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IMPROVED LOCAL CONVERGENCE ANALYSIS FOR A THREE POINT METHOD OF CONVERGENCE ORDER 1.839...

Ioannis K. Argyros, Yeol Je Cho, and Santhosh George

Abstract. In this paper, we present a local convergence analysis of a three point method with convergence order 1.839... for approximating a locally unique solution of a nonlinear operator equation in setting of Banach spaces. Using weaker hypotheses than in earlier studies, we obtain: larger radius of convergence and more precise error estimates on the distances involved. Finally, numerical examples are used to show the advantages of the main results over earlier results.

1. Introduction

In this paper, we are concerned with the problem of approximating a solution x^* of the nonlinear equation:

$$F(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on a subset D of a Banach space X with values in a Banach space Y.

Using mathematical modeling [3], many problems in computational sciences and other disciplines can be brought in a form like the problem (1). In general, the solutions of the equation (1) can not be found in closed form. Therefore, iterative methods are used for obtaining approximate solutions of the problem (1). In particular, the practice of Numerical Functional Analysis for finding such solution is essentially connected to Newton-like methods [1–26].

The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give some conditions ensuring the convergence of the iterative procedure, while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1–26].

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In this paper, we study the local convergence of the method defined as follows: for each $n \geq 0$,

$$x_{n+1} = x_n - A_n^{-1} F(x_n), \tag{2}$$

where x_{-2}, x_{-1}, $x_0 \in D$ are initial points,

$$A_n = [x_n, x_{n-1}; F] + [x_{n-2}, x_n; F] - [x_{n-2}, x_{n-1}; F],$$

$[x, y; F] \in L(X, Y)$ denotes a divided difference of order one for operator F at the point $x, y \in D$ and $[x, y, x; F]$ denotes a divided difference of order two (see [3, 4, 9, 22]).

The local as well as the semi-local convergence of the method (2) was given in [25]. Studies on this and similar methods were given in [6, 7, 10, 21]. The convergence order is $1.839\ldots$ and the method (2) is a useful alternative to higher order methods such as the method of tangent hyperbolas (Halley) or the method of tangent parabolas (Euler-Chebysheff) [3,4,9,11,12,15,25]. However, these methods are very expensive since they require the evaluation of the second Fréchet-derivative at each step. Discretized versions of these methods such as Ulm's method use divided differences of order two [3, 9, 22]. That is why the method (2) is very useful.

Let $U(x, \rho)$ and $\overline{U}(x, \rho)$ stand, respectively, for the open and closed ball in X with center $x \in X$ and radius $\rho > 0$. The local convergence of method (2) was studied in [25] under the conditions $(\mathcal{C})$:

($\mathcal{C}_1$) There exists $x^* \in D$ such that $F'(x^*) = 0$ and $F'(x^*)^{-1} \in L(Y, X)$;

($\mathcal{C}_2$) There exist constants $p \geq 0$ and $q \geq 0$ such that, for each $x, y, u, v \in D$,

$$\|F'(x^*)^{-1}([x, y; F] - [u, v; F])\| \leq p(\|x - u\| + \|y - v\|);$$

($\mathcal{C}_3$) $\|F'(x^*)^{-1}([u, x, y; F] - [v, x, y; F])\| \leq q\|u - v\|$;

($\mathcal{C}_4$) $\bar{U}(x^*, r) \subseteq D$, where r is the radius of the convergence ball given by

$$r = \frac{2}{3p + \sqrt{9p^2 + 24q}}. \tag{3}$$

Example 1.1. Let $X = Y = \mathbb{R}$, $D = [-\frac{5}{2}, \frac{1}{2}]$. Define a function F on D by

$$F(x) = x^3 \log x^2 + x^5 - x^4$$

for all $x \in D$. Then

$$\begin{aligned} F'(x) &= 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2, \\ F''(x) &= 6x \log x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \log x^2 + 60x^2 = 24x + 22. \end{aligned}$$

Now, we are interested in enlarging the radius of convergence for the method (2) under weaker hypotheses.

The advantages denoted by $(\mathcal{A})$ will be: more initial guesses; less computational steps in order to achieve a desired accuracy and application of the method (2) in cases not covered in earlier studies. Below, we list our conditions $(\mathcal{H})$:

($\mathcal{H}_1$) ($\mathcal{H}_1$)=($\mathcal{C}_1$);

($\mathcal{H}_2$) There exist constants $c_1 \geq 0$, $c_2 \geq 0$, $c_3 \geq 0$ and $q \geq 0$ such that, for each $x, y, u, v \in D$,

$$\|F'(x^*)^{-1}([x, x^*; F] - [x, x; F])\| \leq c_1\|x - x^*\|,$$
$$\|F'(x^*)^{-1}([x^*, x^*; F] - [x, x^*; F])\| \leq c_2\|x - x^*\|,$$
$$\|F'(x^*)^{-1}([x, x^*; F] - [x, y; F])\| \leq c_3\|y - x^*\|;$$

($\mathcal{H}_3$) ($\mathcal{H}_3$)=($\mathcal{C}_3$);

($\mathcal{H}_4$)
$$\bar{U}(x^*, R) \subseteq D,$$
where
$$R = \frac{2}{c_1 + c_2 + c_3 + \sqrt{(c_1 + c_2 + c_3)^2 + 24q}}. \tag{4}$$

It turns out (see the proof of the main local convergence result Theorem 2.1 in Section 2) that the ($\mathcal{H}$) and not the ($\mathcal{C}$) conditions are really needed in the proof of Theorem 4.1 in [25, p. 87]. In other words, the condition ($\mathcal{C}_2$) is never used at this general form. Moreover, notice that

$$c_1 \leq p, \ \ c_2 \leq p, \ \ c_3 \leq p, \ \ c_1 \leq c_3, \ \ c_2 \leq c_3 \tag{5}$$

hold in general and $\frac{p}{c_1}, \frac{p}{c_2}, \frac{p}{c_3}, \frac{c_3}{c_1}$ and $\frac{c_2}{c_2}$ can be arbitrarily large [3,4,8]. In view of (3), (4) and (5), we have

$$r \leq R. \tag{6}$$

Moreover, the strict inequality may hold in (6) if $c_1 < p$ or $c_2 < p$ or $c_3 < p$.

The rest of the paper is organized as follows: In Section 2, we present the local convergence analysis of the method (2) under the ($\mathcal{H}$) conditions, whereas, in the concluding Section 3, we present some numerical examples.

2. Local convergence

In this section, we present the local convergence of the method (2) under the ($\mathcal{H}$) conditions in this Section.

Theorem 2.1. *Suppose that the* ($\mathcal{H}$) *conditions hold. Then the sequence* $\{x_n\}$ *generated by the method* (2) *for* $x_{-2}, x_{-1}, x_0 \in U(x^*, R)$ *is well defined, remains in* $U(x^*, R)$ *for each* $n \geq 0$ *and converges to* x^*. *Moreover, the following estimates hold*: *for each* $n \geq 0$,

$$\|x_{n+1} - x^*\| \leq e_n < R, \tag{7}$$

where $e_n = \frac{\Gamma_n}{\Theta_n}$ *with*

$$\Gamma_n = [c_1\|x_n - x^*\| + q(\|x_n - x^*\| + \|x_{n-2} - x^*\|) \times (\|x_n - x^*\| + \|x_{n-1} - x^*\|)]\|x_n - x^*\|$$

and

$$\Theta_n = 1 - [(c_2+c_3)\|x_n - x^*\| + q(\|x_n - x^*\| + \|x_{n-2} - x^*\|) \times \|x_{n-1} - x^*\|].$$

Furthermore, x^* is the unique solution of the equation (1) *in $U(x^*, \frac{1}{c_2})$ (for $c_2 \neq 0$) which is bigger than $U(x^*, R)$.*

Proof. Let $x, y, z \in U(x^*, R)$. Define the operator T by

$$A = [x, y; F] + [z, x; F] - [z, y; F]. \tag{8}$$

Using the condition $(\mathcal{H}_3)$, (8), the second and third hypotheses in $(\mathcal{H}_2)$, we have in turn

$$\begin{aligned}
&\|F'(x^*)^{-1}(A - F'(x^*))\| \\
&= \|F'(x^*)^{-1}([x^*, x^*; F] - [x, x^*; F] + [z, x^*; F] - [z, x; F] + [x, x^*; F] \\
&\quad - [x, y; F] - [z, x^*; F] + [z, y; F])\| \\
&\leq |F'(x^*)^{-1}([x^*, x^*; F] - [x, x^*; F])\| + \|F'(x^*)^{-1}([z, x^*; F] - [z, x; F])\| \\
&\quad + \|F'(x^*)^{-1}([x, x^*, y; F] - [z, x^*, y; F])(x^* - y)\| \\
&\leq (c_2 + c_3)\|x - x^*\| + q\|x - z\|\|x^* - y\| \\
&< (c_2 + c_3)R + q(\|x - x^*\| + \|x^* - z\|)\|x^* - y\| \\
&< (c_2 + c_3)R + 2qR^2 < 1 \qquad (9)
\end{aligned}$$

by the choice of R. It follows from (9) and the Banach Lemma on invertible operators [3,9,18,22] that $A^{-1} \in L(Y, X)$ and, for $x = x_n$, $z = x_{n-2}$, $y = x_{n-1}$,

$$\|A^{-1}F'(x^*)\| \leq \frac{1}{1 - [(c_2+c_3)\|x_n - x^*\| + q(\|x_n - x^*\| + \|x_{n-2} - x^*\|)\|x_{n-1} - x^*\|]}. \tag{10}$$

Suppose that $x_k, x_{k-1}, x_{k-2} \in U(x^*, R)$ for each $k \leq n$. Then it follows that $A_k = A(x_k, x_{k-1}, x_{k-2})$ is invertible. Therefore, using the method (2) and the condition $(\mathcal{H}_1)$, it follows that

$$\begin{aligned}
&\|x_{k+1} - x^*\| \\
&= \|x_k - x^* - A_k^{-1}(F(x_k) - F(x^*))\| \\
&= \| - A_k^{-1}([x_n, x^*; F] - A_k)(x_k - x^*)\| \\
&\leq \|A_k^{-1}F'(x^*)\|\|F'(x^*)^{-1}([x_k, x^*; F] - A_k)\|\|x_k - x^*\|. \qquad (11)
\end{aligned}$$

Next, using (11), the first condition in $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, we have in turn

$$\begin{aligned}
&\|F'(x^*)^{-1}([x_k, x^*; F] - A_k)\| \\
&= \|F'(x^*)^{-1}([x_k, x^*; F] - [x_k, x_k; F] \\
&\quad + [x_k, x_k; F] - [x_k, x_{k-1}; F] - [x_{k-2}, x_k; F] + [x_{k-2}, x_{k-1}; F])\| \\
&= \|F'(x^*)^{-1} \left(([x_k, x^*; F] - [x_k, x_k; F]\right)
\end{aligned}$$

$$+ \, ([x_k, x_k, x_{k-1}; F] - [x_{k-2}, x_k, x_{k-1}; F])(x_k - x_{k-1}))\, \|$$
$$\leq \|F'(x^*)^{-1}([x_k, x^*; F] - [x_k, x_k; F])\|$$
$$+ \|F'(x^*)^{-1}([x_k, x_k, x_{k-1}; F] - [x_{k-2}, x_k, x_{k-1}; F])\|\|x_k - x_{k-1}\|$$
$$\leq c_1\|x_k - x^*\| + q\|x_k - x_{k-2}\|\|x_k - x_{k-1}\|$$
$$\leq c_1\|x_k - x^*\| + q(\|x_k - x^*\| + \|x_{k-2} - x^*\|)$$
$$\times (\|x_k - x^*\| + \|x_{k-1} - x^*\|). \tag{12}$$

In view of (10)-(12), we arrive at

$$\|x_{k+1} - x^*\| \leq e_k < \frac{c_1R + 4qR^2}{1 - ((c_2 + c_3)R + 2qR^2)}R = R$$

by the choice of R which shows (7). It follows from (7) that

$$\|x_{k+1} - x^*\| < \|x_k - x^*\| < R,$$

which shows $x_{k+1} \in U(x^*, R)$ and $\lim_{k\to\infty} x_k = x^*$.

Finally, to show the uniqueness part, let $y^* \in U(x^*, \frac{1}{c_2})$ be a solution of the equation $F(x) = 0$. Then, using the second hypotheses in $(\mathcal{H}_2)$, we have

$$\|F'(x^*)^{-1}(F'(x^*) - [y^*, x^*; F])\| \leq c_2\|y^* - x^*\| < 1. \tag{13}$$

It follows from (13) and the Banach Lemma on invertible operators that the operator $[y^*, x^*; F]$ is invertible. Then, from the identity

$$[y^*, x^*; F](y^* - x^*) = F(y^*) - F(x^*) = 0,$$

we deduce that $x^* = y^*$. It follows from (4) that $R \leq \frac{1}{c_2}$. This completes the proof. □

Remark 2.2. (1) It follows from $(\mathcal{C}_2)$ that

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq 2p\|x - y\|$$

for each x, $y \in D$. Then, the radius of convergence r is smaller that the radius of convergence r_N [3,8,9,22] for Newton's method defined as follows: for each $n \geq 0$,

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{14}$$

where x_0 is an initial point,

$$r_N = \frac{1}{3p}, \;\; r \leq r_N. \tag{15}$$

Notice that, if $q = 0$, then $r = r_N$.

(2) The corresponding error bounds in [25] are:

$$\|x_n - x^*\| \leq \bar{e}_n < r, \tag{16}$$

where

$$\bar{e}_n = \frac{p(\|x_n - x^*\| + q(\|x_{n-2} - x^*\| + \|x_n - x^*\|)(\|x_{n-1} - x^*\| + \|x_n - x^*\|)}{[1 - (2p\|x_n - x^*\| + q(\|x_{n-1} - x^*\| + \|x_{n-2} - x^*\|)\|x_{n-1} - x^*\|}.$$

Notice that

$$e_n \leq \bar{e}_N. \tag{17}$$

Moreover, the strict inequality holds if $c_1 < p$ or $c_2 < p$ or $c_3 < p$. In this case we may also have $r_N < r < R$ (see also the numerical examples).

(3) The ideas of this paper can also be used to improve the semi-local convergence analysis given in [25]. However, we leave this task to the motivated reader.

3. Numerical examples

We present a numerical example in this section.

Example 3.1. Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0,1)$, $x^* = (0,0,0)^T$. Define a function F on D for $w = (x,y,z)^T$ by

$$F(w) = \Big(e^x - 1, \frac{e-1}{2}y^2 + y, z\Big)^T.$$

Then the Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define the divided differences $[x,y;F]$ and $[x,y,z;F]$ by

$$[x,y;F] = \frac{1}{2}(F'(x) + F'(y))$$

and

$$[x,y,z;F](y-z) = [x,y;F] - [x,z;F].$$

Then the conditions $(\mathcal{C}_1)$, $(\mathcal{H}_1)$, $(\mathcal{C}_2)$, $(\mathcal{H}_2)$, $(\mathcal{C}_3)$, $(\mathcal{H}_3)$ are satisfied for $x^* = 0$, $F'(x^*) = F'(x^*)^{-1} = 1$, $p = \frac{e}{2}$, $q = 0$ and $c_1 = c_2 = c_3 = \frac{e-1}{2}$. Notice that

$$c_1 < p, \;\; c_2 < p, \;\; c_3 < p$$

and

$$\begin{aligned} r_N = r = \frac{2}{3e} &= 0.24525296 \\ &< 0.387984471 = R = \frac{2}{3(e-1)} \\ &< 1 - 163953414 = \frac{1}{c_1}. \end{aligned}$$

Therefore, x^* is unique in D.

Next, we compare the error bounds e_n (see (7) with $\bar{e}_n$ (see (16)). Choose

$$\begin{aligned} &x_{-2} = (0.244, 0.244, 0.244)^T, \;\; x_{-1} = (0.242, 0.242, 0.242)^T, \\ &x_0 = (0.24, 0.24, 0.24)^T. \end{aligned}$$

Then we obtain the following table.

Table 1. Comparison table

n	(7)	(16)
2	7.6744e-04	0.0276

It follows from Table 1 that our error estimates are more precise than the corresponding ones in [25].

Example 3.2. Returning back to the motivational example at the introduction of this study, we have

$$c_1 = c_2 = c_3 = p = \frac{96.662907}{2}, \quad q = \frac{146.6629073}{2}.$$

Then we have

$$r = R = 0.006896819962870 < r_N = 0.0081432042892163.$$

Notice that $r = R$ in this case by (3) and (4).

Example 3.3. Let $X = Y = C[0,1]$ (the space of continuous functions defined on $[0,1]$) be equipped with the max norm. Let $D = \overline{U}(0,1)$ and define a function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{18}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta$$

for each $\xi \in D$. Then it follows that $x^* = 0$, $c_1 = c_2 = c_3 = \frac{7.5}{2}$, $p = q = 7.5$. So, we have

$$\begin{aligned} r_N &= 0.0444444444444 \\ &< r = 0.0485479622541332 \\ &< R = 0.06954362549881253. \end{aligned}$$

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Ioannis K. Argyros
Department of Mathematical Sciences
Cameron University
Lawton, OK 73505, USA
Email address: argyros@cameron.edu

Yeol Je Cho
Department of Mathematics Education
Gyeongsang National University
Jinju 52828, Korea
and
School of Mathematical Sciences
University of Electronic Science and Technology of China
Chengdu, Sichuan 611731, P. R. China
Email address: yjcho@gnu.ac.kr

Santhosh George
Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka
Karnataka 575025, India
Email address: sgeorge@nitk.ac.in