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ISOMETRY IN MECHANISM DESIGN

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Abstract—Isometric designs are those that carry the same linear dimensions. Unique specification of the motion requirements of a mechanism between skew axes, with the help of a suitable set of conventions, removes all alternative designs isometric to the original mechanism, with one exception: an isometric design satisfying exactly the same motion requirements is available but with the algebraic sign of the axis distance changed. The transformation is one of reflection and is called an opposite isometry. Extending the use of unique specification of motion requirements to the special case of two-position design with identical specification of motion derivatives at the two positions (as in the case of a crank-rocker design), one is left with an alternative isometric design, without change of sign of axis distance. The device consists of a change of direction of motion of the mechanism, coupled with an interchange of mechanism positions that correspond to the two distinct positions on the curve of motion relationship. There is, in this case, what is called a direct isometry. The two transformations are shown to be useful: (i) in reducing the extent of design cataloguing, and (ii) in reducing the area of search for suitable mechanism designs.

1. INTRODUCTION

Let us suppose that a mechanism has been designed to provide a certain motion conversion between two non-parallel and non-intersecting shafts. The question can then be asked if it is possible to have alternative solutions that have the same linear dimensions. Such alternatives may be referred to as *isometric mechanisms*. These are obviously of considerable practical significance since they can be expected to be arrived at without further calculation or construction.

2. GEOMETRIC PRELIMINARY

A geometric transformation that preserves all distances is referred to as an *isometry* [1]. Any *rigid body motion* represents what is called a *direct isometry* (or proper isometry). The only other form of isometry is the *opposite isometry* (or improper isometry), obtained by *reflection* of the geometric configuration about any plane.†

Opposite isometry is used in Section 4 of this paper and direct isometry in Section 5.

3. UNIQUE SPECIFICATION OF THE DESIGN REQUIREMENT

The following conventions are introduced.

(a) The positive direction of the input shaft axis is the direction in which the input shaft is seen rotating clockwise.

The positive direction of the output shaft axis is the direction in which the output shaft is seen rotating clockwise.

†Reflection about a point is also valid for the three-dimensional case but not for the two-dimensional case.

These positive directions will be referred to as the input and output axis vectors.

(b) The *axis angle* δ is the true angle $\leq 180°$ between the input and output axis vectors. This is considered positive. δ is thus restricted to be between 0 and $+180°$.

(c) The *axis distance* d is measured from the input axis to the output axis. Consider the rotation of the input axis vector into coincidence with the output axis vector through the axis angle δ, already defined. The direction along the common perpendicular to the axes in which this rotation appears clockwise is the positive direction of the axis distance d. The quantity d may thus be positive or negative. The case where d is positive is illustrated in Fig. 1.

If there is choice in the direction of rotation of the output (ϕ), keeping input rotation direction same, this is to be treated as an alternative specification of design requirements.

The importance of the unique specification of design requirements lies in the elimination of unnecessary calculation that leads to repeated designs. Consider, for example, the question of determining designs to generate

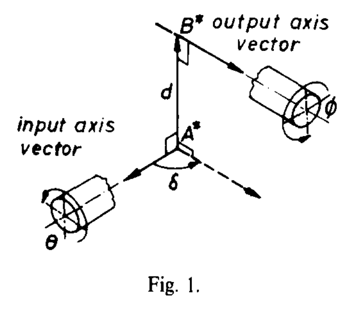

Fig. 1.

the same given function $\phi = f(\theta)$ at various axis angles δ. The conventions given above enable us to limit ourselves to varying δ between 0 and $+180°$ and eliminate the range of 180–360°.

The conventions, on the other hand, allow the axis distance d to have both positive and negative values. This raises a different problem, considered in the next section.

4. ISOMETRIC CONVERSION OF A MECHANISM TO ALTERNATIVE SIGN OF AXIS DISTANCE d

The device put forward in this section is valid for any mechanism and motion, between two skew shafts.

The conventions that may be followed in describing shaft rotations, axis angle and axis distance are, strictly speaking, immaterial for the present section as well as Section 5.

It is supposed that a mechanism design for the given specifications is already at hand (e.g. from a catalogue of designs), except that the axis distance d does not have the appropriate algebraic sign. In other words, the output shaft axis may be "above" the input shaft axis while it is desired to be "below".

To convert the mechanism to one that satisfies this requirement too, it is merely necessary to *reflect* every point† of the mechanism (including the input and output shafts and the frame) about a plane. The result is essentially the same (an opposite isometry), whatever reflecting plane is made use of. The opposite isometry may be described in the following way. In any position of the mechanism consider any four points A, B, C, D of the mechanism, not lying in a plane before and after the transformation: the sense of running through A, B, C in that order, as viewed, from D, changes from a clockwise one to a counter-clockwise one or *vice versa*.

For the purpose of further discussion, it is convenient to introduce a definite co-ordinate system. The foot of the common perpendicular on the input axis is taken as the origin. This is the point A^* in Fig. 1. The **Ox** axis coincides with the input axis vector. The **Oz** axis coincides with the positive direction of the axis distance d (it may thus be in the direction $\boldsymbol{A^*B^*}$ or $\boldsymbol{B^*A^*}$). The **Oy** axis is then decided by the right-handedness of the co-ordinate system (see Fig. 2).

†This reflection should be considered carried out in every position of the mechanism, so that the motion itself is reflected.

‡It may be remembered that the motion itself is reflected.

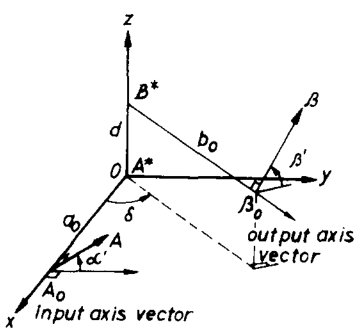

Fig. 2.

Reverting back to the transformation under consideration, the plane xOy can be chosen as the plane of reflection, for convenience of illustration. Both the input axis vector and the output axis vector reverse their directions‡, so that the axis angle remains the same. x and y axes reverse their directios while z does not. The axis distance d thus changes sign.

Suppose further that arbitrary vectors $\boldsymbol{A_0A}$ and $\boldsymbol{B_0B}$ on the input and output shafts (perpendicular to the respective shaft axes) are used to define the angular positions of the input and output links. Let A_0 and B_0 be on the respective axes (Fig. 2). Let the vector $\boldsymbol{A_0A}$ make the angle α' with the **Oy** direction: positive clockwise when viewed along **Ox**. An identical definition of the angle β' made by $\boldsymbol{B_0B}$ is obtained by rotating the output axis vector back through δ into coincidence with the input axis vector. The distances A^*A_0 and B^*B_0 will be denoted a_0 and b_0 respectively, positive in the direction of the axis vectors.

The further effects of the transformation can now be described as follows: (a) α' and β' change by 180°, and (b) a_0 and b_0 change their signs. Other dimensions of the mechanism are maintained in magnitude, but appropriately adjust in sign where needed.

If we wish to describe the transformation without permitting any change in the co-ordinate system, we may proceed as follows: (a) reflect on the xOy plane, and (b) rotate about z-axis through 180°. Alternatively, we may simply reflect about the origin O. The result is what is sometimes called a central inversion[1]. The transformation is illustrated in Fig. 3 for the RSSR mechanism, where A is chosen at the centre of the input side spheric pair and B at the centre of the output side spheric pair. The coupler length AB does not change.

Table 1 summarizes the transformation. If the transformation is repeated once, we are clearly back with the original design.

If the original mechanism represents an optimal design in some way, the question arises whether the trans-

Table 1.

(1) Original	d	a_0	b_0	α'	β'
(2) Reflection of 1	$-d$	$-a_0$	$-b_0$	$180°+\alpha'$	$180°+\beta'$

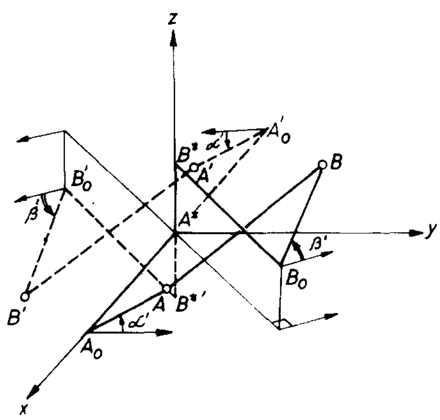

Fig. 3.

formed design is again an optimal one. We note that not only the design but also the specification of motion requirements undergo the same transformation. Thus the query may be answered in the affirmative.

In conclusion of this section, it may be emphasized that the result is quite general and applicable to gears, three-dimensional cams and linkages on skew axes. The method is applicable to motion transfer between parallel axes too, but the result may be ragarded as trivial. Figure 4 gives the example of a planar four-bar mechanism.

It may be noted in passing that the problem of unique description of a given mechanism, considered admirably by Savage and Hall[2], is different from the present one since we are actually considering conversion of one mechanism design to another. The content of the previous section is however in the same line. It may further be pointed out that opposite isometry is also significant in a different context[3]: two single universal joints which are oppositely isometric produce a double joint with uniform motion transmission.

5. DUPLICATE MODE OF OPERATION OR POSITION INTERCHANGE

The device of this section is valid for any mechanism between two skew shafts, under the following conditions:

(a) The motion requirement refers only to two positions of the mechanism, and

(b) The motion derivative requirements at these two positions are identical. Even derivatives change sign.

Figure 5 illustrates the conditions. The derivatives at 1 and 2 must be the same upto the required order. For example, if the first order derivatives are being prescribed at 1 and 2, they should be the same. The most important case of this nature arises when both the derivatives are zero: the design of a crank-rocker mechanism for prescribed oscillation angle ϕ_0 and quick return ratio $\theta_0/(360° - \theta_0)$.

The broken lines in Fig. 5 indicate the method of obtaining an alternative solution: reverse the direction of rotation of the input and output links.

The discussion will be carried out in relation to the crank-rocker design, in view of its importance.

Limit positions 1 *and* 2: These are the positions where the output rocker comes to rest before reversing its direction of motion while the input crank continues to rotate in the same direction. (The intermediate stationary positions obtainable from the more complex linkages are left out of consideration). The numbering of these two limit positions 1 and 2 follows the following convention:

(a) Half-oscillation $1 \rightarrow 2$ is faster than half-oscillation $2 \rightarrow 1$.

(b) The corresponding positions of the input crank are also accordingly numbered 1, 2.

In defining the output axis vector, the rotation of the output rocker from position-1 to 2 is considered. This is the rocker oscillation angle ϕ_0, lying between 0 and $+360°$.

The crank rotation angle θ_0 is from position-1 to 2 and thus θ_0 lies between $0°$ and $+180°$, in view of the particular numbering of the limit positions.

It is supposed that a mechanism design for the given specifications is already at hand. We now proceed to:

(i) Name position 1 of the mechanism as position 2 and vice-versa.

(ii) Reverse the direction of motion of the mechanism. This process of position interchange coupled with reversal of motion direction is successful because the two positions are indistinguishable as far as motion requirements at these two positions are concerned (see Figs. 5 and 6).

It will be noticed that we are dealing with the same "mechanism" but with a different way of utilizing it to satisfy the same specifications. The input axis vector and the output axis vector have both reversed their directions, so that the axis angle δ is unaltered. The sign of axis distance d is also unaltered. The distances a_0 and b_0, defined in the previous section, change sign.

The values of the input and output position variables α' and β' in position 1 are represented as α and β respectively. They change respectively to $[\alpha] = 180° - (\alpha + \theta_0)$ and $[\beta] = 180° - (\beta + \phi_0)$.

The results of this transformation are summarized in Table 2. This amounts to "refitting" the mechanism on the diametrically opposite side of the common perpendicular to the input and output shaft axes and redefining position 1. It can also be interpreted as rotation of the original mechanism about z-axis through 180° and changing the direction of rotation.

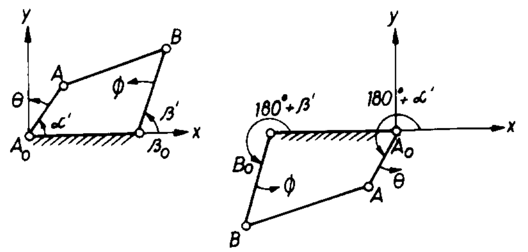

Fig. 4.

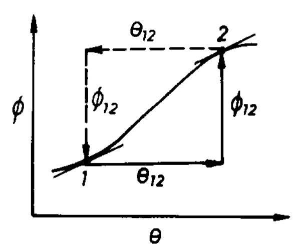

Fig. 5.

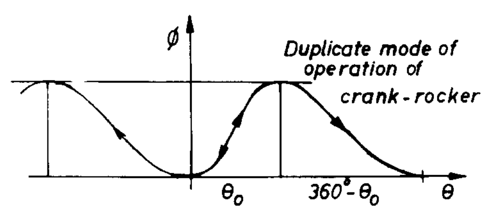

Fig. 6.

Table 2.

(a) Original	d	a_0	b_0	α	β
(b) position interchange in (a)	d	$-a_0$	$-b_0$	$180° - (\alpha + \theta_0)$	$180° \ (\beta\phi_0)$

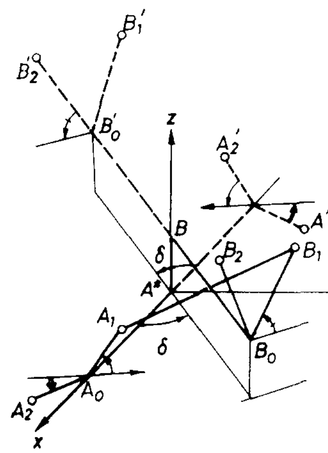

Fig. 7.

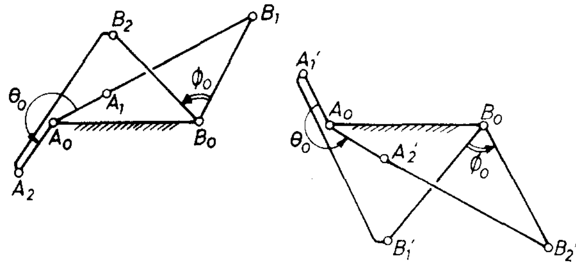

Fig. 8.

The transformation is illustrated in Fig. 7 for the RSSR mechanism and in Fig. 8 for planar four-bar mechanism.

6. COMBINATION OF THE TWO TRANSFORMATIONS

If the conditions mentioned in the previous section are valid, we may apply the transformation of Section 4 (reflection) and that of Section 5 (position interchange) in succession. Table 3 is the result.

It will now be shown that there are no other possibilities beyond these four, when the two transformations are applied. A second successive application of the same transformation merely nullifies the first application. But the solution 4 in Table 3 was obtained in the two ways indicated, so that application of either transformation to it will merely lead us back to either solution 2 or 3, as the case may be.

7. AN APPLICATION OF TABLE 1

Let us suppose that the input and output link position variables α' and β' are design parameters that can be varied to obtain different designs of a mechanism (e.g. the RSSR mechanism) satisfying a specified motion relationship. Table 1 enables us to restrict the range of variation of (α', β'), even when the sign of the axis distance d is specified (along with the value of the axis angle δ). This is because we can use the table later to convert the obtained design to satisfy the appropriate sign of d.

Table 1 changes α' and β' to $180° + \alpha'$ and $180° + \beta'$ respectively. The latter angles can be replaced by $\alpha' - 180°$ and $\beta' - 180°$ respectively, as needed, by subtracting 360°. It is thus possible to consider the four alternatives $(180° + \alpha', 180° + \beta')$, $(\alpha' - 180°, 180° + \beta')$, $(180° + \alpha', \beta' - 180°)$ and $(\alpha' - 180°, \beta' - 180°)$. These changes are applied successively to the square region defined by $\alpha' = 0°$–$360°$, $\beta' = 0°$–$360°$ (Fig. 9a).

Overlapping areas are eliminated at each stage before the next alternative is considered. Use of the first alternative $(180° + \alpha', 180° + \beta')$ transforms the region ABCD to the broken line square indicated in Fig. 9(a). The overlapping area KECH can be eliminated and the remaining region is shown in Fig. 9(b). (Actually there is the choice to eliminate AGKF instead.)

Use of the second alternative $(\alpha' - 180°, 180° + \beta')$ transforms the region ABEKHD to the broken line region indicated in Fig. 9(b). Elimination of the area FKHD retains the rectangle shown in Fig. 9(c).

It is easy to see that no further reduction is possible by the application of the remaining alternatives.

It is thus necessary to consider variation of α' *from* 0 *to* $+360°$ *and* β' *from* 0 *to* $+180°$ *only.* (one can alternatively restrict α' to 180° instead of β'. It is also evident that the lower limit of either variable need not be 0°).

The above fact was taken advantage of in the generation of optimal designs of RSSR crank-rocker

Table 3.

(1) Original	d	a_0	b_0	α	β
(2) Reflection of 1	$-d$	$-a_0$	$-b_0$	$180° + \alpha$	$180° + \beta$
(3) Position interchange in 1	d	$-a_0$	$-b_0$	$180° - (\alpha + \theta_0)$	$180° - (\beta + \phi_0)$
(4) Position interchange in 2 OR Reflection of 3.	$-d$	a_0	b_0	$-(\alpha + \theta_0)$	$-(\beta + \phi_0)$

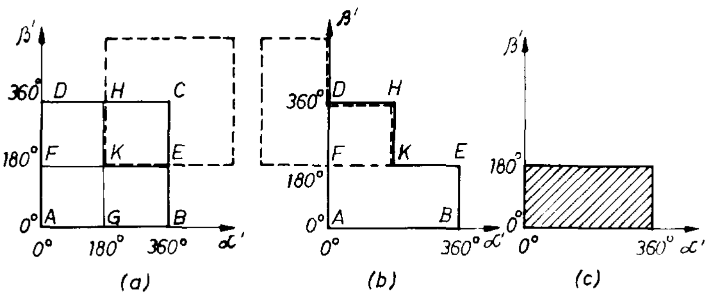

Fig. 9.

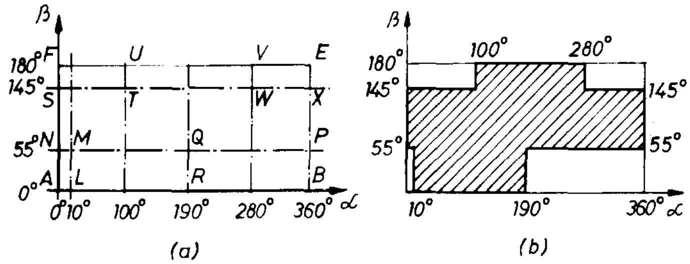

Fig. 10.

mechanisms (in a work of the authors, yet to be published).

8. AN APPLICATION OF TABLE 3

Let us consider, for illustration, the case of $\theta_0 = 160°$ (i.e. a time ratio of $160/200 = 0.8$) and an angle of oscillation $\phi_0 = 70°$. (It may be remembered in passing that $\theta_0 > 180°$ need not be considered, in view of the convention adopted in Section 5). The relevant part of Table 3 takes the following shape:

(1) α	β
(2) $180° + \alpha$	$180° + \beta$
$\alpha - 180°$	$\beta - 180°$
(3) $20° - \alpha$	$110° - \beta$
$380° - \alpha$	$470° - \beta$
(4) $-160° - \alpha$	$-70° - \beta$
$200° - \alpha$	$290° - \beta$
$560° - \alpha$	$650° - \beta$

Additional alternatives have been added under each transformation, as was done in the previous section. How far such alternatives should be added will become clear in the current example.

Transformation 2 has already been considered in the previous section and we are left with the rectangular region shown in Fig. 10(a) after its application.

We proceed to transformation 3. Consider successively the alternatives $(20° - \alpha,\ 110° - \beta)$ and $(380° - \alpha,\ 110° - \beta)$. It will become clear shortly that $470° - \beta$ need not be considered. Transformation $(20° - \alpha)$ has the effect of reflecting the rectangular region ABEF about a line distant $20/2 = +10°$ from the β-axis (Fig. 10a). Transformation $(110°\text{-}\beta)$ has the effect of reflecting the rectangular region ABEF about a line distant $110/2 = +55°$ from the α-axis (Fig. 10a). The region ALMN can thus be eliminated. Transformation $(380° - \alpha)$ reflects the remaining region about a line $+190°$ from the β-axis (Fig. 10a). The region BPQR can thus be eliminated.

A similar application of transformation-4 eliminates the regions FSTU and EVWX. The alternatives $-160° - \alpha$, $-70° - \beta$ and $650° - \beta$, entered in the table, involve reflection about lines outside the rectangular region ABEF and thus create no overlaps.

The reduced region of (α, β) variation finally obtained is indicated in Fig. 10(b).

9. CONCLUSION

The paper has shown how an isometric alternative of a designed mechanism can be obtained, for motion between skew shafts, if the axis distance can be reversed in sign. A catalogue or design chart need contain only one of the two mechanisms, the other being obtained easily from Table 1 when needed.

The alternatives increase to three when appropriate two-position designs are considered, including crank-rocker design. One of these has the same sign of axis distance as the original. A catalogue or design chart need contain only one of the four mechanisms, the others being obtained easily from Table 3 when needed.

Further application of the information to reduce the region of variation of the input and output link position variables as free design parameters has also been shown.

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ISOMETRIE IN DER GETRIEBESYNTHESE

K. Lakshminarayana und L. V. Balaji Rao

Kurzfassung - Isometrische Konstruktionen haben gleiche lineare Dimensionen.

Einheitliche Angabe der Bewegungsbedingungen eines Getriebes zwischen zwei sich kreuzenden Achsen mit Hilfe zweckmäßiger Konventionen läßt alle alternative Konstruktionen, die zum ursprünglichen Getriebe isometrisch sind, entfernen. Es gibt aber eine Ausnahme: ein isometrisches Getriebe, das genau die gleiche Bewegung erfüllt, ist vorhanden, aber mit veränderten algebraischen Zeichen des Achsabstandes. Die Transformation ist eine Reflektion und wird eine uneigentliche oder gegensätzliche Isometrie genannt.

Die einheitliche Angabe der Bewegungsbedingungen läßt sich über den Sonderfall von der Zwei-Lagen-Konstruktion, mit identischen Bewegungsableitungen in den zwei Lagen, erweitern. Damit verbleibt ausschließlich eine alternative isometrische Konstruktion mit unverändertem Zeichen des Achsabstandes. Die Einrichtung besteht aus einer Änderung der Getriebebewegungsrichtung, verbunden mit einem Austausch der zwei Getriebelagen. Hier findet man eine direkte oder eigentliche Isometrie.

Die Transformationen lassen sich verschiedenartig zur Verminderung der Zahl von katalogisierten Lösungen und zur Verminderung des Suchgebietes benutzen.