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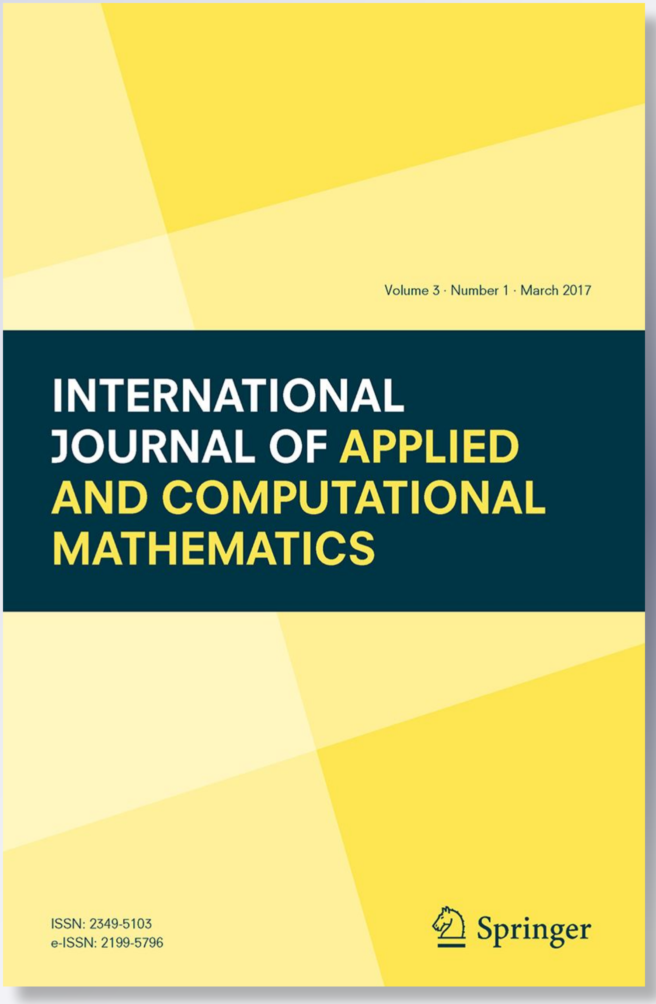

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ORIGINAL PAPER

Extending the Mesh Independence For Solving Nonlinear Equations Using Restricted Domains

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Abstract The mesh independence principle states that, if Newton's method is used to solve an equation on Banach spaces as well as finite dimensional discretizations of that equation, then the behaviour of the discretized process is essentially the same as that of the initial method. This principle was inagurated in Allgower et al. (SIAM J Numer Anal 23(1):160–169, 1986). Using our new Newton–Kantorovich-like theorem and under the same information we show how to extend the applicability of this principle in cases not possible before. The results can be used to provide more efficient programming methods.

Keywords Newton's method · Banach space · Operator equation · Mesh independence

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Introduction

We are interested in the problem of locating a solution x^* of equation

$$F(x) = 0. \tag{1}$$

Here, X, Y denote Banach spaces, $D \subseteq X$ is open, convex and $F : D \subseteq X \longrightarrow Y$ is differentiable in the sense of Fréchet.

Newton's method is defined for $n = 0, 1, 2, \ldots,$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{2}$$

converges quadratically to x^* under certain conditions. However the iterates cannot be found easily in general. That is why we introduce a family of discretized equations

$$P_h(y) = 0 \tag{3}$$

indexed by some positive real number h with $P_h : X_h \longrightarrow Y_h$ and X_h, Y_h being of finite dimension. Define the discretization on X by bounded linear operators $L_h : X \to X_h$, and introduce the family of discretized iterations by

$$y_0^h = L_h(x_0),\ \ y_{n+1}^h = y_n^h - P_h'(y_n^h)^{-1}P_h(y_n^h)\ \ (n \geq 0). \tag{4}$$

In the elegant paper [3] they showed the relationship between distances $\|x_{n+1} - x_n\|$, $\|y_{n+1}^n - y_n^n\|$, $\|x_n - x^*\|$, $\|y_n^h - y_h^*\|$ $(n \geq 0)$ and the connection between the two iterations.

One of the basic assumptions was the Lipschitz continuity of operators F', $P_h'(h > 0)$. Here instead we use a combination of Lipschitz and center-Lipschitz conditions. This way the error bounds are improved, the minimum n for which $\|x_n - x^*\| \leq \varepsilon$ holds can be smaller and the radius of convergence larger [2,3,7,9,12,13,16–22,25,26,28,30–33]. Other studies can be found in [1–33].

Mesh Independence Principle

Let $U(v, \xi)$ and $\bar{U}(v, \xi)$ stand respectively for the open and closed balls in X with center $v \in X$ and of radius $\xi > 0$. Let $x_0 \in D$ and $R > 0$. Define

$$R_0 = \sup\{t \in [0, R) : U(x_0, t) \subseteq D\}$$

We will need the following semilocal and local convergence theorems.

Theorem 1 *Let $F : U(x_0, R_0) \subseteq X \to Y$ be a differentiable operator in the sense of Fréchet. Assume the existence of parameters $n > 0$, $\ell_0 > 0$ such that*

$$F'(x_0)^{-1} \in L(Y, X), \tag{1}$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta \tag{2}$$

and

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \ell_0\|x - x_0\|. \tag{3}$$

Moreover, assume the existence of $\ell > 0$ such that

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq \ell\|x - y\| \text{ for all } x, y \in U(x_0, R_0) \cap U\left(x_0, \frac{1}{\ell_0}\right) := U, \tag{4}$$

$$L\eta \leq 1 \tag{5}$$

and

$$t^* \leq R_0, \tag{6}$$

where

$$\begin{aligned} t^* &= \lim_{n\to\infty} t_n \leq t^{**} \\ t^{**} &= \left[1 + \frac{\ell_0\eta}{2(1-\alpha)(1-\ell_0\eta)}\right]\eta \leq 2b\eta, \\ L &= \frac{1}{4}\left(\sqrt{\ell_0\ell} + 4\ell_0 + \sqrt{\ell\ell_0 + 8\ell_0^2}\right) \\ b &= \frac{1}{2}\left[1 + \frac{1}{2(1-\alpha)}\right] \end{aligned} \tag{7}$$

and

$$\alpha = \frac{2\ell}{\ell + \sqrt{\ell^2 + 8\ell_0\ell}}.$$

$$\begin{aligned} t_0 &= 0,\ t_1 = \eta,\ t_2 = t_1 + \frac{\ell_0(t_1-t_0)^2}{2(1-\ell_0 t_1)} \\ t_{n+2} &= t_{n+1} + \frac{\ell(t_{n+1}-t_n)^2}{2(1-\ell_0 t_{n+1})}\ (n \geq 1). \end{aligned} \tag{8}$$

Then, $\lim_{n\longrightarrow\infty} x_n = x^* \in \bar{U}(x_0, t^*)$ *for some* x^*, $F(x^*) = 0$, *so that the following items hold*

$$\|x_{n+2} - x_{n+1}\| \leq \frac{\ell\|x_{n+1} - x_n\|^2}{2[1 - \ell_0\|x_{n+1} - x_0\|]} \leq t_{n+2} - t_{n+1} \tag{9}$$

and

$$\|x_n - x^*\| \leq t^* - t_n. \tag{10}$$

Moreover, the solution x^* *is unique in* $\bar{U}(x_0, t^*)$, *and if there exists* $R > t^*$ *such that*

$$U(x_0, R) \subseteq D \tag{11}$$

and

$$\ell_0(t^* + R) \leq 2, \tag{12}$$

then, the solution x^* *is unique in* $U(x_0, R)$.

Proof Simply notice that the iterates x_n remain in U which is a more precise location than $U(x_0, R_0)$ used in [17], since $U \subseteq U(x_0, R_0)$. Based on this observation the proof is analogous to the one in [17].

For x^* such that $F(x^*) = 0$, let

$$R_1 = \sup\{t \in [0, R) : U(x^*, t) \subseteq D\}.$$

□

Theorem 2 *Let $F : D \subseteq X \to Y$ differentiable in the sense of Fréchet. Assume: there exist a simple zero $x^* \in D$, of equation $F(x) = 0$ and parameter $\gamma_0 > 0$ such that*

$$\|F'(x^*)^{-1}[F'(x) - F'(x^*)]\| \leq \gamma_0 \|x - x^*\| \text{ for all } x \in U(x^*, R_1), \tag{13}$$

Moreover suppose that there exists $\gamma > 0$ such that

$$\|F'(x^*)^{-1}[F'(x) - F'(y)]\| \leq \gamma \|x - y| \text{ for all } x, y \in U_1 := U(x^*, R_1) \cap U\left(x^*, \frac{1}{\gamma_0}\right), \tag{14}$$

and

$$\gamma_1 \leq R_1, \tag{15}$$

where

$$\gamma_1 = \frac{2}{2\gamma_0 + \gamma}. \tag{16}$$

Then, $\lim_{n \longrightarrow \infty} x_n = x^* \in \bar{U}(x_0, \gamma_1)$, *$x^*$ is the only solution in $\bar{U}(x^*, \gamma_1)$ and*

$$\|x_{n+1} - x^*\| \leq \frac{\gamma \|x_n - x^*\|^2}{2[1 - \gamma_0 \|x_n - x^*\|]}. \tag{17}$$

Proof Notice that iterates remain in U_1. This is a better location for the iterates x_n than $U(x^*, R_1)$ used in [4]. Then, the proof follows exactly as the corresponding one in [4]. □

Remark 3 The preceding results improve the corresponding ones in [17] and [4] which in turn improved the corresponding ones in [2,3]. Indeed, we have:

Semilocal Convergence (Theorem 1)
The Lipschitz condition corresponding to (4) and used in [2,3,17] is given by: there exists $\ell_1 > 0$ such that

$$\|F'(x_0^*)^{-1}[F'(x) - F'(y)]\| \leq \ell_1 \|x - y\| \text{ for each } x, y \in U(x_0, R_0). \tag{2.4$'$}$$

Then, the conclusions of Theorem 1 were obtained as in [17] using ℓ_1 instead of ℓ. Notice however that

$$\ell_0 \leq \ell$$

and in particular

$$\ell \leq \ell_1,$$

hold. If $\ell = \ell_1$ our results reduce to the corresponding ones in [17]. But if $\ell < \ell_1$, then the new results have the following advantages over the ones in [17]:

i. Weaker sufficient convergence criteria. Indeed, the old criteria are given by

$$L_1 \eta \leq 1, \tag{2.5$'$}$$

where

$$L_1 = \frac{1}{4}\left(\sqrt{\ell_0 \ell_1} + 4\ell_0 + \sqrt{\ell_1 \ell_0 + 8\ell^2}\right).$$

Notice that

$$L_1\eta \leq 1 \implies L\eta \leq 1$$

but not necessarily vice versa, unless if $\ell_0 = \ell_1$.

ii More precise error estimates on the distances $\|x_{n+1}-x_n\|$, $\|x_n - x^*\|$. Indeed the majorizing sequence given in [17] is defined by

$$u_0 = 0,\ u_1 = \eta,\ u_2 = u_1 + \frac{\ell_0(u_1 - u_0)^2}{2(1-\ell_0 u_1)},$$
$$u_{n+2} - u_{n+1} + \frac{\ell_1(u_{n+1} - u_n)^2}{2(1-\ell_0 u_{n+1})} \quad \text{for each } \ n = 1, 2, \ldots.$$

Then, a simple inductive argument shows that

$$t_n \leq u_n$$
$$t_{n+1} - t_n \leq u_{n+1} - u_n$$
$$\text{and } \ t^* \leq u^* = \lim_{n\to\infty} u_n.$$

Strict inequality holds in the first two inequalities, if $\ell < \ell_1$ and $n = 3, 4, \ldots.$ The last inequality shows that the information on the location of the solution is more precise, since $t^* \leq u^*$.

Local Convergence (Theorem 2)

The Lipschitz condition corresponding to (14) and used in [4] is given by: there exists $\bar{\gamma} > 0$ such that

$$\|F'(x^*)^{-1}[F'(x) - F'(y)]\| \leq \bar{\gamma}\|x - y\| \text{ for each } x, y \in U(x^*, R_1). \tag{2.14$'$}$$

Then, again in view of (14) and (2.14′), we get that

$$\gamma \leq \bar{\gamma}$$

hold. The radius of convergence in [4] is given by

$$\bar{\gamma}_1 := \frac{2}{2\gamma_0 + \bar{\gamma}}.$$

Then, we have

$$\bar{\gamma}_1 \leq \gamma_1$$

and, if $\gamma < \bar{\gamma}$, then

$$\bar{\gamma}_1 < \gamma_1.$$

The corresponding error bound in [4] using $\bar{\gamma}$ instead of γ is given by

$$\|x_{n+1} - x^*\| \leq \frac{\bar{\gamma}\|x_n - x^*\|^2}{2[1-\gamma_0\|x_n - x^*\|]}. \tag{2.17$'$}$$

In view of (17) and (2.17′) we deduce that the new error bounds are more precise than the old ones leading to fewer iterations in order to obtain a certain desired error tolerance. Finally, notice that no additional computational effort is required because if we find ℓ_1, we also find special cases ℓ_0 and ℓ. The same is true for the constants $\bar{\gamma}$, γ_0 and γ.

Definition 4 As in [3,12] let $S^* \subseteq X$ be such that

$$x^* \in S^*,\ x_n \in S^*,\ x_n - x^* \in S^*,\ \ x_{n+1} - x_n \in S^*,\ \ n \geq 0. \tag{18}$$

Consider the family

$$\{P_h, L_h, \hat{L}_h\},\ \ h > 0, \tag{19}$$

where

$$P_h : D_h \rightarrow \hat{Y}_h, \tag{20}$$

$$L_h : X \rightarrow X_h,\ \ \hat{L}_h : Y \rightarrow \hat{Y}_h \tag{21}$$

such that

$$L_h(S^* \cap U^*) \subseteq D_h. \tag{22}$$

The discretization family (19) is Lipschitz-center, Lipschitz uniform if there exist $\rho > 0,\ \ell_0 = \ell_0(h) > 0$ so that

$$\overline{U}(L_h(x^*), \rho) \subseteq D_h, \tag{23}$$

$$\|P_h'(u) - P_h'(L_h(x^*))\| \leq \ell_0 \|u - L_h(x^*)\|,\ \ u \in \overline{U}(L_h(x^*), \rho) \tag{24}$$

and $\ell = \ell(h) > 0$ such that

$$\|P_h'(u) - P_h'(v)\| \leq \ell \|u - v\|,\ \ u, v \in \overline{U}\left(L_h(x^*), \rho) \cap U(L_h(x^*), \frac{1}{\ell_0}\right) := U_h. \tag{25}$$

Moreover (19) is: bounded if there exists a constant $q > 0$ so that

$$\|L_h(u)\| \leq q\|u\|,\ \ u \in S^*, \tag{26}$$

stable: if there exists a $\sigma > 0$ such that

$$\|P_h'(L_h(u))^{-1}\| \leq \sigma,\ \ u \in S^* \cap U^*, \tag{27}$$

and consistent of order p :, if there exist $c_0 > 0,\ c_1 > 0,\ c_2 > 0$ so that

$$\|\hat{L}_h(F(x^*)) - P_h(L_h(x^*))\| \leq c_0 h^p, \tag{28}$$

$$\|\hat{L}_h(F(x)) - P_h(L_h(x))\| \leq c_1 h^p,\ \ x \in S^* \cap U^*, \tag{29}$$

and

$$\|\hat{L}_h(F'(x))(y) - P_h'(L_h(x))L_h(y)\| \leq c_2 h^p, \tag{30}$$

$x \in S^* \cap U^*,\ \ y \in S^*$. We can show the following result for (3) and (4).

Theorem 5 *Let $F : D \subseteq X \rightarrow Y$ be an operator satisfying hypotheses of Theorem 2 such that a Lipschitz, center-Lipschitz uniform discretization (19) exists which is bounded, stable and consistent of order p. Then* *(a) Equation* (3) *has a solution which is locally unique with*

$$y_h^* = L_h(x^*) + O(h^p), \tag{31}$$

for each h such that

$$0 < h \le h_0 = \max\left\{\left(\frac{\rho}{2bc_0\sigma}\right)^{1/p}, \left(\frac{1}{c_0\sigma^2 L}\right)^{1/p}\right\};' \tag{32}$$

(b) There exist $h_1 \in (0, h_0]$, $r_1 \in (0, r^*]$ *such that Newton's method* (4) *converges to* y_h^*; *and for all* $k \ge 0$

$$y_k^h = L_h(x_k) + O(h^p), \tag{33}$$

$$P_h(y_k^h) = \hat{L}_h(F(x_k)) + O(h^p) \tag{34}$$

$$y_k^h - y_h^* = L_h(x_k - x^*) + O(h^p) \tag{35}$$

for each $h \in (0, h_1]$ *and* $x_0 \in U(x^*, r_1)$.

Proof We showed in Theorem 1 that when

$$\alpha(h) = L\sigma \| P_h'(L_h(x^*))^{-1} P_h(L_h(x^*))\| \le 1, \tag{36}$$

$$r(h) \le 2b \| P_h'(L_h(x^*))^{-1} P_h(L_h(x^*))\| \le \rho, \tag{37}$$

then Eq. (3) has a solution y_h^* which is unique in $\overline{U}(L_h(x^*), r(h))$. Using (36), (27), (28) and (32) we get in turn

$$\begin{aligned}\alpha(h) &\le L\sigma^2 \| P_h(L_h(x^*))\| \\ &= L\sigma^2 \| P_h(L_h(x^*)) - \hat{L}_h(F(x^*))\| \\ &\le L\sigma^2 c_0 h^p \le 1\end{aligned} \tag{38}$$

and

$$r(h) \le 2bc_0 h^p \le \rho \tag{39}$$

which hold by the choice of h given by (32). Hence (31) follows from

$$\|y_h^* - L_h(x^*)\| \le r(h) \le 2b\sigma c_0 h^p. \tag{40}$$

By Theorem 2 Newton's method (4) converges to y_h^* if

$$\|L_h(x_0) - y_h^*\| < \frac{2}{(2\ell_0 + \ell)\| P_h'(y_h^*)^{-1}\|}, \tag{41}$$

and

$$\overline{U}(y_h^*, \|L_h(x_0) - y_h^*\| \subseteq \overline{U}(L_h(x^*), \rho). \tag{42}$$

Estimate (42) holds, if

$$\|y_h^* - L_h(x^*)\| + \|L_h(x_0) - y_h^*\| \le \rho. \tag{43}$$

By (26) and (38) we can have

$$\begin{aligned}\|L_h(x_0)-y_h^*\| &\le \|L_h(x_0)-L_h(x^*)\|+\|L_h(x^*)-y_h^*\|\\ &\le q\|x_0-x^*\|+2b\sigma c_0 h^p. \end{aligned} \tag{44}$$

Therefore (42) holds if

$$q\|x_0-x^*\|+4b\sigma c_0 h^p \le \rho. \tag{45}$$

Using the identity and the Banach perturbation Lemma [7,9,17,29]

$$P_h'(y_h^*)=P_h'(L_h(x^*))[I-P_h'(L_h(x^*))^{-1}(P_h'(L_h(x^*))-P_h'(y_h^*))] \tag{46}$$

we get

$$\|P_h'(y_h^*)^{-1}\| \le \frac{\|P_h'(L_h(x^*))^{-1}\|}{1-\ell_0\|P_h'(L_h(x^*))^{-1}\|\|L_h(x^*)-y_h^*\|} \tag{47}$$

$$\le \frac{\sigma}{1-2b\ell_0\sigma^2 c_0 h^p}. \tag{48}$$

Hence (41) holds if

$$q\|x_0-x^*\|+4b\sigma c_0 h^p < \frac{2(1-2b\ell_0 c_0\sigma^2 h^p)}{(2\ell_0+\ell)\sigma}. \tag{49}$$

Choose

$$h_2=\min\left\{\left(\frac{\rho}{8bc_0\sigma}\right)^{1/p},\left(\frac{1}{4bc_0\sigma(1+\ell_0\sigma)}\right)^{1/p}\right\}; \tag{50}$$

and

$$r_2=\min\left\{\frac{\rho}{2q},\frac{1}{(2\ell_0+\ell)q\sigma}\right\}. \tag{51}$$

Then (41) and (42) hold for each $h\in(0,h_2]$ and $x_0\in U(x^*,r_2)$. That is for these choices of h and x_0, Newton's method (4) converges to y_h^*. Define

$$h_1=\min\left\{h_2,\left[\frac{1}{4\sigma^2(c_1+c_2)(2\ell_0+\ell)}\right]^{1/p}\right\} \tag{52}$$

$$r_1=\min\left\{r_2,\frac{1}{4\ell\sigma q}\right\}. \tag{53}$$

With the above choice equation in λ

$$\frac{\sigma}{1-\ell_0\sigma\lambda}\left[\frac{\ell}{2}\lambda^2+2\ell q\|x_0-x^*\|\lambda+(c_1+c_2)h^p\right]=\lambda \tag{54}$$

is quadratic and has a positive solution, which satisfies

$$d\le 4\sigma(c_1+c_2)h^p. \tag{55}$$

We now show using induction on n that for $h\in(0,h_1)$, $x_0\in \overline{U}(x^*,r_1)$, and all $n\ge 0$

$$\|y_n^h-L_h(x_n)\|\le d \tag{56}$$

holds.

For $n = 0$ (56) holds. Assume (56) holds for $n = 0, 1, \ldots, k$. Using (4) we obtained the identity

$$\begin{aligned} y_{k+1}^h - L_h(x_{k+1}) = P_h'(y_k^h)^{-1}\{&[P_h'(y_k^h)(y_k^h - L_h(x_k)) \\ &- P_h(y_k^h) + P_h(L_h(x_k))] \\ &+ [(P_h'(y_k^h) - P_h'(L_h(x_k)))L_h(F'(x_k)^{-1}F(x_k))] \\ &+ [P_h'(L_h(x_k))L_h(F'(x_k)^{-1}F(x_k)) - \hat{L}_h(F(x_k))] \\ &+ [\hat{L}_h(F(x_k)) - P_h(L_h(x_k))]\}. \end{aligned} \tag{57}$$

As in (47) we get

$$\|P_h'(y_k^h)^{-1}\| \le \frac{\sigma}{1 - \ell_0\sigma\|y_k^h - L_h(x_k)\|} \le \frac{\sigma}{1 - \ell_0\sigma d}. \tag{58}$$

We can get in turn

$$\begin{aligned} \left\| P_h'(y_k^h)(y_k^h - L_h(x_k)) - P_h(y_k^h) + P_h(L_h(x_k)) \right\| &\le \frac{\ell}{2}\left\| y_k^h - L_h(x_k) \right\|^2 \\ &\le \frac{\ell}{2} d^2, \end{aligned} \tag{59}$$

$$\begin{aligned} \|(P_h'(y_k^h) - P_h'(L_h(x_k)))L_h(F'(x_k)^{-1}F(x_k))\| &\le \ell q\|y_k^h - L_h(x_k)\|\|x_{k+1} - x_k\| \\ &\le 2\ell q d\|x_0 - x^*\|, \end{aligned} \tag{60}$$

(since $\|x_{k+1} - x^*\| \le \|x_k - x^*\|$)

$$\|P_h'(L_h(x_k))L_h(F'(x_k)^{-1}F(x_k)) - \hat{L}_h(F(x_k))\| \le c_2 h^p, \tag{61}$$

and

$$\|\hat{L}_h(F(x_k)) - P_h(L_h(x_k))\| \le c_1 h^p. \tag{62}$$

By (55) and (57)–(62) we get

$$\|y_{k+1}^h - L_h(x_{k+1})\| \le d \le 4\sigma(c_1 + c_2)h^p, \tag{63}$$

where d satisfies (54). Moreover by the Lipschitz continuity of P_h' there exists b such that

$$\|P_h'(x)\| \le b, \quad x \in U_h. \tag{64}$$

Therefore we can have

$$\begin{aligned} \|P_h(y_k^h) - \hat{L}_h(F(x_k))\| &\le \|P_h(y_k^h) - P_h(L_h(x_k))\| \\ &\quad + \|P_h(L_h(x_k)) - \hat{L}_h(F(x_k))\| \\ &\le b\|y_k^h - L_h(x_k)\| + c_1 h^p \\ &\le 4\sigma b(c_1 + c_2)h^p + c_1 h^p = c_3 h^p \end{aligned} \tag{65}$$

where $c_3 = 4\sigma b(c_1 + c_2) + c_1$. Furthermore by (40), (56) and (63) we get

$$\begin{aligned} \|y_k^h - y_h^* - L_h(x_k - x^*)\| &\le \|y_k^h - L_h(x_k)\| + \|y_h^* - L_h(x^*)\| \\ &\le 4\sigma b(c_1 + c_2)h^p + 2b\sigma c_0 h^p = ch^p, \end{aligned} \tag{66}$$

where $c = 2\sigma(bc_0 + 2c_1 + 2c_2)$. □

The following result is the second part of the mesh independence principle.

Theorem 6 *Suppose: hypotheses of Theorem 5 hold; there exists* $\mu > 0$ *such that*

$$\liminf_{h>0} \|L_h(x)\| \geq \mu\|x\| \quad \text{for } x \in S^* \tag{67}$$

Let $\bar{r} \in (0, r_1]$ *and each fixed* $\varepsilon > 0$, $x_0 \in \overline{U}(x^*, \bar{r})$. *Then,* $\bar{h} = \bar{h}(\varepsilon, h_1]$ *can be obegintheoained such that*

$$|\mu| \leq 1 \tag{68}$$

for each $h \in (0, \bar{h}]$, *where* $\mu = \min\{n \geq 0, \|x_n - x^*\| < \varepsilon\} - \min\{n \geq 0, \|y_n^h - y_h^*\| < \varepsilon\}$.

Proof Let k be the unique integer satisfying

$$\|x_{k+1} - x^*\| < \varepsilon \leq \|x_k - x^*\|, \tag{69}$$

and $h_3 > 0$ (depending on x_0) such that

$$\|L_h(x_k - x^*)\| \geq \mu\|x_k - x^*\| \quad \text{for all } h \in (0, h_3). \tag{70}$$

Define

$$\bar{r} = \max\left\{r_1, \frac{\beta}{2\sigma q(\ell + \beta\ell_0)}\right\}, \quad \beta = \min\left\{\frac{1}{q}, \mu, 2q\right\}, \tag{71}$$

and

$$\bar{h} = \min\left\{h_1, h_3, \left[\frac{\beta}{2\sigma c(\ell + \beta\ell_0)}\right]^{1/p}, \left(\frac{\mu\varepsilon}{2c}\right)^{1/p}\right\}. \tag{72}$$

By (66) and (72) we can get

$$\|y_{k+1}^h - y_h^*\| \leq \|L_h(x_{k+1} - x^*)\| + ch^p \leq q\varepsilon + \frac{\beta\varepsilon}{2} < 2q\varepsilon. \tag{73}$$

Moreover from Theorem 2 we get

$$\begin{aligned} \|y_{k+1}^h - y_h^*\| &\leq \frac{\ell\sigma\|y_{k+1}^h - y_h^*\|^2}{2[1 - \ell_0\sigma\|y_{k+1}^h - y_h^*\|]} \\ &\leq \frac{\ell\sigma\|y_0^h - y_h^*\|}{2[1 - \ell_0\sigma\|y_0^h - y_h^*\|]}\|y_{k+1}^h - y_h^*\| \\ &< \frac{\ell\sigma(q\bar{r} + c\bar{h}^p)}{1 - \ell_0\sigma(q\bar{r} + c\bar{h}^p)} \\ &\leq \lambda q\varepsilon < \varepsilon. \end{aligned} \tag{74}$$

By (66) and (70)

$$\varepsilon \leq \|x_k - x^*\| \leq \frac{1}{\mu}\|L_h(x_k - x^*)\| \leq \frac{1}{\mu}(\|y_k^h - y_h^*\| + c\bar{h}^p), \tag{75}$$

or

$$\|y_k^h - y_h^*\| \geq \mu\varepsilon - c\bar{h}^p \geq \mu\varepsilon - \frac{\mu\varepsilon}{2} = \frac{\mu\varepsilon}{2}. \tag{76}$$

Furthermore if $\|y_{k-1}^h - y_h^*\| < \varepsilon$, we get

$$\|y_k^h - y_h^*\| < \frac{1}{2}\beta\varepsilon \leq \frac{\mu\varepsilon}{2} \tag{77}$$

contradicting (76). Hence we get

$$\|y_{k-1}^h - y_h^*\| \geq \varepsilon. \tag{78}$$

The result now follows from (69), (74) and (78). □

Remark 7 The preceding results reduce to the corresponding ones in [3], when

$$\ell = \ell_0 \ and \ \ c_0 = c_1. \tag{79}$$

Note though that (79) and

$$c_0 \leq c_1 \tag{80}$$

hold. In case (74) or (80) hold as strict inequalities then it is clear that our smallest integer n_1 satisfying $\|x_n - x^*\| < \varepsilon$ is smaller than the corresponding integer n_2 given in the references mentioned above. Hence we require less computational steps to achieve the same error tolerance ε than before. The ratios in relationships (33)–(35) are also finer.

Note that the improvements made through our Theorem 1-6 are achieved under the same hypotheses as before. The rest of the works on the mesh independence principle listed in the references can also be improved along the same lines.

Remark 8 If (67) is replaced by the stronger but standard in most discretization studies condition

$$\lim_{h\to 0} \|L_h(x)\| = \|x\| \ \text{ uniformly for } \ x \in S^*, \tag{81}$$

then Theorem 6 still holds but $\bar{h}_1$ does not depend on x_0. Note also that (68) follows from (81).

Remark 9 As already noted in [2,3,7,9,11–23,25–33] the local results obtained here can be used to provide a more efficient programming for projection iteration methods such as Arnoldi's, the generalized minimum residual iteration method(GMRES), the generalized conjugate residual iteration method (GCR), for combined Newton/finite-difference projection iteration methods. Moreover, the results can be useful to solve mesh independence problems where the trapezoidal method, the box method and allocation iterations for boundary value problems are involved.

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