Applied Mathematics E-Notes, 19(2019), 220-227 © ISSN 1607-2510
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Local Convergence For A Quadrature Based Third-Order Method Using Only The First Derivative[*]

Ioannis Konstantinos Argyros[†], Santhosh George[‡]

Received 16 April 2018

Abstract

We extend the applicability of a method for approximating a locally unique solution of a nonlinear equation. The convergence analysis in earlier work was based on Taylor expansions and hypotheses reaching up to the second derivative of the function involved, although only the first derivative appears in the method. In this study, we use only hypotheses on the first derivative of the involved function. Numerical examples are also presented in this study.

1 Introduction

Ujevic in [7], considered an iterative method for approximating a solution of the nonlinear equation

$$F(x) = 0,$$

where $F : D \subseteq S \to S$ is a differentiable nonlinear function, S is $\mathbb{R}$ or $\mathbb{C}$ and D is a subset of S. In this paper we study the method in [7] using only hypotheses on the first derivative of the function. Precisely, we present the local convergence analysis of the following method defined for each $n = 0, 1, 2, \cdots$, by

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - 4\alpha F'(x_n)^{-1}F(x_n)(3F(x_n) - 2F(y_n))^{-1}F(x_n), \end{aligned} \tag{1}$$

where, $x_0 \in D$ is an initial point and $\alpha \in S$ is a parameter. The method (1) was studied in [7] for $S = \mathbb{R}$ and $\alpha \in (0, 1)$. The convergence of the method was shown using Taylor expansions and hypotheses reaching up to the second derivative of the function F. Moreover, method (1) was compared favorably to existing methods. However, the hypotheses on the second derivative limit the applicability of method (1). As a motivational example, let us define function F on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

[*]Mathematics Subject Classifications: 65D10, 65D99.

[†]Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
[‡]Mathematical and Computational Sciences, National Institute of Technology Karnataka, India

Choose $x^* = 1$. We have that

$$F'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$F''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x$$

and

$$F'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

The results in [7] cannot be used to show the convergence of method (1) on the above example, since the first and second derivatives of function F have zeros on D. But, our results can apply (see Example 3.2). Hence, we extend the applicability of method (1). We also find the computational order of convergence (COC) or the approximate computational order of convergence that do not require the usage of higher order derivatives (see Remark 2.2 part 4) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds based on Lipschitz constants and a uniqueness result not given in [7]. Special cases and numerical examples are presented in the concluding Section 3.

2 Local Convergence

Let $L_0 > 0$, $L > 0$, $M \geq 1$ and $\alpha \in S$. Define functions g_1, p and h_p on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{1}{2(1 - L_0 t)}(Lt + 2M|1-\alpha|),$$

$$p(t) = \frac{1}{3}(\frac{3L_0}{2}t + 2Mg_1(t)),$$

$$h_p(t) = p(t) - 1$$

and parameters r_1 and r_A by

$$r_1 = \frac{2(1 - M|1-\alpha|)}{2L_0 + L} \text{ and } r_A = \frac{2}{2L_0 + L}.$$

Suppose that

$$M|1-\alpha| < 1. \tag{2}$$

By definition of the functions and (2) we have that $0 < r_1 < r_A, g_1(r_1) = 1$ and for each $t \in [0, r_1)$, $0 \leq g_1(r_1) < 1$. Moreover, suppose that

$$\frac{2}{3}M^2|1-\alpha| < 1. \tag{3}$$

Then, we get by (3) that $h_p(0) = \frac{2}{3}M^2|1-\alpha| - 1 < 0$ and $h_p(t) \to +\infty \; as \; t \to \frac{1}{L_0}^-$. It follows by the intermediate value theorem that function h_p has zeros in the interval

$(0, \frac{1}{L_0})$. Denote by r_p the smallest such zeros. Define functions g_2 and h_2 on the interval $(0, r_p)$ by

$$g_2(t) = \frac{1}{2(1-L_0t)}(Lt + \frac{2M^2(|3-4\alpha| + 2g_1(t))}{3(1-p(t))})$$

and

$$h_2(t) = g_2(t) - 1.$$

Further, suppose that

$$\frac{M^2(|3-4\alpha| + 2M|1-\alpha|)}{3(1-\frac{2}{3}M^2|1-\alpha|)} < 1. \tag{4}$$

In view of (4), we have that $h_2(0) < 0$ and $h_2(t) \to +\infty$ *as* $t \to r_p^-$. Denote by r_2 the smallest zero of function h_2 on the interval $(0, r_p)$. Set

$$r = \min\{r_1, r_2\}. \tag{5}$$

Then, we have

$$0 < r < r_A < \frac{1}{L_0} \tag{6}$$

and for each $t \in [0, r)$

$$0 \leq g_1(t) < 1, \tag{7}$$

$$0 \leq p(t) < 1 \tag{8}$$

and

$$0 \leq g_2(t) < 1. \tag{9}$$

Let $U(v, \rho), \bar{U}(v, \rho)$ stand, respectively for the open and closed balls in S with center $v \in S$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (1) using the preceding notation.

THEOREM 2.1. Let $F : D \subset S \to S$ be a differentiable function. Suppose that there exist $x^* \in D$ and $L_0 > 0$ such that

$$F(x^*) = 0, \;\; F'(x^*) \neq 0 \tag{10}$$

and the center Lipschitz condition holds

$$|F'(x^*)^{-1}(F'(x) - F'(x^*)))| \leq L_0|x - x^*|. \tag{11}$$

Further, suppose that there exist $L > 0$ and $M \geq 1$ and $\alpha \in S$ satisfying (2)–(4) and for each $x, y \in D_0 = D \cap U(x^*, \frac{1}{L_0})$

$$|F'(x^*)^{-1}(F'(x) - F'(y))| \leq L|x - y|, \tag{12}$$

$$|F'(x^*)^{-1}F'(x)| \leq M \tag{13}$$

and

$$\bar{U}(x^*, r) \subseteq D$$

where the radius of convergence r is given by (5). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold

$$|y_n - x^*| \leq g_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r, \tag{14}$$

and

$$|x_{n+1} - x^*| \leq g_2(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|, \tag{15}$$

where the $''g''$ functions are defined previously. Furthermore, for $T \in [r, \frac{2}{L_0})$ the limit point x^* is the only solution of equation $F(x) = 0$ in $D_1 = \bar{U}(x^*, T) \cap D$.

PROOF. We shall show estimates (14) and (15) using mathematical induction. By hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (6) and (11), we get that

$$|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \leq L_0|x_0 - x^*| < L_0 r < 1. \tag{16}$$

It follows from (16) and the Banach Lemma on invertible functions [2, 3], $F'(x_0) \neq 0$ and

$$|F'(x_0)^{-1}F'(x^*))| \leq \frac{1}{1 - L_0|x_0 - x^*|}. \tag{17}$$

We also have that y_0 is well-defined by the second sub-step of method (1) for $n = 0$. We can write by (10) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{18}$$

Notice that $|x^* + \theta(x_0 - x^*) - x^*| = \theta|x_0 - x^*| < r$, so $x^* + \theta(x_0 - x^*) \in U(x^*, r)$. Then, by (13) and (18), we have that

$$|F'(x^*)^{-1}F(x_0)| \leq M|x_0 - x^*|. \tag{19}$$

Using the first substep of method (1) for $n = 0$, (6), (7), (12), (17) and (19) we obtain that

$$\begin{aligned}
& |y_0 - x^*| \\
= \;& |(x_0 - x^* - F'(x_0)^{-1}F(x_0)) + (1 - \alpha)F'(x_0)^{-1}F(x_0)| \\
\leq \;& |F'(x_0)^{-1}F'(x^*))| |\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta| \\
& + |1 - \alpha||F'(x_0)^{-1}F'(x^*)||F'(x^*)^{-1}F(x_0)| \\
\leq \;& \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} + \frac{M|1 - \alpha||x_0 - x^*|}{1 - L_0|x_0 - x^*|} \\
= \;& g_1(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < r,
\end{aligned} \tag{20}$$

which shows (14) for $n = 0$ and $y_0 \in U(x^*, r)$. Next, we must show that $3F(x_0) - 2F(y_0) \neq 0$ for $x_0 \neq x^*$. Using (20), we obtain in turn that

$$\begin{aligned}
& |(3F'(x^*)(x_0 - x^*))^{-1}[3(F(x_0) - F(x^*) - F'(x^*)(x_0 - x^*)) - 2F(y_0)]| \\
\leq\ & \frac{|x_0 - x^*|^{-1}}{3}[\frac{3L_0}{2}|x_0 - x^*|^2 + 2M|y_0 - x^*|] \\
\leq\ & \frac{|x_0 - x^*|^{-1}}{3}[\frac{3L_0}{2}|x_0 - x^*| + 2Mg_1(|x_0 - x^*|)]|x_0 - x^*| \\
=\ & p(|x_0 - x^*|) < p(r) < 1.
\end{aligned} \tag{21}$$

By (21), $3F(x_0) - 2F(y_0) \neq 0$ and

$$|(3F(x_0) - 2F(y_0))^{-1}F'(x^*)| \leq \frac{1}{3|x_0 - x^*|(1 - p(|x_0 - x^*|))}. \tag{22}$$

We also have that x_1 is well defined by the second substep of method (1) and (22). Then, using (6), (9), (17), (19) (for $y_0 = x_0$), (20), (22) and the approximation

$$\begin{aligned}
& x_1 - x^* \\
=\ & x_0 - x^* - F'(x_0)^{-1}F(x_0) \\
& + F'(x_0)^{-1}F(x_0)[1 - 4\alpha(3F(x_0) - 2F(y_0))^{-1}F(x_0)] \\
=\ & x_0 - x^* - F'(x_0)^{-1}F(x_0) \\
& + F'(x_0)^{-1}F(x_0)(3F(x_0) - 2F(y_0))^{-1}[(3 - 4\alpha)F(x_0) - 2F(y_0)],
\end{aligned}$$

we get in turn that

$$\begin{aligned}
|x_1 - x^*| \leq\ & \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} + \frac{M^2|x_0 - x^*|(|3 - 4\alpha||x_0 - x^*| + 2|y_0 - x^*|)}{3(1 - L_0|x_0 - x^*|)|x_0 - x^*|(1 - p(|x_0 - x^*|))} \\
\leq\ & \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} + \frac{M^2|x_0 - x^*|(|3 - 4\alpha| + 2g_1(|x_0 - x^*|))}{3(1 - L_0|x_0 - x^*|)(1 - p(|x_0 - x^*|))} \\
=\ & g_2(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < r,
\end{aligned}$$

which shows (15) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding estimates we arrive at estimates (14) and (15). Then, it follows from the estimate $|x_{k+1} - x^*| \leq c|x_k - x^*| < r, c = g_2(|x_0 - x^*|) \in [0, 1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. The uniqueness part has been shown in [5].

REMARK 2.2.

(i) In view of (10) and the estimate

$$\begin{aligned}
|F'(x^*)^{-1}F'(x)| &= |F'(x^*)^{-1}(F'(x) - F'(x^*)) + I| \\
&\leq 1 + |F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq 1 + L_0|x - x^*|
\end{aligned}$$

condition (13) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

(ii) The results obtained here can be used for operators F satisfying autonomous differential equations [2] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

(iii) In [2, 3] we showed that $r_A = \frac{2}{2L_0+L}$ is the convergence radius of Newton's method:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \ \ foreach \ \ n = 0, 1, 2, \cdots \tag{23}$$

under the conditions (11) and (12). It follows from the definition of r in (5) that the convergence radius r of the method (1) cannot be larger than the convergence radius r_A of the second order Newton's method (23). As already noted in [2, 3] r_A is at least as large as the convergence radius given by Rheinboldt [15]

$$r_R = \frac{2}{3L}.$$

The same value for r_R was given by Traub [17]. In particular, for $L_0 < L$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \ \ as \ \ \frac{L_0}{L} \to 0.$$

That is the radius of convergence r_A is at most three times larger than Rheinboldt's.

(iv) It is worth noticing that method (1) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [7]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{|x_{n+1} - x^*|}{|x_n - x^*|}\right) / \ln\left(\frac{|x_n - x^*|}{|x_{n-1} - x^*|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}\right) / \ln\left(\frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F. Note also that the computation of ξ_1 does not require the usage of the solution x^*.

3 Numerical Examples

We present numerical examples in this section.

EXAMPLE 3.1. Let $D = (-\infty, +\infty)$. Define function F of D by

$$F(x) = \sin(x).$$

Then we have for $x^* = 0$ that $L_0 = L = 1$, $M = 1$. Then, the parameters are:

$$\alpha = 0.6256,\ r_1 = 0.41666666666666666666666666666667,$$

$$r_A = 0.66666666666666666666666666666667,\ r_p = 0.53137303340311403676565760179074,$$
$$r_2 = 0.26492754861221623485789677943103 = r.$$

EXAMPLE 3.2. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 96.662907$, $M = 1.0631$. Then, the parameters are:

$$\alpha = 0.6417,\ r_1 = 0.0042695902413662084606826851995 72,$$

$$r_A = 0.0068968199628702108600064590859802,\ r_p = 0.0052580209390285324172475966975071,$$
$$r_2 = 0.0022430651885094708697376830741632 = r.$$

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