to our work in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] for such criteria. The same advantages are obtained, if we specialize the "L" functions to "gamma" functions [21]. Our new technique of the restricted convergence region can be used to other iterative methods. Examples where (3.9) and (3.11) hold as strict inequalities can be found in [11, 12, 13, 14, 15]. The local convergence analysis can be improved along the same lines (see also [5, 12, 13, 14, 15]).*

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