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Iterative regularization methods for ill-posed operator equations in Hilbert scales

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Abstract. In this paper we report on a method for regularizing a nonlinear ill-posed operator equation in Hilbert scales. The proposed method is a combination of Lavrentiev regularization method and a Modified Newton's method in Hilbert scales . Under the assumptions that the operator F is continuously differentiable with a Lipschitz-continuous first derivative and that the solution of (1.1) fulfils a general source condition, we give an optimal order convergence rate result with respect to the general source function.

1 Introduction

Let X be a real Hilbert space. In this study we are concerned with the problem of approximately solving the operator equation

$$F(x) = y, \tag{1.1}$$

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where $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator (i.e.,$\langle F(u) - F(v), u - v\rangle \geq 0$, $\forall u, v \in D(F)$). We shall use the notations $\langle ., .\rangle$ and $\|.\|$ for the inner product and the corresponding norm in the Hilbert spaces X. The equation (1.1) is, in general, ill-posed, in the sense that a unique solution that depends continuously on the data does not exist.

In the following, we always assume the existence of an x_0-MNS ([7])for exact data y. Recall that ([22, 23, 28]) a solution $\hat{x}$ of (1.1) is said to be an x_0-minimal norm (x_0-MNS) solution of (1.1) if

$$F(\hat{x}) = y$$

and

$$\|x_0 - \hat{x}\| = \min_{x \in D(F)} \{\|x - x_0\| : F(x) = y\}. \tag{1.2}$$

Further we assume throughout that X is a real Hilbert space, $y^\delta \in X$ are the available noisy data with

$$\|y - y^\delta\| \leq \delta \tag{1.3}$$

and $\|F'(x)\|_{X \to X} \leq M$ for all $x \in D(F)$. Since (1.1) is ill-posed, regularization methods are to be employed for obtaining a stable approximate solution for (1.1). See, for example [7]- [16], [19]– [29] for various regularization methods for ill-posed operator equations.

In [29], Vasin and George considered the sequence $\{x^\delta_{n,\alpha}\}$ defined iteratively by

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - R_\beta(x_0)^{-1}[F(x^\delta_{n,\alpha}) - y^\delta + \alpha(x^\delta_{n,\alpha} - x_0)] \tag{1.4}$$

where $x^\delta_{0,\alpha} := x_0$ is an initial guess and $R_\beta(x_0) := F'(x_0) + \beta I$, with $\beta > \alpha$ for obtaining an approximation of $\hat{x}$. Here α is the regularization parameter chosen appropriately depending on the inexact data y^δ and the error level δ satisfying (1.3). For this we used the adaptive parameter selection procedure suggested by Pereverzev and Schock [21]. In order to improve the error estimate available in [29], in this paper we consider the Hilbert scale variant of (1.4).

Let $L : D(L) \subset X \to X$, be a linear, unbounded, self-adjoint, densely defined and strictly positive operator on X. We consider the Hilbert scale $(X_r)_{r\in\Re}$ (see [13]-[20], [25], [27]) generated by L for our analysis. Recall ([17]- [20])that the space X_t is the completion of $D := \cap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle u, v\rangle_t := \langle L^t u, L^t v\rangle, \qquad u, v \in D. \tag{1.5}$$

Moreover, if $\beta \leq \gamma$, then the embedding $X_\gamma \hookrightarrow X_\beta$ is continuous, and therefore the norm $\|.\|_\beta$ is also defined in X_γ and there is a constant $c_{\beta,\gamma}$ such that

$$\|x\|_\beta \leq c_{\beta,\gamma}\|x\|_\gamma,\ x \in X_\gamma.$$

Usually $(X_r)_{r\in\Re}$ are the Sobolev spaces of various kinds (see [16], Example 1).

In this paper we consider the sequence $\{x^\delta_{n,\alpha,s}\}$ in order to obtain stable approximate solution to (1.1), defined iteratively by

$$x^\delta_{n+1,\alpha,s} = x^\delta_{n,\alpha,s} - R_\beta(x_0)^{-1}[F(x^\delta_{n,\alpha,s}) - y^\delta + \alpha L^s(x^\delta_{n,\alpha,s} - x_0)], \tag{1.6}$$

where $x^\delta_{0,\alpha,s} := x_0$ is an initial guess and $R_\beta(x_0) := F'(x_0) + \beta L^s$, with $\beta > \alpha$ for obtaining an approximation for $\hat{x}$. Here also α is the regularization parameter chosen appropriately depending on the inexact data y^δ and the error level δ satisfying (1.3). For this we use the adaptive parameter selection procedure suggested by Pereverzev and Schock [21].

This paper is organized as follows. In section 2 convergence analysis of the proposed iterative method is given. Error bounds under an a priori and under the balancing principle are given in section 3. Finally the paper ends with conclusion in section 4.

2 The Method and Convergence Analysis

In the earlier papers such as [11, 23, 24] etc., the authors used the following Assumption:

Assumption 2.1. (cf.[24], Assumption 3 (A3)) There exists a constant $K \geq 0$ such that for every $x, u \in D(F)$ and $v \in X$ there exists an element $\Phi(x,u,v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x,u,v), \|\Phi(x,u,v)\| \leq K\|v\|\|x-u\|$.

The hypotheses of Assumption 2.1 may not hold or may be very expensive or impossible to verify in general. In particular, as it is the case for well-posed nonlinear equations the computation of the Lipschitz constant K even if this constant exists is very difficult. Moreover, there are classes of operators for which Assumption 2.1 is not satisfied but the iterative method converges (see the numerical examples).

In the present paper, we use the following weaker Assumption.

Assumption 2.2. There exists a constant $k_0 \geq 0$ such that for every $x \in D(F)$ and $v \in X$ there exists an element $\Phi(x,x_0,v) \in X$ such that $[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x,x_0,v), \|\Phi(x,x_0,v)\| \leq k_0\|v\|\|x-x_0\|$.

Note that

$$k_0 \leq K$$

holds in general and $\frac{K}{k_0}$ can be arbitrary large (see Example 4.3). The advantages of the new approach are:

(1) Assumption 2.2 is weaker than Assumption 2.1. Notice that there are classes of operators that satisfy Assumption 2.2 but do not satisfy Assumption 2.1 (see the first two numerical examples);
(2) The computational cost of finding the constant k_0 is less than that of constant K, even when $K = k_0$;
(3) The sufficient convergence criteria are weaker;
(4) The error estimate in this paper is better than that of [29];
(5) The information on the location of the solution is more precise;

and

(6) The convergence domain of the iterative method is larger.

These advantages are also very important in computational mathematics since they provide under less computational cost a wider choice of initial guesses for iterative method and the computation of fewer iterates to achieve a desired error tolerance.

In this section, we consider the iterative method defined in (1.6) for approximating the zero $x_{\alpha,s}^{\delta}$ of the equation,

$$F(x) + \alpha L^s(x - x_0) = y^\delta \tag{2.1}$$

and then we show that $x_{\alpha,s}^{\delta}$ is an approximation to the solution $\hat{x}$ of (1.1).

Usually, for the analysis of regularization methods in Hilbert scales, an assumption of the form (cf.[25], [27])

$$\|F'(\hat{x})x\| \sim \|x\|_{-b}, \quad x \in X \tag{2.2}$$

on the degree of ill-posedness is used. In this paper instead of (2.2) we require only a weaker assumption;

$$d_1\|x\|_{-b} \leq \|F'(x_0)x\| \leq d_2\|x\|_{-b}, \qquad x \in D(F), \tag{2.3}$$

for some reals b, d_1, and d_2.

Note that (2.3) is simpler than that of (2.2). Next, we define f and g by

$$f(t) = min\{d_1^t, d_2^t\}, \quad g(t) = max\{d_1^t, d_2^t\}, \quad t \in \mathbb{R}, \ |t| \leq 1.$$

Let $B_s := L^{-s/2}F'(x_0)L^{-s/2}$. One of the crucial result for proving the results in this paper is the following Proposition.

PROPOSITION 2.3. (See. [14], Proposition 3.1) For $s > 0$ and $|\nu| \leq 1$,

$$f(\nu/2)\|x\|_{\frac{-\nu(s+b)}{2}} \leq \|B_s^{\nu/2}x\| \leq g(\nu/2)\|x\|_{\frac{-\nu(s+b)}{2}}, \qquad x \in X.$$

Let $\psi_2(s) := \frac{g(\frac{-s}{2(s+b)})}{f(\frac{s}{2(s+b)})}$, $\overline{\psi_2(s)} := \frac{g(\frac{s}{2(s+b)})}{f(\frac{s}{2(s+b)})}$.

LEMMA 2.4. Let Proposition 2.3 hold. Then for all $h \in X$, the following hold;

(a)

$$\|(F'(x_0)+\beta L^s)^{-1}F'(x_0)h\| \leq \overline{\psi_2(s)}\|h\|$$

(b)

$$\|(F'(x_0)+\beta L^s)^{-1}L^s h\| \leq \frac{\overline{\psi_2(s)}}{\beta}\|h\|$$

(c)

$$\|(F'(x_0)+\beta L^s)^{-1}h\| \leq \psi_2(s)\beta^{\frac{-b}{(s+b)}}\|h\|$$

Proof. Observe that by Proposition 2.3,

$$\begin{aligned}
\|(F'(x_0)+\beta L^s)^{-1}F'(x_0)h\| &= \|L^{-s/2}(L^{-s/2}F'(x_0)L^{-s/2}+\beta I)^{-1}L^{-s/2} \\
&\quad F'(x_0)L^{-s/2}L^{s/2}h\| \\
&\leq \frac{1}{f(\frac{s}{2(s+b)})}\|B_s^{\frac{s}{2(s+b)}}(B_s+\beta I)^{-1}B_sL^{s/2}h\| \\
&\leq \frac{1}{f(\frac{s}{2(s+b)})}\|(B_s+\beta I)^{-1}B_s\|\|B_s^{\frac{s}{2(s+b)}}L^{s/2}h\| \\
&\leq \frac{g(\frac{s}{2(s+b)})}{f(\frac{s}{2(s+b)})}\|L^{s/2}h\|_{-s/2} \\
&\leq \frac{g(\frac{s}{2(s+b)})}{f(\frac{s}{2(s+b)})}\|h\|.
\end{aligned}$$

This proves (a). To prove (b) and (c) we observe that

$$\delta\| = \|(F'(x_0)+\beta L^s)^{-1}(F(x_0)-y^\delta)\|$$

$$\begin{aligned}
\|(F'(x_0)+\beta L^s)^{-1}L^sh\| &\leq \|L^{-s/2}(L^{-s/2}F'(x_0)L^{-s/2}+\beta I)^{-1}L^{s/2}h\| \\
&\leq \frac{1}{f(\frac{s}{2(s+b)})}\|B_s^{\frac{s}{2(s+b)}}(B_s+\beta I)^{-1}L^{s/2}h\| \\
&\leq \frac{1}{f(\frac{s}{2(s+b)})}\|(B_s+\beta I)^{-1}B_s^{\frac{s}{2(s+b)}}L^{s/2}h\| \\
&\leq \frac{g(\frac{s}{2(s+b)})}{f(\frac{s}{2(s+b)})}\beta^{-1}\|h\| \\
&\leq \overline{\psi_2(s)}\beta^{-1}\|h\| \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned} {}^{\delta}\| &= \|(F'(x_0)+\beta L^s)^{-1}(F(x_0)-y^\delta)\| \\ \|(F'(x_0)+\beta L^s)^{-1}h\| &\leq \|L^{-s/2}(L^{-s/2}F'(x_0)L^{-s/2}+\beta I)^{-1}L^{-s/2}h\| \\ &\leq \frac{1}{f(\frac{s}{2(s+b)})}\|B_s^{\frac{s}{2(s+b)}}(B_s+\alpha I)^{-1}L^{-s/2}h\| \\ &\leq \frac{1}{f(\frac{s}{2(s+b)})}\|(B_s+\beta I)^{-1}B_s^{\frac{s}{(s+b)}}B_s^{\frac{-s}{2(s+b)}}L^{-s/2}h\| \\ &\leq \frac{g(\frac{-s}{2(s+b)})}{f(\frac{s}{2(s+b)})}\beta^{\frac{-b}{(s+b)}}\|h\| \\ &\leq \psi_2(s)\beta^{\frac{-b}{(s+b)}}\|h\|. \end{aligned} \tag{2.5}$$

□

Let

$$G(x) = x - R_\beta(x_0)^{-1}[F(x)-y^\delta+\alpha L^s(x-x_0)]. \tag{2.6}$$

Note that with the above notation $G(x_{n,\alpha,s}^\delta) = x_{n+1,\alpha,s}^\delta$.

First we prove that $x_{n,\alpha,s}^\delta$ converges to the zero $x_{\alpha,s}^\delta$ of

$$F(x)+\alpha L^s(x-x_0) = y^\delta \tag{2.7}$$

and then we prove that $x_{\alpha,s}^\delta$ is an approximation for $\hat{x}$.

Hereafter we assume that $\|\hat{x}-x_0\| < \rho$ where

$$\rho < \frac{1}{\psi_1(s)M}\left(\frac{\beta^{\frac{b}{s+b}}[1-\overline{\psi_2(s)}(\frac{\beta-\alpha}{\beta})]^2}{4k_0\overline{\psi_2(s)}^2} - \psi(s)\frac{\delta_0}{\alpha_0^{\frac{a}{2(s+a)}}}\right)$$

with $\delta_0 < \frac{\beta^{\frac{b}{s+b}}[1-\overline{\psi_2(s)}(\frac{\beta-\alpha}{\beta})]^2}{4k_0\psi(s)\overline{\psi_2(s)}^2}\alpha_0^{\frac{-a}{2(s+a)}}$. Let

$$\gamma_\rho := \psi_2(s)\beta^{\frac{-b}{(s+b)}}[M\rho+\delta_0].$$

and we define

$$q = \overline{\psi_2(s)}[k_0r+\frac{\beta-\alpha}{\beta}],\ \ r\in(r_1,r_2) \tag{2.8}$$

where

$$r_1 = \frac{[1-\overline{\psi_2(s)}(\frac{\beta-\alpha}{\beta})] - \sqrt{[1-\overline{\psi_2(s)}(\frac{\beta-\alpha}{\beta})]^2-4k_0\overline{\psi_2(s)}\gamma_\rho}}{2k_0\overline{\psi_2(s)}}$$

and

$$\begin{aligned} r_2 = \min\{&\frac{1}{k_0\overline{\psi_2(s)}}, \frac{1}{k_0}[\frac{1}{\overline{\psi_2(s)}}-\frac{\beta-\alpha}{\beta}], \\ &\frac{[1-\overline{\psi_2(s)}(\frac{\beta-\alpha}{\beta})] + \sqrt{[1-\overline{\psi_2(s)}(\frac{\beta-\alpha}{\beta})]^2-4k_0\overline{\psi_2(s)}\gamma_\rho}}{2k_0\overline{\psi_2(s)}}\}. \end{aligned}$$

REMARK 2.5. Note that for $r \in (r_1, r_2)$ we have $q < 1$ and $\gamma_\rho < \frac{\gamma_\rho}{1-q} \leq r$.

THEOREM 2.6. Let $r \in (r_1, r_2)$ and Assumption 2.2 be satisfied. Then the sequence $(x^\delta_{n,\alpha,s})$ defined in (1.6) is well defined and $x^\delta_{n,\alpha,s} \in B_r(x_0)$ for all $n \geq 0$. Further $(x^\delta_{n,\alpha,s})$ is a complete sequence in $B_r(x_0)$ and hence converges to $x^\delta_{\alpha,s} \in \overline{B_r(x_0)}$ and $F(x^\delta_{\alpha,s}) + \alpha L^s(x^\delta_{\alpha,s} - x_0) = z^\delta_\alpha$. Moreover, the following estimate holds for all $n \geq 0$,

$$\|x^\delta_{n,\alpha,s} - x^\delta_{\alpha,s}\| \leq \frac{\gamma_\rho q^n}{1-q}. \tag{2.9}$$

Proof Let G be as in (2.6). Then for $u, v \in B_r(x_0)$,

$$\begin{aligned}
G(u) - G(v) &= u - v - R_\beta(x_0)^{-1}[F(u) - y^\delta + \alpha L^s(u - x_0)] \\
&\quad + R_\beta(x_0)^{-1}[F(v) - y^\delta + \alpha L^s(v - x_0)] \\
&= R_\beta(x_0)^{-1}[R_\beta(x_0)(u-v) - (F(u) - F(v))] \\
&\quad + \alpha R_\beta(x_0)^{-1} L^s(v-u) \\
&= R_\beta(x_0)^{-1}[F'(x_0)(u-v) - (F(u) - F(v)) + \beta L^s(u-v)] \\
&\quad + \alpha R_\beta(x_0)^{-1} L^s(v-u) \\
&= R_\beta(x_0)^{-1}[F'(x_0)(u-v) - (F(u) - F(v)) + (\beta - \alpha) L^s(u-v)] \\
&= R_\beta(x_0)^{-1} \int_0^1 [F'(x_0) - F'(v + t(u-v)] dt (u-v) \\
&\quad + R_\beta(x_0)^{-1}(\beta - \alpha) L^s(u-v)].
\end{aligned}$$

Thus by Assumption 2.2 and Lemma 2.4 we have

$$\|G(u) - G(v)\| \leq q\|u - v\|. \tag{2.10}$$

Now we shall prove that $x^\delta_{n,\alpha,s} \in B_r(x_0)$, for all $n \geq 0$. Note that

$$\begin{aligned}
\|x^\delta_{1,\alpha,s} - x_0\| &= \|(F'(x_0) + \beta L^s)^{-1}(F(x_0) - y^\delta)\| \\
&\leq \|L^{-s/2}(L^{-s/2} F'(x_0) L^{-s/2} + \beta I)^{-1} L^{-s/2} \\
&\quad (F(x_0) - y^\delta)\| \\
&\leq \frac{1}{f(\frac{s}{2(s+b)})} \|B_s^{\frac{s}{2(s+b)}} (B_s + \alpha I)^{-1} L^{-s/2} \\
&\quad (F(x_0) - y^\delta)\| \\
&\leq \frac{1}{f(\frac{s}{2(s+b)})} \|(B_s + \beta I)^{-1} B_s^{\frac{s}{(s+b)}} B_s^{\frac{-s}{2(s+b)}} \\
&\quad L^{-s/2}(F(x_0) - y^\delta)\| \\
&\leq \frac{g(\frac{-s}{2(s+b)})}{f(\frac{s}{2(s+b)})} \beta^{\frac{-b}{(s+b)}} \|F(x_0) - y^\delta\| \\
&\leq \psi_2(s) \beta^{\frac{-b}{(s+b)}} [\|F(x_0) - F(\hat{x})\| \\
&\quad + \|y - y^\delta\|] &(2.11)\\
&\leq \psi_2(s) \beta^{\frac{-b}{(s+b)}} [M\rho + \delta_0] = \gamma_\rho. &(2.12)
\end{aligned}$$

Assume that $x^{\delta}_{k,\alpha,s} \in B_r(x_0)$, for some k. Then

$$\begin{aligned}
\|x^{\delta}_{k+1,\alpha,s} - x_0\| &= \|x^{\delta}_{k+1,\alpha,s} - x^{\delta}_{k,\alpha,s} + x^{\delta}_{k,\alpha,s} - x^{\delta}_{k-1,\alpha,s} \\
&\quad + \cdots + x^{\delta}_{1,\alpha,s} - x_0\| \\
&\leq \|x^{\delta}_{k+1,\alpha,s} - x^{\delta}_{k,\alpha,s}\| + \|x^{\delta}_{k,\alpha,s} - x^{\delta}_{k-1,\alpha,s}\| \\
&\quad + \cdots + \|x^{\delta}_{1,\alpha,s} - x_0\| \\
&\leq (q^k + q^{k-1} + \cdots + 1)\gamma_\rho \\
&\leq \frac{\gamma_\rho}{1-q} \leq r.
\end{aligned}$$

So $x^{\delta}_{k+1,\alpha,s} \in B_r(x_0)$ and hence, by induction $x^{\delta}_{n,\alpha,s} \in B_r(x_0)$, $\forall n \geq 0$. Next we shall prove that $(x^{\delta}_{k+1,\alpha,s})$ is a Cauchy sequence in $B_r(x_0)$.

$$\begin{aligned}
\|x^{\delta}_{n+m,\alpha,s} - x^{\delta}_{n,\alpha,s}\| &\leq \sum_{i=0}^{m} \|x^{\delta}_{n+i+1,\alpha,s} - x^{\delta}_{n+i,\alpha,s}\| \qquad (2.13)\\
&\leq \sum_{i=0}^{m} q^{n+i}\gamma_\rho \\
&\leq \frac{q^n}{1-q}\gamma_\rho. \qquad (2.14)
\end{aligned}$$

Thus $(x^{\delta}_{n,\alpha,s})$ is a complete sequence in $B_r(x_0)$ and hence converges to some $x^{\delta}_{\alpha,s} \in \overline{B_r(x_0)}$. Now by $n \to \infty$ in (1.6) we obtain $F(x^{\delta}_{\alpha,s}) + \alpha L^s(x^{\delta}_{\alpha,s} - x_0) = y^{\delta}$.

□

3 Error Bounds Under Source Conditions

We use the following assumption to obtain the error estimate for $\|\hat{x} - x^{\delta}_{\alpha,s}\|$.

Assumption 3.1. There exists a continuous, strictly monotonically increasing function $\varphi : (0, \|B_s\|] \to (0, \infty)$ such that the following conditions hold:

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0$,
- $$\sup_{\lambda > 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha) \qquad \forall \lambda \in (0, \|B_s\|]$$

  and
- there exists $w \in X$ with $\|w\| \leq E_2$, such that

  $$B_s^{\frac{s}{2(s+b)}} L^{s/2}(x_0 - \hat{x}) = \varphi(B_s)w$$

  operator $G(x, x_0)$ $\|G(x, x_0)\| \leq k_1$.

REMARK 3.2. If $x_0 - \hat{x} \in X_t$ i.e., $\|x_0 - \hat{x}\|_t \leq E$ for some positive constant E and $0 \leq t \leq s + b$ then the above assumption is satisfied. This can be seen as follows.

$$\begin{aligned} B_s^{\frac{s}{2(s+b)}} L^{s/2}(\hat{x}-x_0) &= B_s^{\frac{t}{s+b}} B_s^{\frac{(s-2t)}{(2s+2b)}} L^{s/2}(\hat{x}-x_0) \\ &= \varphi(B_s)w \end{aligned}$$

where $\varphi(\lambda)=\lambda^{t/(s+b)}$, $w=B_s^{\frac{s-2t}{2(s+b)}} L^{s/2}(\hat{x}-x_0)$ and $\|w\|\le g(\frac{s-2t}{2(s+b)})E := E_2$.

THEOREM 3.3. Suppose $x_{\alpha,s}^{\delta}$ is the solution of (2.7) and Assumptions 2.2 and 3.1 hold. Then

$$\|\hat{x}-x_{\alpha,s}^{\delta}\| \le \frac{1}{1-\overline{\psi_2(s)}k_0 r}\Big(\frac{1}{f(\frac{s}{2(s+b)})}\varphi(\alpha)+\psi_2(s)\alpha^{\frac{-b}{s+b}}\delta\Big).$$

Proof. Note that $(F(x_{\alpha,s}^{\delta})-y^{\delta})+\alpha L^s(x_{\alpha,s}^{\delta}-x_0)=0$, so

$$\begin{aligned} (F'(x_0)+\alpha L^s)(x_{\alpha,s}^{\delta}-\hat{x}) &= (F'(x_0)+\alpha L^s)(x_{\alpha,s}^{\delta}-\hat{x}) \\ &\quad -(F(x_{\alpha,s}^{\delta})-y^{\delta})-\alpha L^s(x_{\alpha,s}^{\delta}-x_0) \\ &= \alpha L^s(x_0-\hat{x}) \\ &\quad +F'(x_0)(x_{\alpha,s}^{\delta}-\hat{x})-[F(x_{\alpha,s}^{\delta})-y^{\delta}] \\ x_{c\alpha})-y]] & \\ &= \alpha L^s(x_0-\hat{x}) \\ &\quad +F'(x_0)(x_{\alpha,s}^{\delta}-\hat{x})-[F(x_{\alpha,s}^{\delta})-F(\hat{x})+F(\hat{x})-y^{\delta}] \\ &= \alpha L^s(x_0-\hat{x})-(F(\hat{x})-y^{\delta}) \\ &\quad +F'(x_0)(x_{\alpha,s}^{\delta}-\hat{x})-[F(x_{\alpha,s}^{\delta})-F(\hat{x})]. \end{aligned}$$

Thus

$$\begin{aligned} \|x_{\alpha,s}^{\delta}-\hat{x}\| &\le \|\alpha(F'(x_0+\alpha L^s)^{-1}L^s(x_0-\hat{x})\|+\|(F'(x_0)+\alpha L^s)^{-1} \\ &\quad (F(\hat{x})-y^{\delta})\|+\|(F'(x_0)+\alpha L^s)^{-1}[F'(x_0)(x_{\alpha,s}^{\delta}-\hat{x}) \\ &\quad -(F(x_{\alpha,s}^{\delta})-F(\hat{x}))]\| \\ &\le \|\alpha(F'(x_0)+\alpha L^s)^{-1}L^s(x_0-\hat{x})\| \\ &\quad +\psi_2(s)\alpha^{\frac{-b}{s+b}}\delta+\Gamma \end{aligned} \tag{3.1}$$

where $\Gamma := \|(F'(x_0)+\alpha L^s)^{-1}\int_0^1[F'(x_0)-F'(\hat{x}+t(x_{\alpha,s}^{\delta}-\hat{x})](x_{\alpha,s}^{\delta}-\hat{x})dt\|$. Note that by Assumption 3.1, we obtain

$$\begin{aligned} & \|\alpha(F'(x_0)+\alpha L^s)^{-1}L^s(x_0-\hat{x})\| \\ &= \|\alpha L^{-s/2}(B_s+\alpha)^{-1}L^{s/2}(x_0-\hat{x})\| \\ &\le \frac{1}{f(\frac{s}{2(s+b)})}\|\alpha(B_s+\alpha)^{-1}B_s^{\frac{s}{2(s+b)}}L^{s/2}(x_0-\hat{x})\| \\ &\le \frac{1}{f(\frac{s}{2(s+b)})}\sup_{\lambda\in\sigma(F'(x_0))}\frac{\alpha\varphi_1(\lambda)}{\lambda+\alpha} \\ &\le \frac{1}{f(\frac{s}{2(s+b)})}\varphi(\alpha) \end{aligned} \tag{3.2}$$

and by Assumption 2.2, and Lemma 2.4 we obtain

$$\Gamma \leq \|(F'(x_0)+\alpha L^s)^{-1}\int_0^1 [F'(x_0)-F'(\hat{x}+t(x^\delta_{\alpha,s}-\hat{x})]$$
$$(x^\delta_{\alpha,s}-\hat{x})dt\|$$

$$x_0)(x^\delta_{c,\alpha}-\hat{x})dt\| \tag{3.3}$$

$$\leq \overline{\psi_2(s)}k_0 r\|x^\delta_{\alpha,s}-\hat{x}\| \tag{3.4}$$

and hence by (3.2), (3.3) and (3.1) we have

$$\|x^\delta_{\alpha,s}-\hat{x}\| \leq \frac{1}{1-\overline{\psi_2(s)}k_0 r}\left(\frac{1}{f(\frac{s}{2(s+b)})}\varphi(\alpha)+\psi_2(s)\alpha^{\frac{-b}{s+b}}\delta\right).$$

□

The following Theorem is a consequence of Theorem 2.6 and Theorem 3.3.

THEOREM 3.4. Let $x^\delta_{n,\alpha,s}$ be as in (1.6) with $\alpha=\alpha$ and $\delta\in(0,\delta_0]$, assumptions in Theorem 2.6 and Theorem 3.3 hold. Then

$$\|\hat{x}-x^\delta_{n,\alpha,s}\| \leq \frac{\gamma_\rho}{1-q}q^n+\frac{1}{1-\overline{\psi_2(s)}k_0 r}\left(\frac{1}{f(\frac{s}{2(s+b)})}\varphi(\alpha)+\psi_2(s)\alpha^{\frac{-b}{s+b}}\delta\right).$$

THEOREM 3.5. Let $x^\delta_{n,\alpha,s}$ be as in (1.6) with $\alpha=\alpha$ and $\delta\in(0,\delta_0]$, and assumptions in Theorem 3.4 hold. Let

$$n_k := \min\{n : q^n \leq \alpha^{\frac{-b}{s+b}}\delta\}.$$

Then

$$\|\hat{x}-x^\delta_{n_k,\alpha,s}\| \leq \frac{\max\{\frac{1}{f(\frac{s}{2(s+b)})},\psi_2(s)+\frac{\gamma_\rho}{1-q}\}}{1-\overline{\psi_2(s)}k_0 r}(\varphi(\alpha)+\alpha^{\frac{-b}{s+b}}\delta).$$

The error estimate $\varphi(\alpha)+\alpha^{\frac{-b}{s+b}}\delta$ in Theorem 3.5 attains minimum for the choice $\alpha:=\alpha(\delta,s,b)$ which satisfies $\varphi(\alpha)=\alpha^{\frac{-b}{s+b}}\delta$. Clearly $\alpha(\delta,s,b)=\varphi^{-1}(\psi^{-1}_{s,b}(\delta))$, where

$$\psi_{s,b}(\lambda)=\lambda[\varphi^{-1}(\lambda)]^{\frac{b}{s+b}}, \quad 0<\lambda\leq\|B_s\| \tag{3.5}$$

and in this case

$$\|\hat{x}-x^\delta_{\alpha,s}\| \leq C_s\psi^{-1}_{s,b}(\delta), \tag{3.6}$$

where $C_s = \frac{\max\{\frac{1}{f(\frac{s}{2(s+b)})},\psi_2(s)+\frac{\gamma_\rho}{1-q}\}}{1-\overline{\psi_2(s)}k_0 r}$. The above error estimate has at least optimal order with respect to δ, s and b (cf. [7]).

3.1 Adaptive Scheme and Stopping Rule

In this paper we consider the adaptive scheme suggested by Pereverzev and Schock in [21] modified suitably, for choosing the parameter α which does not involve even the regularization method in an explicit manner.

$$n_i := \min\{n : q^n \leq \alpha_i^{\frac{-b}{s+b}}\delta\}.$$

Let $i\in\{0,1,2,\cdots,N\}$ and $\alpha_i=\mu^i\alpha_0$ where $\mu=\eta^{(1+s/b)}, \eta>1$ and $\alpha_0=\delta^{(1+s/b)}$. Let

$$l := max\{i : \varphi(\alpha_i) \le \alpha_i^{\frac{-b}{s+b}}\delta\} < N \tag{3.7}$$

and

$$k := max\{i : \|x^\delta_{n_i,\alpha_i,s} - x^\delta_{n_j,\alpha_j,s}\| \le 4\alpha_j^{\frac{-b}{s+b}}\delta, j = 0,1,2,\cdots,i\}. \tag{3.8}$$

The proof of the following Theorem is Analogous to the proof of Theorem 3.10 in [15], so we omit the proof.

THEOREM 3.6. ([15, Theorem 3.10]) Let l be as in (3.7), k be as in (3.8), $\psi_{s,a}$ be as in (3.5) and $x^\delta_{n_k,\alpha_k,s}$ be as in (1.6) with $\alpha = \alpha_k$, $n = n_k$. Then $l \le k$; and

$$\|\hat{x} - x^\delta_{n_k,\alpha_k,s}\| \le C_s(2 + \frac{4\eta}{\eta - 1})\eta\psi^{-1}_{s,a}(\delta)$$

where C_s is as in (3.6).

4 Numerical Examples

We present numerical examples to illustrate the mathematical results.

EXAMPLE 4.1. Let $X = Y = \mathbb{R}$, $D = [0,\infty), x_0 = 1$ and define function F on D by

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}} + c_1x + c_2, \tag{4.1}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. Hence, Assumption 2.1 is not satisfied. However central Lipschitz condition Assumption 2.2 holds for $k_0 = 1$. Indeed, we have

$$\begin{aligned}\|F'(x) - F'(x_0)\| &= |x^{1/i} - x_0^{1/i}| \\ &= \frac{|x - x_0|}{x_0^{\frac{i-1}{i}} + \cdots + x^{\frac{i-1}{i}}}\end{aligned}$$

so

$$\|F'(x) - F'(x_0)\| \le k_0|x - x_0|.$$

EXAMPLE 4.2. We consider the integral equations

$$u(s) = f(s) + \lambda\int_a^b G(s,t)u(t)^{1+1/n}dt, \quad n \in \mathbb{N}. \tag{4.2}$$

Here, f is a given continuous function satifying $f(s) > 0, s \in [a,b], \lambda$ is a real number, and the kernel G is continuous and positive in $[a,b] \times [a,b]$. For example, when $G(s,t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$\begin{aligned} u'' &= \lambda u^{1+1/n} \\ u(a) &= f(a), u(b) = f(b). \end{aligned}$$

These type of problems have been considered in [1]- [5]. Equation of the form (4.2) generalize equations of the form

$$u(s) = \int_a^b G(s,t)u(t)^n dt \tag{4.3}$$

studied in [1]-[5]. Instead of (4.2) we can try to solve the equation $F(u) = 0$ where

$$F : \Omega \subseteq C[a,b] \to C[a,b], \Omega = \{u \in C[a,b] : u(s) \geq 0, s \in [a,b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b G(s,t)u(t)^{1+1/n} dt.$$

The norm we consider is the max-norm. The derivative F' is given by

$$F'(u)v(s) = v(s) - \lambda(1+\frac{1}{n}) \int_a^b G(s,t)u(t)^{1/n} v(t)dt, \ \ v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b] = [0,1], G(s,t) = 1$ and $y(t) = 0$. Then $F'(y)v(s) = v(s)$ and

$$\|F'(x) - F'(y)\| = |\lambda|(1+\frac{1}{n}) \int_a^b x(t)^{1/n} dt.$$

If F' were a Lipschitz function, then

$$\|F'(x) - F'(y)\| \leq L_1 \|x - y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n} dt \leq L_2 \max_{x \in [0,1]} x(s), \tag{4.4}$$

would hold for all $x \in \Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t) = \frac{t}{j}, \ \ j \geq 1, \ \ t \in [0,1].$$

If these are substituted into (4.4)

$$\frac{1}{j^{1/n}(1+1/n)} \leq \frac{L_2}{j} \Leftrightarrow j^{1-1/n} \leq L_2(1+1/n), \ \ \forall j \geq 1.$$

This inequality is not true when $j \to \infty$. Therefore, condition (4.4) is not satisfied in this case. Hence Assumption 2.1 is not satisfied. However, condition Assumption 2.2 holds. To show this, let $x_0(t) = f(t)$ and $\gamma = \min_{s \in [a,b]} f(s), \alpha > 0$ Then for $v \in \Omega$,

$$\begin{aligned}\|[F'(x) - F'(x_0)]v\| &= |\lambda|(1+\frac{1}{n}) \max_{s \in [a,b]} |\int_a^b G(s,t)(x(t)^{1/n} - f(t)^{1/n})v(t)dt| \\ &\leq |\lambda|(1+\frac{1}{n}) \max_{s \in [a,b]} G_n(s,t)\end{aligned}$$

where $G_n(s,t) = \frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n} + x(t)^{(n-2)/n} f(t)^{1/n} + \cdots + f(t)^{(n-1)/n}} \|v\|$. Hence,

$$\begin{aligned}\|[F'(x) - F'(x_0)]v\| &= \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} \max_{s \in [a,b]} \int_a^b G(s,t)dt \|x - x_0\| \\ &\leq k_0 \|x - x_0\|,\end{aligned}$$

where $k_0 = \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} N$ and $N = \max_{s \in [a,b]} \int_a^b G(s,t)dt$. Then Assumption 2.2 holds for sufficiently small λ.

In the last example, we show that $\frac{K}{k_0}$ can be arbitrarily large in certain nonlinear equation.

EXAMPLE 4.3. Let $X = D(F) = \mathbb{R}, x_0 = 0$, and define function F on $D(F)$ by

$$F(x) = d_0 x + d_1 + d_2 \sin e^{d_3 x}, \tag{4.5}$$

where $d_i, i = 0,1,2,3$ are given parameters. Then, it can easily be seen that for d_3 sufficiently large and d_2 sufficiently small, $\frac{K}{k_0}$ can be arbitrarily large.

EXAMPLE 4.4. (see section 4.3 in [24]) Let $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$F(u) := \int_0^1 k(t,s) u^3(s) ds, \tag{4.6}$$

where

$$k(t,s) = \begin{cases} (1-t)s, 0 \le s \le t \le 1 \\ (1-s)t, 0 \le t \le s \le 1 \end{cases}.$$

Then for all $x(t), y(t) : x(t) > y(t)$: (see section 4.3 in [24])

$$\langle F(x) - F(y), x - y \rangle = \int_0^1 \left[\int_0^1 k(t,s)(x^3 - y^3)(s) ds \right] (x-y)(t) dt \ge 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3 \int_0^1 k(t,s)(u(s))^2 w(s) ds. \tag{4.7}$$

Note that for $u, v > 0$,

$$\begin{aligned} (F'(v) - F'(u))w &= 3 \int_0^1 k(t,s) \frac{(v(s))^2 - (u(s))^2}{u(s)^2} w(s) ds \\ &:= F'(u)\Phi(v,u,w). \end{aligned}$$

where $\Phi(v,u,w) = \frac{(v(s))^2 - (u(s))^2}{u(s)^2}$.

Observe that

$$\Phi(v,u,w) = \frac{(u(s)+v(s))(v(s)-u(s))}{(u(s))^2}.$$

So Assumption 2.2 satisfies with $k_0 \ge \| \frac{u(s)+v(s)}{u(s)^2} \|$.

In our computation, we take $y(t) = \frac{t - t^{11}}{110}$ and $y^\delta = y + \delta$. Then the exact solution

$$\hat{x}(t) = t^3.$$

We take $L : D \subset L^2(0,1) \to L^2(0,1)$ as

$$Lx = \sum_{k=1}^{\infty} k \langle x, e_k \rangle e_k \text{ with } e_k(t) = \sqrt{2} \sin(k\pi t)$$

and

$$x_0(t) = t^3 + t$$

Table 1 Error tabulated in each iteration.

n	k	n_k	δ	α_k	$\|x_k-\hat{x}\|$	$\frac{\|x_k-\hat{x}\|}{\delta^{1/2}}$
8	2	3	0.1016	0.3428	1.0127	1.3479
16	2	3	0.1004	0.3388	1.0970	1.4621
32	2	3	0.1001	0.3378	1.1387	1.5183
64	2	3	0.1000	0.3376	1.1596	1.5463
128	2	3	0.1000	0.3375	1.1699	1.5601
256	2	3	0.1000	0.3375	1.1750	1.5669
512	2	3	0.1000	0.3375	1.1776	1.5704
1024	2	3	0.1000	0.3375	1.1789	1.5721

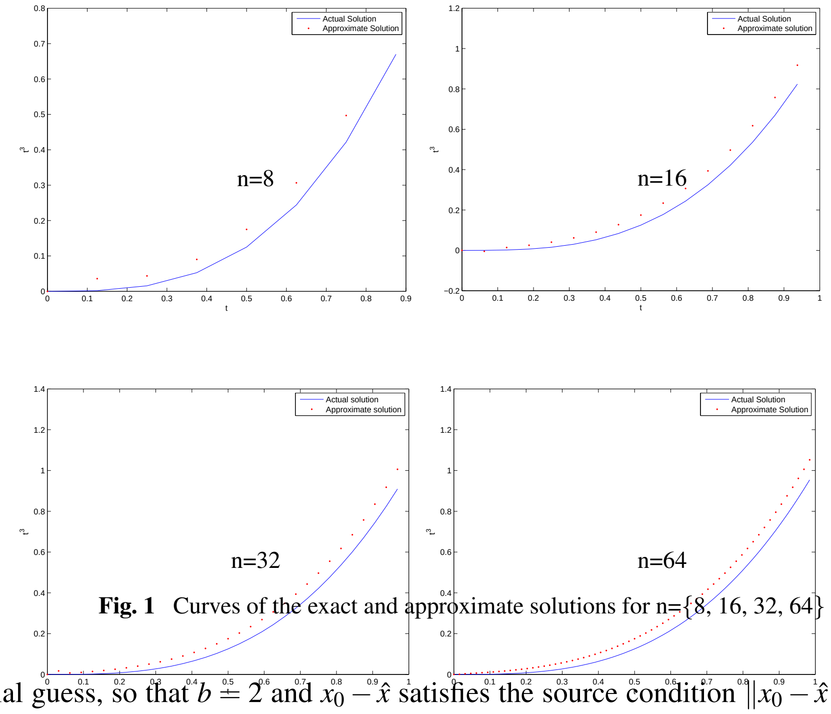

Fig. 1 Curves of the exact and approximate solutions for n={8, 16, 32, 64}

as our initial guess, so that $b = 2$ and $x_0 - \hat{x}$ satisfies the source condition $\|x_0 - \hat{x}\|_t \leq E$ $t \in [0, 1/2)$ (see [18, Proposition 5.3]). Thus we expect to obtain the rate of convergence $O(\delta^{\frac{1}{8}})$. dimensional subspace $F'(x_0)$

We choose $\alpha_0 = (1.5)\delta$, $\mu = 1.5$ and $q = 0.51$. The results of the computation are presented in Table 1. The plots of the exact solution and the approximate solution obtained are given in Figures 1 and 2.

The last column of the Table 1 shows that the error $\|x_k - \hat{x}\|$ is of $O(\delta^{\frac{1}{8}})$.

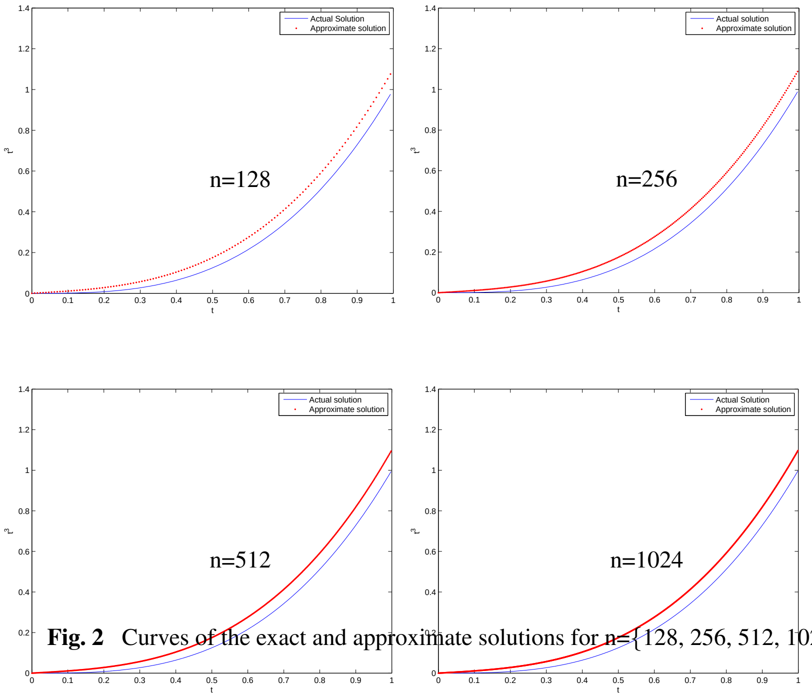

Fig. 2 Curves of the exact and approximate solutions for n={128, 256, 512, 1024}

5 Conclusion

In this paper we present an iterative regularization method for obtaining an approximate solution of an ill-posed operator equation $F(x) = y$ in the Hilbert scale setting where F is a nonlinear monotone operator. It is assumed that the available data is y^δ in place of exact data y. We considered the Hilbert space $(X_t)_{t\in\mathbb{R}}$ generated by L for the analysis where $L : D(L) \to X$ is a linear, unbounded, self-adjoint, densely defined and strictly positive operator on X. For choosing the regularization parameter α we used the adaptive scheme of Pereverzev and Schock (2005).

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