Inverse linear multistep methods for the numerical solution of initial value problems of second order differential equations

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ABSTRACT

In this paper inverse linear multistep methods for the numerical solution of second order differential equations are presented. Local accuracy and stability of the methods are defined and discussed. The methods are applicable to a class of special second order initial value problems, not explicitly involving the first derivative. The methods are not convergent, but yield good numerical results if applied to problems they are designed for. Numerical results are presented for both the linear and nonlinear initial value problems.

1. INTRODUCTION

Second order differential equations arise in a wide variety of important physical problems. Two special cases of the second order initial value problem

$$F(t, y, y'') = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \qquad (1.1)$$

are

$$y'' = f(t, y), \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \qquad (1.2)$$

and

$$y = g(t, y''), \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \qquad (1.3)$$

Sometimes (1.2) can be transformed into (1.3). For example in the case when

$f(t, y) = \eta y, \quad \eta \neq 0$

the corresponding problem (1.3) is given by

$g(t, y'') = \eta^{-1} f(t, y)$.

On the other hand, the simple integration problem (1.2) defined by

$f(t, y) = \phi(t)$

cannot be transformed into (1.3).
To obtain the numerical solution of (1.2), we make use of linear multistep methods given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (1.4)$$

When conventional methods of the form (1.4), with $k > 2$, are used to solve the initial value problem (1.2), the time increment must be limited to a value proportional to the reciprocal of the largest eigen value of the Jacobian of (1.2). Any attempt to use a larger increment results in the calculations becoming unstable and producing erroneous results. The Störmer-Cowell linear multistep methods of order greater than two when used to solve (1.2) are found to be unstable for large step sizes (see Stiefel and Bettis [4]). Recently, Alfeld [1] has developed a class of linear multistep methods known as inverse linear multistep methods for solving a stiff system of first order differential equations. In this paper, to obtain the numerical solution of (1.3), we consider linear multistep methods of the form

$$h^{-2} \sum_{j=0}^{k-1} \hat{\alpha}_j y_{n+j} = \sum_{j=0}^{k} \hat{\beta}_j f_{n+j} \qquad (1.5)$$

where

$\hat{\beta}_k = 1$ and $y_{n+j} = g(t_{n+j}, f_{n+j})$.

The methods (1.5) are referred to as inverse linear multistep methods (ILMMs). They are explicit when applied to initial value problems of the form (1.3).

2. LOCAL ACCURACY

With the ILMM (1.5), we associate the difference operator

$$T[z(t_n), h] = \sum_{j=0}^{k} [h^{-2} \hat{\alpha}_j z(t_{n+j}) - \hat{\beta}_j z''(t_{n+j})] \qquad (2.1)$$

with $\hat{\alpha}_k = 0$, where $z(t)$ is an arbitrarily often differentiable test function. Proceeding similarly as for explicit linear multistep methods (see Lambert [3]), we expand $T[z(t_n), h]$, collect terms and obtain

$$T[z(t_n), h] = h^{-2}[\hat{C}_0 z(t_n) + \hat{C}_1 h z^{(1)}(t_n) + \hat{C}_2 h^2 z^{(2)}(t_n) + \ldots] \qquad (2.2)$$

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where

$$\hat{C}_0 = \hat{a}_0 + \hat{a}_1 + \ldots\ldots + \hat{a}_k$$

$$\hat{C}_1 = \hat{a}_1 + 2\hat{a}_2 + \ldots + k\,\hat{a}_k$$

$$\vdots$$

$$\hat{C}_q = \frac{1}{q!}\left(\hat{a}_1 + 2^q \hat{a}_2 + \ldots\ldots + k^q \hat{a}_k\right) - \frac{1}{(q-2)!}\left(\hat{\beta}_1 + 2^{q-2}\hat{\beta}_2 + \ldots\ldots + k^{q-2}\hat{\beta}_k\right);$$

$$q = 2, 3, 4, \ldots$$

Definition 2.1

The ILMM (1.5) is said to be of order p if in (2.2)

$$\hat{C}_0 = \hat{C}_1 = \ldots = \hat{C}_{p+2} = 0, \quad \hat{C}_{p+3} \neq 0 \tag{2.3}$$

$\hat{C}_{p+3}$ is known as the error constant of (1.5). The method (1.5) is consistent if it is of order $p \geqslant 1$. A necessary condition for $p \geqslant 1$ is $k \geqslant 4$.

3. STABILITY OF ILMMs

Applying the method (1.5) to the test equation

$$y'' = -\lambda^2 y, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{3.1}$$

we obtain

$$\hat{\pi}(\xi, \bar{H}) = \hat{\rho}(\xi) + \bar{H}^2 \hat{\sigma}(\xi) \tag{3.2}$$

where

$$\hat{\rho}(\xi) = \sum_{j=0}^{k-1} \hat{a}_j \xi^j$$

$$\hat{\sigma}(\xi) = \sum_{j=0}^{k} \hat{\beta}_j \xi^j$$

$$\bar{H}^2 = \lambda^2 h^2.$$

We refer to the polynomials $\hat{\rho}(\xi)$ and $\hat{\sigma}(\xi)$ as the first and the second characteristic polynomials and to $\hat{\pi}(\xi, \bar{H})$ as the stability polynomial of (1.5).

Definition 3.1

The ILMM (1.5) is said to be absolutely stable for a given $\bar{H}^2 \in C$, if for that $\bar{H}^2$, all the roots ξ_m of (3.2) satisfy $|\xi_m| < 1$ for $m = 1, 2, \ldots, k$. The set $R = \{\bar{H}^2 \in C / \text{absolutely stable for } \bar{H}^2\}$ is called the region of absolute stability.

The Störmer-Cowell linear multistep methods, when applied to the test equation (3.1) possess absolute stability intervals $0 < \bar{H}^2 < \bar{H}_0^2$. The values of $\bar{H}_0^2$ for k = 2, 3, 4 and 5 are given in table 1.

In (3.2), the degree of $\hat{\rho}(\xi) \leqslant k-1 < k$ whereas the degree of $\hat{\sigma}(\xi)$ is k. So one of the roots of $\hat{\pi}(\xi, \bar{H}) = 0$ tends to infinity as $\bar{H}$ tends to zero. This shows that the ILMM (1.5) is always unstable for small values of $\bar{H}$. Hence the method (1.5) is non-convergent as $\bar{H}$ tends to zero.

Since the zeros of the stability polynomial (3.2) tend to those of $\hat{\sigma}(\xi)$ as $\bar{H}$ tends to infinity, we are led to seek ILMMs whose second characteristic polynomial possesses only zeros of modulus less than unity i.e., it is a Schur polynomial.

Definition 3.2

The ILMM (1.5) is said to be *infinite-stable* if $\hat{\sigma}(\xi)$ is a Schur polynomial.

Definition 3.3

The ILMM (1.5) is said to be *strongly infinite-stable* if $\hat{\sigma}(\xi) = \xi^k$.

A strongly infinite-stable ILMM has the form

$$f_{n+k} = h^{-2} \sum_{j=0}^{k-1} \hat{a}_j y_{n+j} \tag{3.3}$$

We note that the concept of infinite-stability is in a way dual to the concept of zero-stability. Zero-stability deals with the case that $\bar{H}$ tends to zero whereas infinite-stability deals with the case when $\bar{H}$ tends to infinity.

For a strongly infinite-stable ILMM we have k-parameters $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{k-1}$ at our disposal and can thus expect to be able to attain an order k − 3. Also we can expect a consistent strongly infinite-stable ILMM to have a step number $k \geqslant 4$.

The following theorem gives the maximum attainable order of infinite-stable or strongly infinite-stable ILMMs.

Theorem 3.1

(a) Let $\hat{\sigma}(\xi)$ be a polynomial of degree k (with $\hat{\beta}_k = 1$). Then there exists a unique polynomial of degree k − 1, such that the ILMM defined by $\hat{\rho}(\xi)$ and $\hat{\sigma}(\xi)$ has order at least k − 3.

(b) The maximum order of an infinite-stable ILMM is k − 3. For each $k \geqslant 4$, there exists a strongly infinite-stable ILMM of order k − 3.

We omit the proof of this theorem, as it follows closely the lines of Henrici's proof ([2], pp. 304-307) on the maximum order of zero-stable linear multistep methods, with necessary modifications being adopted as suggested in [1] for ILMMs of first order system of differential equations.

4. SPECIFICATION OF ILMMs

The following is a list of inverse linear k-step methods of order k − 3 for k = 4, 5, 6 with free parameters that give complete control over the coefficients and thus over the zeros of $\hat{\sigma}(\xi)$. The error constants in terms of the free parameters and the maximum value of $\lambda^2 h^2 = \bar{H}^2_{min}$ are given for which the methods are strongly infinite-stable.

k = 4

$$f_{n+4} = -\hat{\beta}_3 f_{n+3} - \hat{\beta}_2 f_{n+2} - \hat{\beta}_1 f_{n+1} - \hat{\beta}_0 f_n$$
$$+ [(3+2\hat{\beta}_3 + \hat{\beta}_2 - \hat{\beta}_0) y_{n+3} + (-8 - 5\hat{\beta}_3 - 2\hat{\beta}_2 + \hat{\beta}_1$$
$$+ 4\hat{\beta}_0) y_{n+2} + (7 + 4\hat{\beta}_3 + \hat{\beta}_2 - 2\hat{\beta}_1 - 5\hat{\beta}_0) y_{n+1}$$
$$+ (-2 - \hat{\beta}_3 + \hat{\beta}_1 + 2\hat{\beta}_0) y_n]/h^2 \quad (4.1)$$

Order of the method : 1

Error constant : $\hat{C}_4 = \frac{1}{12}(-35 - 11\hat{\beta}_3 + \hat{\beta}_2 + \hat{\beta}_1 - 11\hat{\beta}_0)$

$\bar{H}^2_{min} = 20.$

k = 5

$$f_{n+5} = -\hat{\beta}_4 f_{n+4} - \hat{\beta}_3 f_{n+3} - \hat{\beta}_2 f_{n+2} - \hat{\beta}_1 f_{n+1} - \hat{\beta}_0 f_n$$
$$+ [(71 + 35\hat{\beta}_4 + 11\hat{\beta}_3 - \hat{\beta}_2 - \hat{\beta}_1 + 11\hat{\beta}_0) y_{n+4}$$
$$+ (-236 - 104\hat{\beta}_4 - 20\hat{\beta}_3 + 16\hat{\beta}_2 + 4\hat{\beta}_1 - 56\hat{\beta}_0) y_{n+3}$$
$$+ (294 + 114\hat{\beta}_4 + 6\hat{\beta}_3 - 30\hat{\beta}_2 + 6\hat{\beta}_1 + 114\hat{\beta}_0) y_{n+2}$$
$$+ (-164 - 56\hat{\beta}_4 + 4\hat{\beta}_3 + 16\hat{\beta}_2 - 20\hat{\beta}_1 - 104\hat{\beta}_0) y_{n+1}$$
$$+ (35 + 11\hat{\beta}_4 - \hat{\beta}_3 - \hat{\beta}_2 + 11\hat{\beta}_1 + 35\hat{\beta}_0) y_n]/12\, h^2 \quad (4.2)$$

Order of the method : 2

Error constant : $\hat{C}_5 = \frac{1}{144}(-540 - 110\hat{\beta}_4 + 12\hat{\beta}_3 - 12\hat{\beta}_1 + 120\hat{\beta}_0)$

$\bar{H}^2_{min} = \frac{200}{3}$

k = 6

$$f_{n+6} = -\hat{\beta}_5 f_{n+5} - \hat{\beta}_4 f_{n+4} - \hat{\beta}_3 f_{n+3} - \hat{\beta}_2 f_{n+2} - \hat{\beta}_1 f_{n+1} - \hat{\beta}_0 f_n$$
$$+ [(116 + 45\hat{\beta}_5 + 10\hat{\beta}_4 - \hat{\beta}_3 + \hat{\beta}_1 - 10\hat{\beta}_0) y_{n+5}$$
$$+ (-461 - 154\hat{\beta}_5 - 15\hat{\beta}_4 + 16\hat{\beta}_3 - \hat{\beta}_2 - 6\hat{\beta}_1 + 61\hat{\beta}_0) y_{n+4}$$
$$+ (744 + 214\hat{\beta}_5 - 4\hat{\beta}_4 - 30\hat{\beta}_3 + 16\hat{\beta}_2 + 14\hat{\beta}_1 - 156\hat{\beta}_0) y_{n+3}$$
$$+ (-614 - 156\hat{\beta}_5 + 14\hat{\beta}_4 + 16\hat{\beta}_3 - 30\hat{\beta}_2 - 4\hat{\beta}_1 + 214\hat{\beta}_0) y_{n+2}$$
$$+ (260 + 61\hat{\beta}_5 - 6\hat{\beta}_4 - \hat{\beta}_3 + 16\hat{\beta}_2 - 15\hat{\beta}_1 - 154\hat{\beta}_0) y_{n+1}$$
$$+ (-45 - 10\hat{\beta}_5 + \hat{\beta}_4 - \hat{\beta}_2 + 10\hat{\beta}_1 + 45\hat{\beta}_0) y_n]/12\, h^2 \quad (4.3)$$

Order of the method : 3

Error constant :

$$\hat{C}_6 = \frac{1}{180}(-812 - 137\hat{\beta}_5 + 13\hat{\beta}_4 - 2\hat{\beta}_3 - 2\hat{\beta}_2 + 13\hat{\beta}_1 - 137\hat{\beta}_0)$$

$\bar{H}^2_{min} = \frac{560}{3}$

5. COMPARISON OF ELMMs AND ILMMs

	ELMM	ILMM
Minimum step number of consistent method :	2	2
Minimum step number of zero-stable (infinite-stable) consistent method:	2	4
Max. order of zero-stable (or infinite-stable) k-step method :	k	k-3

6. IMPLEMENTATION OF THE ILMMs

The implementation of the ILMMs (1.5) to IVPs of the form (1.3) can be carried out as follows. Assume $y_0, y_1, \ldots, y_{k-1}$ are computed by some other method and thus $f_j = f(t_j, y_j), j = 0(1)k-1$ are known. We compute f_{n+k} from the ILMM (1.5) and then y_{n+k} is obtained from $y_{n+k} = g(t_{n+k}, f_{n+k})$.

A class of problems to which the ILMMs (1.5) can successfully be adopted is given by

$$y = \frac{[z'' - \phi(t, y)]}{-\lambda^2} + z(t) \quad (6.1)$$

where λ is large and z(t) is the exact solution.

If $\phi(t, y)$ in (6.1) is nonlinear, then having computed f_{n+k} $(= Z''_{n+k})$ by the ILMM (1.5), we adopt the Picard iteration technique given by

$$y^{(i+1)}_{n+k} = \frac{f_{n+k} - \phi(t_{n+k}, y^{(i)}_{n+k})}{-\lambda^2} + z(t_{n+k}) \quad (6.2)$$

to find y_{n+k}. We take the initial approximation $y^{(0)}_{n+k}$ as y_{n+k-1} which is the known calculated value at the previous step point.

Remark

We note that the implicit linear multi-step methods are iterative in character when applied to nonlinear IVPs. The iteration scheme (6.2) shows that the ILMMs (1.5) are not iterative in character.

7. NUMERICAL RESULTS

We solve the following linear and nonlinear IVPs by the method (4.1) with $\hat{\beta}_0 = \hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta}_3 = 0$. The initial values y_0, y_1, y_2, y_3 are taken from the exact solution.

Example 1

Consider the linear problem (see [1])

$$y(t) = \frac{[y''(t) + \cos t]}{-\lambda^2} + \cos t \qquad (7.1)$$

where $\lambda^2 = 10^4$ with the exact solution

$y(t) = [y(0) - 1] \cos 100\,t + y'(0) \sin 100\,t + \cos t$

for $t_0 = 0$. The absolute errors at $t = 100$ for the following initial conditions are tabulated in table 2.

(i) $y(0) = 1, \quad y'(0) = 0$ (7.1a)

(ii) $y(0) = 1 + \epsilon, \quad y'(0) = 0, \quad \epsilon = 0.001$ (7.1b)

We also observed that the method is unstable for $\lambda^2 = 100$ and $h = 0.1$ since in this case $\lambda^2 h^2 = 1 < \bar{H}^2_{min}$.

Example 2

Consider the nonlinear problem

$$y = \frac{y'' - e^{2y}}{-\lambda^2} - \log^{(1+t)} \qquad (7.2)$$

with initial conditions $y(0) = 0$, $y'(0) = -1$. The exact solution of (7.2) is $y(t) = -\log^{(1+t)}$. We have tested the method for $\lambda^2 = 100$ with $h = 0.5$ and the absolute errors at $t = 20, 40, 60, 80$ and 100 are tabulated in table 3. We used the Picard iteration technique given by

$$y^{(i+1)}_{n+k} = \frac{f_{n+k} - e^{2y^{(i)}_{n+k}}}{-\lambda^2} - \log^{(1+t_{n+k})} \qquad (7.3)$$

with $y^{(0)}_{n+k} = y_{n+k-1}$.

The iteration is stopped when

$\left| y^{(i+1)}_{n+k} - y^{(i)}_{n+k} \right| < 10^{-8}$. We observed that the desired accuracy is obtained within only two iterations.

8. CONCLUSIONS

The ILMMs can be directly applied to problems of the form (1.3). The methods are explicit when applied to problems of the form (1.3) and thus avoid iteration techniques which are inherent in implicit linear multistep methods. The methods are infinite-stable which makes them computationally superior to the classical Störmer-Cowell methods, using large step sizes.

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TABLE 1. Absolute stability intervals of Störmer-Cowell methods $0 < \bar{H}^2 < \bar{H}_0^2$

k	2	3	4	5
Explicit Störmer	4	3	2	$\frac{240}{199}$
Implicit Cowell	6	6	$\frac{60}{11}$	$\frac{60}{13}$

TABLE 2. Approximate solution and absolute errors at $t = 100$

h	(7.1a)		(7.1b)	
	y_n	$\lvert y_n - y(t_n)\rvert$	y_n	$\lvert y_n - y(t_n)\rvert$
0.1	8.623(–01)	2.219(–06)	8.614(–01)	9.544(–04)
0.5	8.623(–01)	1.245(–05)	8.614(–01)	9.646(–04)

TABLE 3. Approximate solution and absolute errors for the nonlinear problem (7.2)

t	y_n	$\lvert y_n - y(t_n)\rvert$
20	- 3.044	4.507 (-05)
40	- 3.714	1.189 (-05)
60	- 4.111	5.381 (-06)
80	- 4.394	3.031 (-06)
100	- 4.615	1.968 (-06)