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# Kantorovich-Like Convergence Theorems for Newton's Method Using Restricted Convergence Domains

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**ABSTRACT**

The convergence set for Newton's method is small in general using Lipschitz-type conditions. A center-Lipschitz-type condition is used to determine a subset of the convergence set containing the Newton iterates. The rest of the Lipschitz parameters and functions are then defined based on this subset instead of the usual convergence set. This way the resulting parameters and functions are more accurate than in earlier works leading to weaker sufficient semi-local convergence criteria. The novelty of the paper lies in the observation that the new Lipschitz-type functions are special cases of the ones given in earlier works. Therefore, no additional computational effort is required to obtain the new results. The results are applied to solve Hammerstein nonlinear integral equations of Chandrasekhar type in cases not covered by earlier works.

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## 1. Introduction

We are seeking solutions $x^*$ of equation

$$F(x) = 0, \tag{1.1}$$

where $: \Omega \subseteq B_1 \to B_2$. is a differentiable operator in the sense of Fréchet, $B_1$, $B_2$ are Banach spaces and $\Omega$ is a convex set.

A plethora of problems in optimization, control theory, signal and image processing, inverse and ill-posed, least squares, mathematical physics, mathematical chemistry, mathematical biology, mathematical economics and also problems in engineering can be written like Equation (1.1) using mathematical modeling [1–9, 13, 15, 16]. The solution methods for solving (1.1) are usually iterative since closed form solutions although desirable can be derived only in some cases. The celebrated Newton's method

$$x_{n+1} = x_n - F(x_n)^{-1}F(x_n) \tag{1.2}$$

has been utilized by numerous authors to generate a sequence $\{x_n\}$ approximating $x^*$ [1–27]. Here $F'(x) \in \ell(B_1, B_2)$ stands for the space of all

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bounded linear operators from $B_1$ to $B_2$ and $F'(x)$ denotes the derivative of operator $F(x)$ in the sense of Fréche [4, 7, 16, 21].

Using Kantorovich-like conditions, Rheinboldt [23] presented a convergence theorem for Newton's method. Later an improved result was given by Dennis in [12], Deuflhard and Heindl in [14], Potra in [20, 21].

Miel [17, 18] improved the error estimates given by Rheinboldt [23]. Under stronger conditions than those of Rheinboldt, Moret [19] not only obtained a convergence theorem and error estimates for Newton's method but also, using a numerical example showed that his estimates are sharper than those of Miel. But, no proof was given in [19]. Yamamoto in [26] presented a method for finding error estimates under Dennis conditions and showed that the estimates obtained extend those of Rheinboldt, Dennis and Miel and reduce to Moret's estimates if we replace the conditions by his strong conditions. It was also shown that Moret's results can be derived from Rheinboldt's. Related works can be found in Zabrejko and Nguen [27]; Amat et al. [1], Hernandez et al. [15] Proinov [22], Nashed [27], Chen [10], Chen and Yamamoto [11] where the method of recurrent relations was used. Argyros et al. [2–7] using the technique of recurrent functions presented a unified convergence theory with the following improvements denoted by ($\mathcal{A}$): weaker sufficient semi-local convergence conditions, more precise error estimates on the distances $||x_{n+1}-x_n||$, $||x_n-x^*||$, $(n \geq 0)$, and an at least as tight information on the location of the solution. The above improvements are obtained while using the needed center-Lipschitz instead of the Lipschitz condition commonly used for the derivation of the upper estimates on the norms $||F(x_n)^{-1}F'(x_0)||$ $(n \geq 0)$. This modification leads to more precise majorizing sequences, which in turn result weaker sufficient convergence conditions in most interesting cases (see also Section 4).

The Kantorovich theory is a powerful tool for studying equations. Many papers have been written in this area providing local and semi-local convergence results for Newton-type methods. The convergence criteria are sufficient but not necessary and the convergence domain is small in general. Therefore, it is important to extend the convergence domain without additional conditions and also provide new insight into iterative methods. In particular, in the present paper, we extend the applicability of Newton's method even further than in the preceding works using more precise domains containing the iterates $x_n$ leading to smaller Lipschitz conditions which finally lead to a finer convergence analysis for this method. The novelty of the paper lies in the fact that the improved results are obtained under the same computational effort as in the preceding studies. Indeed, in practice we utilize Lipschitz functions which are tighter and special cases of the functions appeared in the works mentioned previously (see also the

Remarks and the Examples). This approach can be utilized to extend the applicability of other iterative methods using inverses

The rest of the paper is structured as follows: Sections 2 and 3 contain the semi-local convergence of Newton's method. Special cases and applications are given in Section 4 to show that our results can apply to solve equations, where earlier ones cannot.

## 2. Semi-local convergence analysis

Let $U(v,\rho), \bar{U}(v,\rho)$ stand, respectively for the open and closed balls in $B_1$ with center $v \in B_1$ and of radius $\rho > 0$. Let also $x_0 \in D, R > 0$ and $K : [0,R] \to \mathbb{R}_{\geq 0}, L : [0,R] \to \mathbb{R}_{\geq 0}, L_1 : [0,R] \to \mathbb{R}_{\geq 0}$ be continuous non-negative and non-decreasing functions. Suppose that $F'(x_0)^{-1} \in \ell(B_2, B_1)$. Moreover, suppose that there exists $\eta \geq 0$ such that

$$||F'(x_0)^{-1}F(x_0)|| \leq \eta. \tag{2.1}$$

Furthermore, we define functions $\varphi, \psi$ and $\psi_1$ on the interval $[0,R]$ by

$$\varphi(t) = \eta - t + \int_0^t K(s)(t-s)ds$$

$$\psi(t) = \eta - t + \int_0^t L(s)(t-s)ds$$

and

$$\psi_1(t) = \eta - t + \int_0^t L_1(s)(t-s)ds.$$

Define

$$R_0 := \sup\{t \in [0,R) : \varphi'(t) < 1\}. \tag{2.2}$$

Suppose that: the center-Lipschitzian condition holds for each $r \in [0,R], u \in U := \bar{U}(x_0,r) \cap D$

$$||F'(x_0)^{-1}(F'(u) - F'(x_0))|| \leq K(r)||u - x_0||; \tag{2.3}$$

the Lipschitzian condition holds for each $r \in [0,R], u,v \in U_0 = \bar{U}(_0,r) \cap U(x_0,R_0) \cap D$

$$||F'(x_0)^{-1}(F'(u) - F'(v))|| \leq L(r)||u - v|| \tag{2.4}$$

and the Lipschitzian condition holds for each $r \in [0,R], u,v \in U$.

$$||F'(x_0)^{-1}(F'(u) - F'(v))|| \leq L_1(r)||u - v||. \tag{2.5}$$

We have by (2.3) and (2.5) that

$$K(r) \leq L_1(r) \tag{2.6}$$

and by (2.4) and (2.5)

$$L(r) \leq L_1(r), \tag{2.7}$$

since $U_0 \subseteq U$. Moreover the ratio $\frac{L_1}{K}$ can be arbitrarily large [2–7].

The semi-local convergence results for Newton's method in the literature (with the exception of our works that are based on (2.3) and (2.5)) have used only (2.5) [8–27]. Notice that (2.5) implies (2.3) or (2.4) but not necessarily vice versa. In the present study we use (2.3) and (2.4). This way we obtain smaller than $L_1$ functions $K$ and $L$ leading to the advantages $(\mathcal{A})$ of the new semi-local convergence analysis for Newton's method.

Define scalar sequences $\{r_n\}, \{s_n\}$ and $\{t_n\}$ by

$$\begin{aligned}
r_0 &= 0,\ r_1 = \eta,\\
r_2 &= r_1 - \frac{\varphi(r_1)-\varphi(r_0)-\varphi'(r_0)(r_1-r_0)}{\varphi'(r_1)},\\
r_{n+1} &= r_n - \frac{\psi(r_n)-\psi(r_{n-1})-\psi'(r_{n-1})(r_n-r_{n-1})}{\varphi'(r_n)}\\
s_0 &= 0,\ s_{n+1} = s_n - \frac{\psi(s_n)}{\psi'(s_n)}\\
t_0 &= 0,\ t_{n+1} = t_n - \frac{\psi_1(t_n)}{\psi_1'(t_n)}.
\end{aligned} \tag{2.8}$$

Suppose that sequences $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are convergent under some conditions and

$$-\frac{\psi(\beta)-\psi(\alpha)-\psi'(\alpha)(\beta-\alpha)}{\varphi'(\beta)} \leq -\frac{\psi(\gamma)}{\psi'(\gamma)} \tag{2.9}$$

for each $\alpha, \beta, \gamma \in [0, R_0]$ with $\alpha \leq \beta \leq \gamma$ and

$$-\frac{\psi(\gamma)}{\psi'(\gamma)} \leq -\frac{\psi_1(\delta)}{\psi_1'(\delta)} \tag{2.10}$$

for each $\gamma, \delta \in [0, R_0]$ with $\gamma \leq \delta$. Then, a simple inductive argument shows that:

$$r_n \leq s_n, \tag{2.11}$$

$$r_{n+1}-r_n \leq s_{n+1}-s_n, \text{ and } r^* := \lim_{n\to\infty} r_n \leq s^* := \lim_{n\to\infty} s_n \tag{2.12}$$

if (2.9) holds and

$$s_n \leq t_n, \tag{2.13}$$

$$s_{n+1}-s_n \leq t_{n+1}-t_n, \text{ and } s^* := \lim_{n\to\infty} s_n \leq t^* := \lim_{n\to\infty} t_n, \tag{2.14}$$

if (2.10) holds. Inequalities (2.11), (2.13) are strict for $n = 2, 3, \dots$ and first inequalities in (2.12) and (2.14) for $n = 1, 2, \dots$. These results indicate that, if sequence $\{r_n\}$ or sequence $\{s_n\}$ is found to be majorizing for Newton's method, then the earlier results using $\{t_n\}$ as majorizing sequence are improved under the same computational cost, since in practice the computation of function $L_1$ requires the computation of $K$ or $L$ as special cases.

Let $F : D \to B_2$ be $m-$ times differentiable in the sense of Fréchet. Then, for any $u, v \in D$ and any integer $\lambda = 0, 1, 2, ..., m$, we can write

$$T_\lambda(F; u, v) = F(v) - \sum_{k=0}^{\lambda} D^k F(u) \frac{(v-u)^k}{k!}$$

for the remainder of the Taylor expansion of $F$ to order $\lambda$, centered at $x$ and evaluated at $v$. It is well known that, if $\lambda + 1 \le m$ and, if the segment $[u, v] \subseteq D$, then

$$T_\lambda(F; u, v) = \int_{[u,v]} F^{(\lambda+1)}(w) \frac{(v-w)^\lambda}{\lambda!} dw.$$

Next, we state the main semi-local convergence result for Newton's method using the preceding notation.

**Theorem 2.1.** *Let $F : D \subseteq B_1 \to B_2$ be a differentiable operator in the sense of Fréchet, and let $x_0 \in D$ be such that $F'(x_0)^{-1} \in \ell(B_2, B_1)$. Suppose: function $\psi$ defined by (2.1) has a unique zero $s_-$ in the interval $[0, R_0)$, $U(x_0, R_0) \subseteq D$, $\psi(R_0) \le 0$, $K(t) \le L(t), t \in [0, R_0]$, where $R_0$ is defined in (2.2); conditions (2.2) and (2.3) hold. Then, the following assertions hold.*

(a) *The scalar sequence $\{s_n\}$ generated by (2.8) is well defined, remains in $[0, s_-]$ and nondecreasingly convergent to $s_-$.*

(b) *The sequence $\{x_n\}$ generated by Newton's method is well defined, remains in $\bar{U}(x_0, s_-)$ and converges to a unique solution $x^*$ of equation $F(x) = 0$ in $U(x_0, R_0)$. Moreover, the following error bounds hold*

$$||x_{n+1} - x_n|| \le s_{n+1} - s_n \tag{2.15}$$

*and*

$$||x_n - x^*|| \le s_- - s_n. \tag{2.16}$$

*Proof.*

(a) *Notice that $\psi'' \ge 0$. Then, the proof is standard [27].*

(b) *We shall show using induction that (2.15) holds. If $n = 0$, (2.15) holds by (2.1), since*

$$||x_1 - x_0|| = ||F'(x_0)^{-1} F(x_0)|| \le \eta = s_1 - s_0.$$

Suppose that (2.15) holds for all integers $k$ smaller or equal to $n$. Using the induction hypothesis, we get that

$$||x_k - x_0|| = \sum_{i=0}^{k-1} ||x_{i+1} - x_i|| \le \sum_{i=0}^{k-1} (s_{i+1} - s_i) = s_k - s_0 = s_k \le s_-.$$

Then, by (2.3) we get that

$$\begin{aligned} ||F'(x_0)^{-1}\big(F'(x_k) - F'(x_0)\big)|| &\le 1 + \varphi'(||x_k - x_0||) \\ &\le 1 + \varphi'(s_k) < 1. \end{aligned} \tag{2.17}$$

It follows from (2.17) and the Banach lemma on invertible operators [7, 16, 20, 21] that $F'(x_k)^{-1} \in \ell(B_2, B_1)$ and

$$\begin{aligned} ||F'(x_k)^{-1}F'(x_0)|| &\le \frac{1}{1 - ||F'(x_0)^{-1}\big(F'(x_k) - F'(x_0)\big)||} \\ &\le -\frac{1}{\varphi'(||x_k - x_0||)} \le -\frac{1}{\varphi'(s_k)} \\ &\le -\frac{1}{\psi'(s_k)}. \end{aligned} \tag{2.18}$$

Hence, $x_{k+1}$ is well defined. Using the Taylor expansion of $F$ at $x_{k-1}$ and the definition of $x_k$ we can write

$$F(x_k) = F(x_{k+1}) + F'(x_{k-1})(x_k - x_{k-1}) + T_1(F; x_{k-1}, x_k) = T_1(F; x_{k-1}, x_k). \tag{2.19}$$

Then, by (2.19) the first Lemma in [27], the definition of $s_k$ and the induction hypothesis estimate $||x_k - x_{k-1}|| \le s_k - s_{k-1}$, we have in turn

$$\begin{aligned} ||F_1\Big(F'(x_0)^{-1}F; x_{k-1}, x_k\Big)|| &\le T_1(\psi; s_{k-1}, s_k) \\ &= \psi(s_k) - \psi(s_{k-1}) \\ &\quad - \psi'(s_{k-1})(s_k - s_{k-1}) \\ &= \psi(s_k). \end{aligned} \tag{2.20}$$

That is we have

$$||F'(x_0)^{-1}F(x_k)|| \le \psi(s_k). \tag{2.21}$$

Then, in view of (2.18) and (2.21) we get that

$$\begin{aligned} ||x_{k+1} - x_k|| &\le ||F'(x_k)^{-1}F'(x_0)||||F'(x_0)^{-1}F(x_k)|| \\ &\le -\frac{\psi(s_k)}{\psi'(s_k)} = s_{k+1} - s_k. \end{aligned} \tag{2.22}$$

The induction for estimate (2.15) is complete. It follows from (2.22) that $\{x_k\}$ is a complete sequence in a Banach space $B_1$ and such it converges to some $x^* \in \bar{U}(x_0, s_-)$ ($\bar{U}(x_0, s_-)$ is a closed). If $k \to \infty$ in (2.21), we deduce that $F(x^*) = 0$. Estimate (2.16) follows from (2.15) by using standard majorizing techniques [2, 4, 7, 16, 23]. Finally, to show the uniqueness part, let

$y^*$ be a solution of equation $F(x) = 0$ in $U(x_0, R_0)$ and let $q = \frac{||y^* - x_0||}{R_0} < 1$. Using induction we shall show

$$||x_n - y^*|| \leq q^{2^n}(R_0 - s_k) \tag{2.23}$$

leading to $x^* = y^*$, since we showed $\lim_{k\to\infty} x_k = x^*$. Estimate (2.23) is true for $n = 0$. Suppose that it holds for all integer up to $k$. We get in turn as in (2.20) that (since $\psi'' = L$)

$$\begin{aligned} ||T_1\left(F'(x_0)^{-1}F; x_k, x^*\right)|| &\leq T_1(\psi; s_k, s_k + ||x^* - x_k||) \\ &= \int_{s_k}^{s_k + ||x_k - y^*||} \psi''(s)\left(s_k + ||x_k - y^*|| - s\right)ds \\ &\leq q^{2^{k+1}} \int_{s_k}^{R_0} \psi''(s)(R_0 - s)ds \\ &= q^{2^{k+1}}\left(\psi(R_0) - \psi(s_k) - \psi'(s_k)(R_0 - s_k)\right). \end{aligned} \tag{2.24}$$

Using (2.18) and (2.24), we conclude that

$$||x_{n+1} - y^*|| \leq -q^{2^{k+1}} \frac{\psi(R_0) - \psi(s_k) - \psi'(s_k)(R_0 - s_k)}{\psi'(s_k)}. \tag{2.25}$$

But, by hypothesis $\psi(R_0) \leq 0$, (2.25) yields (2.23). □

The uniqueness part can be extended as follows:

**Proposition 2.2.** *Under the hypotheses of Theorem 2.1 further assume that there exists $R_1 \in [0, R_0]$ such that*

$$\int_0^1 \varphi(R_0 + (1-t)R_1)dt < 1, \text{ and } \bar{U}(x_0, R_1) \subseteq D, \tag{2.26}$$

*then, the point $x^*$ is the unique solution of equation $F(x) = 0$ in $\bar{U}(x_0, R_1)$.*

*Proof.* Let $y^*$ be a solution of equation $F(x) = 0$ in $\bar{U}(x_0, R_1)$. Define $M = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$. Then, as in (2.17), we get by (2.26) that

$$||F'(x_0)^{-1}\left(M - F'(x_0)\right)|| \leq \int_0^1 \varphi(R_0 + (1-t)R_1)dt < 1. \tag{2.27}$$

It follows from (2.27) that $M^{-1} \in \ell(B_2, B_1)$. Then, by the identity $0 = F(y^*) - F(x^*) = M(y^* - x^*)$, we conclude that $x^* = y^*$. □

**Remark 2.3.**

(a) If $K = L = L_1$, then Theorem 2.1 reduces to the corresponding one in [27]. Moreover, if $L_0 \leq L_1$, then Theorem 2.1 reduces to our earlier results [2–7]. However, if

$$K \leq L \leq L_1, \tag{2.28}$$

*then the new results improve the older ones. In particular, under the hypotheses of Theorem 2.1: if (2.9) holds then sequence $\{r_n\}$ can replace sequence $\{s_n\}$ in Theorem 2.1 (see also (2.13) and (2.14)). If (2.10) holds and the hypotheses of Theorem 2.1 hold with $R, L_1$ replacing $R_0, L$, respectively (i.e., the hypotheses of Theorem 2.4 in [27]) in Theorem 2.1, then we have the advantages ($\mathcal{A}$) and (2.11)–(2.14). Notice, however, that a direct study of the convergence of scalar sequences $\{r_n\}$ and $\{s_n\}$ can lead to even weaker convergence criteria (see Section 4.). Concluding and in view of the estimate $\varphi(t^*) \leq \psi(t^*) \leq \psi_1(t^*) = 0$, we see that the results in [27] always imply our results but not necessarily vice versa.*

(b) The radius R can be chosen such that R = R0 or by $R = \sup\{t \geq 0 : U(x_0, t) \subseteq D\}$ or some other way.

## 3. Semi-local convergence for $m \geq 2$

We suppose again that condition (2.3) holds in this section, so we can define $R_0$ and keep the Newton iterates $\{x_n\}$ in $U$. Moreover, suppose that for each $i = 2, 3, \ldots, m$ there exist

$$c_i \geq ||F'(x_0)^{-1} F^{(i)}(x_0)|| \tag{3.1}$$

and a continuous non-negative and non-decreasing function $L_m : [0, R] \to \mathbb{R}_{\geq 0}$ satisfying

$$||F'(x_0)^{-1}\left(F^{(m)}(u) - F^{(m)}(v)\right)|| \leq L_m(r)||u - v|| \quad \text{for each } r \in [0, R] \text{ and } u, v \in U_0. \tag{3.2}$$

Furthermore, define function $g_m$ on the interval $[0, R]$ by

$$g_m(t) = \eta - t + c_2 \frac{t^2}{2!} + \ldots + c_m \frac{t^m}{m!} + \int_0^t L_m(s) \frac{(t-s)^m}{m!} ds.$$

Finally, suppose that

$$-\frac{1}{\varphi'(t)} \leq -\frac{1}{g_m'(t)} \quad \text{for each } t \in [0, R_0]. \tag{3.3}$$

Define, scalar sequence $\{s_n\}$ by

$$s_0 = 0, \; s_{n+1} = s_n - \frac{g_m(s_n)}{g_m'(s_n)}. \tag{3.4}$$

Then, we can show the following semi-local convergence result for Newton's method.

**Theorem 3.1.** *Let $F : D \subseteq B_1 \to B_2$ be a $m-$times differentiable operator in the sense of Fréchet for $m \geq 2$, and let $x_0 \in D$ be such that $F'(x_0)^{-1} \in \ell(B_2, B_1)$. Suppose that: hypotheses (2.3), (3.1), (3.2) and (3.3) hold; function $g_m$ has a unique zero $s_-$ in the interval $[0, R_0)$, $U(x_0, R_0) \subseteq D$ and $g_m(R_0) \leq 0$. Then, the following assertions hold.*

(a) *The scalar sequence $\{s_n\}$ defined by (3.4) is non-decreasingly convergent to $s_-$.*

(b) *The sequence $\{x_n\}$ generated by Newton's method is well defined, remains in $\bar{U}(x_0, s_-)$ for each $n = 0, 1, 2, \ldots$ and converges to a unique solution $x^*$ of equation F(x) = 0 in $U(x_0, R)$. Moreover, the following error bounds hold*

$$||x_{n+1} - x_n|| \leq s_{n+1} - s_n$$

*and*

$$||x_n - x^*|| \leq s_- - s_n.$$

*Proof.* Define function $L$ by

$$L(t) = g_m'\prime(t) = c_2 + \ldots + c_m \frac{t^{m-2}}{m!} + \int_0^1 L_m(s) \frac{(t-s)^{m-2}}{(m-2)!} ds$$

and $t_u = ||u - x_0||$. We shall show that the hypotheses of Theorem 2.1 are satisfied with $\psi(t) = g_m(t)$. We have again by the first Lemma in [27] applied to $F^{(m-1)}$ that

$$||F'(x_0)^{-1}\left(F^{(m)}(u) - F^{(m)}(x_0)\right)|| \leq \int_0^{t_u} L_m(s) ds.$$

- Case $m = 2$. We have

$$T_{m-2}\left(F'(x_0)^{-1}F''; x_0, u\right) = F'(x_0)^{-1}\left(F''(u) - F''(x_0)\right).$$

  Then, its norm is bounded above by $\int_0^{t_u} L_m(s) ds = T_{m-2}(g_m'\prime; 0, t_u)$.

- **Case** $m \geq 3$. The integral form of the Taylor remainder of $F''$ to order $m$ – 3 yields to

$$\begin{aligned} T_{m-2}\left(F'(x_0)^{-1}F''; x_0, u\right) &= T_{m-3}\left(F'(x_0)^{-1}F''; x_0, u\right) - F^{(m)}(x_0) \frac{(u-x_0)^{m-2}}{(m-2)!} \\ &= \int_{[x_0, u]} \left(F^{(m)}(w) - F^{(m)}(x_0)\right) \frac{(u-w)^{m-3}}{(m-3)!} dw, \end{aligned}$$

so

$$||T_{m-2}\left(F'(x_0)^{-1}F''; x_0, u\right)|| \leq \int_0^{t_u}\int_0^t L_m(s)\frac{(t_x-t)^{m-3}}{(m-3)!}dtds$$
$$= T_{m-2}\left(g_m'\prime; 0, t_u\right)$$

leading to

$$||T_{m-2}(F'(x_0)^{-1}F''; x_0, u)|| \leq \sum_{i=2}^{m} c_i \frac{t_x^{(i-2}}{(i-2)!}$$
$$+ T_{m-2}(g_m''; 0, t_x) = g_m''(t_u).$$

□

If we take the limit of the previous result when $m$ approaches infinity, we obtain the following result.

**Proposition 3.2.** *Let $F : D \subseteq B_1 \to B_2$ be a $\infty-$times differentiable operator in the sense of Fréchet and let $x_0 \in D$ be such that $F'(x_0)^{-1} \in \ell(B_2, B_1)$. Define function $g_\infty : [0, R_0] \to \mathbb{R}_{\geq 0}$ by*

$$g_\infty(t) = \eta - t + \sum_{i=2}^{\infty} c_i \frac{t^i}{i!}.$$

*Suppose that (2.3), (3.1), (3.2), (3.3) hold (with $g_m$ replaced by $g_\infty$), function $g_\infty$ is well defined and has a unique zero $s_-$ in the interval $[0, R_0), U(x_0, R_0) \subseteq D$ and $g_\infty(R_0) \leq 0$. Then, the following assertions hold.*

(a) *The scalar sequence $\{s_n\}$ defined by*

$$s_0 = 0,\ s_{n+1} = s_n - \frac{g_\infty(s_n)}{g_\infty'(s_n)},$$

*is well defined, remains in $[0, s_-]$ and is nondecreasingly convergent to $s_-$.*

(b) *The sequence $\{x_n\}$ generated by Newton's method is well defined, remains in $\bar{U}(x_0, s_-)$ for each $n = 0, 1, 2, \ldots$ and converges to a unique solution $x^*$ of equation F(x) = 0 in $U(x_0, R)$. Moreover, the following error bounds hold*

$$||x_{n+1} - x_n|| \leq s_{n+1} - s_n$$

*and*

$$||x_n - x^*|| \leq s_- - s_n.$$

*Proof.* Define again function $L$ by $L(t) = g'_\infty\prime(t)$. Using the Taylor expansion of $F''$ at $x_0$, we get for each $x \in U_0$

$$||F'(x_0)^{-1}F''(x)|| \leq L(||x - x_0||). \tag{3.5}$$

Hence, the hypotheses of Theorem 2.1 hold with $\psi(t) = g_\infty(t)$. □

Remarks analogous to Remark 2.3 can be made for the results of this section (see also Section 4).

## 4. Special cases and examples

**Case 4.1** $\boldsymbol{m = 2}$ **(a)** Let $K$, $L$ and $L_1$ be constant functions. Then, the convergence conditions of Theorem 2.1 become:

$$h = L\eta \leq \frac{1}{2} \tag{4.1}$$

which is weaker than the celebrated Newton-Kantorovich condition [12]

$$h_1 = L_1\eta \leq \frac{1}{2}. \tag{4.2}$$

Notice that

$$h_1 \leq \frac{1}{2} \Longrightarrow h \leq \frac{1}{2}. \tag{4.3}$$

The convergence is not true unless, if $L = L_1$. The corresponding majorizing sequences are:

$$r_0 = 0, r_1 = \eta, r_2 = r_1 + \frac{L(r_1 - r_0)^2}{2(1 - Kr_1)},$$

$$r_{n+2} = r_{n+1} + \frac{L(r_{n+1} - r_n)^2}{2(1 - Kr_{n+1})}$$

$$s_0 = 0, s_{n+1} = s_n + \frac{L(s_n - s_{n-1})^2}{2(1 - Ls_n)}$$

and

$$t_0 = 0, t_{n+1} = t_n + \frac{L_1(t_n - t_{n-1})^2}{2(1 - L_1 t_n)}.$$

Notice that $\{r_n\}$ is a majorizing sequence provided that (4.1) holds and $L \leq K$, whereas the other two are majorizing sequences provided that (4.1) and (4.2) hold, respectively. Then, the analogous to (2.9) and (2.10) conditions for (2.11)–(2.14) to hold are respectively

$$L \leq K \tag{4.4}$$

and

$$L \leq L_1 \tag{4.5}$$

(see also the numerical examples). Finally, notice that a direct study for the convergence of these sequences yields even weaker sufficient semi-local convergence conditions in this case (see [5–7]).

- Case $m = \infty$ Let $g_\infty$ be a rational function with numerator of degree 2 and denominator of degree 1. Suppose there exists $\gamma_0 > 0$ such that

$$\gamma_0 \geq \left|\left|\frac{F'(x_0)^{-1}F^{(i)}(x_0)}{i!}\right|\right|^{\frac{1}{i-1}}$$

for each $i \geq 2$ and let $c_i = i!\gamma_0^{i-1}$, so that $g_\infty(t) = \eta - t + \frac{\gamma_0 t^2}{1-\gamma_0 t}$. Set $\alpha_0 = \eta\gamma_0$. Then, according to Proposition 3.2 we should have

$$R_0 < \frac{1}{\gamma_0}$$

and

$$\alpha_0 = \eta\gamma_0 \leq 3-2\sqrt{2}. \tag{4.6}$$

The corresponding $\alpha-$ conditions given by Smale [24] are:

$$R < \frac{1}{\gamma_0}$$

and (4.6). Therefore, the new conditions are weaker, since $R_0 < R$.

- **Case $m = 2$ (b)** Wang's condition [25] is given by

$$||F'(x_0)^{-1}F''(u)|| \leq \frac{2\gamma_1}{(1-\gamma_1||u-x_0||)^3} \text{foreach } u \in U$$

and some $\gamma_1 > 0$. But in our case the condition is

$$||F'(x_0)^{-1}F''(u)|| \leq \frac{2\gamma}{(1-\gamma||u-x_0||)^3} \text{ for each } u \in U$$

and some $\gamma > 0$. Then, again we have:

$$\gamma \leq \gamma_1,$$

whereas the corresponding $\alpha-$ convergence criteria are

$$\alpha = \gamma\eta \leq 3-2\sqrt{2}$$
$$\alpha = \gamma_1\eta \leq 3-2\sqrt{2}.$$

Notice that:

$$\alpha_1 \leq 3{-}2\sqrt{2} \Longrightarrow \alpha \leq 3{-}2\sqrt{2}.$$

The convergence is not true unless, if $\gamma = \gamma_1$. Let us define sequence $\{u_n\}$ by

$$u_n = \frac{s_n{-}s_-}{s_n = s_+}.$$

Then, it is simple algebra to show that

$$u_{n+1} = \frac{s_n{-}s_-{-}\frac{g_\infty(s_n)}{g'_\infty(s_n)}}{s_n - s_- + \frac{g_\infty(s_n)}{g'_\infty(s_n)}} = \frac{1{-}\gamma s_+}{1 - \gamma s_-} u_n^2.$$

Hence, sequence $\{s_n\}$ can be expressed explicitly in terms of $\alpha, \eta$ and $\gamma$.

**Example 4.1.** *Let* $B_1 = B_2 = \mathbb{R}, x_0 = 1, \Omega = \{x : |x{-}x_0| \leq 1{-}\theta\}$, $\theta \in [0, \frac{1}{2})$, $R = 1{-}\theta, R_0 = \frac{1}{L}$. *Define function F on* $\Omega$ *by*

$$F(x) = x^3{-}\theta.$$

*By the hypotheses of Theorem 2.1 we get:*

$$\eta = \frac{1}{3}(1{-}\theta),\ K = 3{-}\theta,\ L_1 = 2(2{-}\theta) \ \text{ and } L = 2\left(1 + \frac{1}{K}\right),$$

*so*

$$K < L < L_1, \ \text{ and } R_0 < R,$$

*(4.4) and (4.5) hold as strict inequalities. The Newton-Kantorovich condition (4.2) is not satisfied, since*

$$\frac{4}{3}(1{-}\theta)(2{-}\theta) > 1 \text{ and } \theta \in \left[0, \frac{1}{2}\right). \tag{4.7}$$

*Hence, there is no assurance that Newton's method (1.2) converges to* $x^* = \sqrt{\theta 3}$, *for* $x_0 = 1$. *However, our condition (4.1) is satisfied for all* $\theta \in I = [0.4619832, \frac{1}{2})$. *Hence, the conclusions of our Theorem 2.1 can apply to solve equation F(x) = 0 for all* $\theta \in I$.

**Example 4.2.** *Assume* $B_1 = B_2 = C[0, 1]$. *Let* $\Omega = \{z \in C[0, 1]; ||z|| \leq R\}$, *such that* $R > 0$, *where* $||.||$ *stands for the norm max. Define F on* $\Omega$ *by [14]:*

$$F(z)(s) = z(s){-}\varsigma(s){-}\tau \int_0^1 G(s, t)z(t)^3 dt, \quad z \in C[0, 1], s \in [0, 1].$$

*Here,* $\varsigma \in C[0, 1]$ *is a given function,* $\tau$ *is a real constant and the kernel G is the Green's function*

$$G(s, t) = \begin{cases} (1{-}s)t, & t \leq s, \\ s(1{-}t), & s \leq t. \end{cases}$$

*Notice that nonlinear integral equation $F(z)(s) = 0$ is of Chandrasekhar type [9]. Then, $F'(z)$ is a linear operator given for each $z \in \Omega$, by*

$$\left[F'(z)(v)\right](s) = v(s) - 3\tau \int_0^1 G(s,t)z(t)^2 v(t)dt, \quad v \in C[0,1], s \in [0,1].$$

*Let us choose $x_0(s) = \varsigma(s) = 1$ to obtain that $||I - F'(x_0)|| \le 3|\tau|/8$, so, if $|\tau| < 8/3$, $F'(x_0)^{-1}$ exists and*

$$||F'(x_0)^{-1}|| \le \frac{8}{8 - 3|\tau|}.$$

*Notice that*

$$||F(x_0)|| \le \frac{|\tau|}{8},$$

*so*

$$\eta = ||F'(x_0)^{-1}F(x_0)|| \le \frac{|\tau|}{8 - 3|\tau|}.$$

*Moreover, for $u, v \in \Omega$ we get*

$$||F'(u) - F'(v)|| \le ||u - v|| \frac{1 + 3|\tau|(||u + v||)}{8} \le ||u - v|| \frac{1 + 6R|\tau|}{8}.$$

*and*

$$||F'(u) - F'(1)|| \le ||u - 1|| \frac{1 + 3|\tau|(||u|| + 1)}{8} \le ||u - 1|| \frac{1 + 3(1 + R)|\tau|}{8}.$$

*Let $\tau = 1.175$ and $R = 2$. Then,we get $\eta = 0.26257\ldots, L_1 = 2.76875\ldots, K = 1.8875\ldots, \frac{1}{K} = 0.529801\ldots, L = 1.47314\ldots$. Notice that (4.4) and (4.5) are satisfied as strict inequalities. Finally, we conclude that condition (4.2) is not satisfied:*

$$\bar{h}^1 = 1.02688\ldots > 1,$$

*whereas condition (4.1) is satisfied: $h_A^1 = 0.986217\ldots < 1$. Therefore, the convergence of the Newton's method is assured by Theorem 2.1.*

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