ARAB JOURNAL OF MATHEMATICAL SCIENCES

Arab J Math Sci 23 (2017) 205–214

Local convergence of a fifth convergence order method in Banach space

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Received 28 June 2016; received in revised form 20 October 2016; accepted 24 October 2016
Available online 12 November 2016

Abstract. We provide a local convergence analysis for a fifth convergence order method to find a solution of a nonlinear equation in a Banach space. In our paper the sufficient convergence conditions involve only hypotheses on the first Fréchet-derivative. Previous works use conditions reaching up to the fourth Fréchet-derivative. This way, the applicability of these methods is extended under weaker conditions and less computational cost for the Lipschitz constants appearing in the convergence analysis. Numerical examples are also given in this paper.

Keywords: High convergence order method; Banach space; Local convergence; Fréchet-derivative; Nonlinear equation

2010 Mathematics Subject Classification: 65D99; 65D10

1. INTRODUCTION

Let $\mathcal{F} : \Omega \subset \mathcal{X} \longrightarrow \mathcal{Y}$ be a Fréchet-differentiable operator, where $\mathcal{X}, \mathcal{Y}$ are Banach space and $\Omega \subset \mathcal{X}$. A lot of problems from many areas can be written like the nonlinear equation

$$\mathcal{F}(x) = 0. \tag{1.1}$$

In this paper, we study the problem of approximating a solution $\alpha^*$of Eq. (1.1). In Numerical Analysis, finding a solution of (1.1) is related to Newton-like methods [1–13]. The Newton-like methods are usually studied using: semi-local and local convergence. The semi-local convergence case is: using the information about an initial guess, to find conditions that ensure the convergence of the iterative procedure; while the local one is, using the information

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2016.10.002
1319-5166

about a solution, to find radii of the convergence balls. There are numerous papers dealing with the convergence of Newton-like methods [1–13].

We are interested in the local convergence analysis of a fifth convergence order method defined for each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned}\beta_n &= \alpha_n - \mathcal{F}'(\alpha_n)^{-1}\mathcal{F}(\alpha_n),\\ \gamma_n &= \beta_n - \mathcal{F}'(\alpha_n)^{-1}(\mathcal{F}'(\alpha_n) - \mathcal{F}'(\beta_n))(\mathcal{F}'(\alpha_n) + \mathcal{F}'(\beta_n))^{-1}\mathcal{F}(\alpha_n),\\ \alpha_{n+1} &= \gamma_n - (\mathcal{F}'(\alpha_n) + \mathcal{F}'(\beta_n))^{-1}(3\mathcal{F}'(\alpha_n) - \mathcal{F}'(\beta_n))\mathcal{F}'(\alpha_n)^{-1}\mathcal{F}(\gamma_n),\end{aligned} \tag{1.2}$$

where α_0 is an initial guess. If $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, we obtain the method whose local convergence was studied in [13]. The convergence in [13] was studied under the assumptions that derivatives $\mathcal{F}^{(i)}, i = 1, 2, 3, 4$ are bounded.

Similar assumptions have been used by several authors [1–13], on other high convergence order methods. These conditions are restrictive. As an academic example, let function f on $\Omega = [-\frac{1}{2}, \frac{5}{2}]$ be defined by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0\\ 0, & x = 0.\end{cases}$$

Let $\alpha^* = 1$. We get that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \;\; f'(1) = 3,\\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x\\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22.\end{aligned}$$

Then, clearly, function f''' is not bounded on Ω. In our paper, we only use conditions on the first Fréchet derivative (see conditions (2.11)–(2.15)). This way we extend the usage of method (1.2).

In the rest of the study: The local convergence of method (1.2) is given in Section 2. The numerical examples are given in Section 3. Some comments are given in the concluding Section 4.

2. Local convergence analysis

We provide the local convergence analysis of method (1.2). Let $B(v, \rho), \bar{B}(v, \rho)$ stand for the open and closed balls, respectively, in $\mathcal{X}$ with center $v \in \mathcal{X}$ and of radius $\rho > 0$.

Let $Ł_0 > 0$, $Ł > 0$ and $\mathcal{M} \geq 1$ be given parameters. The local convergence analysis of method (1.2) is based on some functions. Define functions on the interval $[0, \frac{1}{Ł_0})$ by

$$\begin{aligned} g(t) &= \frac{Łt}{2(1 - Ł_0 t)},\\ g_1(t) &= \frac{1}{1 - Ł_0 t}\left[Ł + \frac{Ł_0 \mathcal{M}(1 + g(t))t}{1 - \frac{Ł_0}{2}(1 + g(t))t}\right] t,\\ &= g(t) + \frac{Ł_0 \mathcal{M}(1 + g(t))t}{2(1 - Ł_0 t)\left(1 - \frac{Ł_0}{2}(1 + g(t))t\right)}\end{aligned}$$

$$\begin{aligned} g_2(t) &= \left[1 + \frac{(Ł_0(3+g(t))t+2)\mathcal{M}}{2\left(1-\frac{Ł_0}{2}(1+g(t))t\right)}(1-Ł_0\, t)\right] g_1(t), \\ &= g_1(t) + \frac{(Ł_0(3+g(t))t+2)\mathcal{M}}{2\left(1-\frac{Ł_0}{2}(1+g(t))t\right)}(1-Ł_0\, t) g_1(t), \\ h_1(t) &= g_1(t) - 1, \\ h_2(t) &= g_2(t) - 1 \end{aligned}$$

and parameter

$$r = \frac{2}{2\,Ł_0 + Ł}. \tag{2.1}$$

We have by the choice of r that

$$0 \le g(t) < 1 \ \text{ for each } \ t \in [0, r). \tag{2.2}$$

Using the definition of function h_1 we get that $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to (\frac{1}{Ł_0})^-$. Using the intermediate value theorem, we deduce that function h_1 has zeros in the interval $(0, \frac{1}{Ł_0})$. Denote by r_1 the smallest such zero. We also have that

$$\begin{aligned} h_1(r) &= g_1(r) - 1 = g(r) - 1 + \frac{Ł_0\, \mathcal{M}(1+g(r))r}{2\left(1-\frac{Ł_0}{2}(1+g(r)r)\right)(1-Ł_0\, r)}, \\ &= \frac{Ł_0\, \mathcal{M}(3+g(r))r}{2(1-Ł_0\, r)^2} > 0, \end{aligned} \tag{2.3}$$

since $g(r) - 1 = 0$ and $r < \frac{1}{Ł_0}$. Then, we have by (2.1)–(2.3) that

$$0 < r_1 < r, \tag{2.4}$$

and

$$0 \le g_1(t) < 1 \ \text{ for each } \ t \in [0, r_1). \tag{2.5}$$

Similarly, we have that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to (\frac{1}{Ł_0})^-$. Then, function h_2 has zeros in the interval $(0, \frac{1}{Ł_0})$. Denote by r_2 the smallest such zero. Then, again we have that

$$\begin{aligned} h_2(r_1) &= g_1(r_1) - 1 + \frac{(Ł_0(3+g(r_1))r_1+2)\mathcal{M}}{2\left(1-\frac{Ł_0}{2}(1+g(r_1))r_1\right)(1-Ł_0\, r_1)} g_1(r_1) \\ &= \frac{(Ł_0(3+g(r_1))r_1+2)\mathcal{M}}{2\left(1-\frac{Ł_0}{2}(1+g(r_1))r_1\right)(1-Ł_0\, r_1)} > 0, \end{aligned} \tag{2.6}$$

since $g_1(r_1) - 1 = 0$, $r_1 < \frac{1}{Ł_0}$ and $\frac{Ł_0}{2}(1+g(r_1))r_1 < \frac{Ł_0}{2}(1+1)r_1 = Ł_0\, r_1 < 1$. Then, we have that

$$r_2 < r_1 < r, \tag{2.7}$$

$$0 \le g(t) < 1, \tag{2.8}$$
$$0 \le g_1(t) < 1, \tag{2.9}$$

and

$$0 \le g_2(t) < 1 \tag{2.10}$$

for each $t \in [0, r_2)$.

The local convergence analysis of method (1.2) using the preceding notation is next.

Theorem 2.1. *Let $\mathcal{F} : \Omega \subseteq \mathcal{X} \to \mathcal{Y}$ be a Fréchet-differentiable operator. Suppose that there exist $\alpha^* \in \Omega$ and parameter $Ł_0 > 0$ such that for each $x \in \Omega$*

$$\mathcal{F}(\alpha^*) = 0, \;\; \mathcal{F}'(\alpha^*)^{-1} \in L(\mathcal{Y}, \mathcal{X}), \tag{2.11}$$
$$\|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(\alpha^*))\| \le Ł_0 \,\|x - \alpha^*\|. \tag{2.12}$$

Further, suppose that there exist $Ł > 0$ and $\mathcal{M} \ge 1$ such that for each $x, y \in \bar{B}(\alpha^*, \frac{1}{Ł_0}) \cap \Omega$

$$\|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(y))\| \le Ł\|x - y\|, \tag{2.13}$$
$$\|\mathcal{F}'(\alpha^*)^{-1}\mathcal{F}'(x)\| \le \mathcal{M} \tag{2.14}$$

and

$$\bar{B}(\alpha^*, r_2) \subseteq \Omega, \tag{2.15}$$

where the radius of convergence r_2 is defined previously. Then, the sequence $\{\alpha_n\}$ given by method (1.2) for $\alpha_0 \in B(\alpha^*, r_2) - \{\alpha^*\}$ is well defined, stays in $B(\alpha^*, r_2)$ for each $n = 0, 1, 2, \ldots$ and $\lim_{n \longrightarrow \infty} \alpha_n = \alpha^*$. Moreover, the following error bounds hold for each $n = 0, 1, 2, \ldots,$

$$\|\beta_n - \alpha^*\| \le g(\|\alpha_n - \alpha^*\|)\|\alpha_n - \alpha^*\| < \|\alpha_n - \alpha^*\| < r_2, \tag{2.16}$$
$$\|\gamma_n - \alpha^*\| \le g_1(\|\alpha_n - \alpha^*\|)\|\alpha_n - \alpha^*\| < \|\alpha_n - \alpha^*\|, \tag{2.17}$$

and

$$\|\alpha_{n+1} - \alpha^*\| \le g_2(\|\alpha_n - \alpha^*\|)\|\alpha_n - \alpha^*\| < \|\alpha_n - \alpha^*\| \tag{2.18}$$

where the "g" functions are defined previously. Furthermore, for $R \in [r_2, \frac{2}{Ł_0})$ α^* is the unique solution of equation $\mathcal{F}(x) = 0$ in $\bar{B}(\alpha^*, R) \cap \Omega$.

Proof. Using (2.13), the definition of r_2 and the hypothesis $\alpha_0 \in U(\alpha^*, r_2) - \{\alpha^*\}$, we have that

$$\|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(\alpha_0) - \mathcal{F}'(\alpha^*))\| \le Ł_0 \,\|\alpha_0 - \alpha^*\| < Ł_0 \, r_2 < 1. \tag{2.19}$$

By (2.19) and the Banach Lemma on invertible operator [2,3], we have that $\mathcal{F}'(\alpha_0)^{-1} \in L(\mathcal{Y}, \mathcal{X})$ and

$$\|\mathcal{F}'(\alpha_0)^{-1}\mathcal{F}'(\alpha^*)\| \le \frac{1}{1 - Ł_0 \,\|\alpha_0 - \alpha^*\|} < \frac{1}{1 - Ł_0 \, r_2}. \tag{2.20}$$

Hence, β_0 is well defined by method (1.2). Using the first sub-step in method (1.2) for $n = 0$, we get that

$$\begin{aligned}\beta_0 - \alpha^* &= \alpha_0 - \alpha^* - \mathcal{F}'(\alpha_0)^{-1}\mathcal{F}(\alpha_0)\\ &= -\mathcal{F}'(\alpha_0)^{-1}\mathcal{F}'(\alpha^*)\int_0^1 \mathcal{F}'(\alpha^*)^{-1}\\ &\quad \times [\mathcal{F}(\alpha^* + t(\alpha_0-\alpha^*)) - \mathcal{F}'(\alpha_0)](\alpha_0-\alpha^*)dt. \end{aligned} \tag{2.21}$$

It follows from (2.13), (2.19) and (2.21) that

$$\begin{aligned}\|\alpha_0 - \alpha^* - \mathcal{F}'(\alpha_0)^{-1}\mathcal{F}(\alpha_0)\| &\le \|\mathcal{F}'(\alpha_0)^{-1}\mathcal{F}'(\alpha^*)\|\\ &\quad \left\|\int_0^1 [\mathcal{F}'(\alpha^* + t(\alpha_0-\alpha^*)) - \mathcal{F}'(\alpha_0)](\alpha_0-\alpha^*)dt\right\|\\ &\le \frac{Ł\|\alpha_0-\alpha^*\|^2}{2(1-Ł_0\,\|\alpha_0-\alpha^*\|)}\\ &\le \frac{Łr_2}{2(1-Ł_0\,r_2)}\|\alpha_0-\alpha^*\| < \|\alpha_0-\alpha^*\| < r_2, \end{aligned} \tag{2.22}$$

which shows (2.16) for $n = 0$. Using (2.14) we have that

$$\mathcal{F}(\alpha_0) = \mathcal{F}(\alpha_0) - \mathcal{F}(\alpha^*) = \int_0^1 \mathcal{F}'(\alpha^* + \theta(\alpha_0-\alpha^*))(\alpha_0-\alpha^*)d\theta$$

so,

$$\|\mathcal{F}'(\alpha^*)^{-1}\mathcal{F}(\alpha_0)\| \le \mathcal{M}\|\alpha_0-\alpha^*\|, \tag{2.23}$$

since $\|\alpha^* - (\alpha^* + \theta(\alpha_0-\alpha^*))\| = |\theta|\|\alpha_0-\alpha^*\| < r_2$., i.e., $\alpha^* + \theta(\alpha_0-\alpha^*) \in B(\alpha^*, r_2)$ for each $\theta \in [0,1]$. Using (2.12) and the definition of r_2, we get in turn that

$$\begin{aligned}\|(2\mathcal{F}'(\alpha^*)^{-1})(\mathcal{F}'(\alpha_0) + \mathcal{F}'(\beta_0) - 2\mathcal{F}'(\alpha^*))\| &\le \frac{1}{2}(\|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(\alpha_0) - \mathcal{F}'(\alpha^*))\|\\ &\quad + \|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(\beta_0) - \mathcal{F}'(\alpha^*))\|)\\ &\le \frac{Ł_0}{2}(\|\alpha_0-\alpha^*\| + \|\beta_0-\alpha^*\|)\\ &< \frac{Ł_0}{2}(\|\alpha_0-\alpha^*\| + \|\alpha_0-\alpha^*\|)\\ &= Ł_0\,r_2 < 1. \end{aligned} \tag{2.24}$$

It follows from (2.24) that $(\mathcal{F}'(\alpha_0) + \mathcal{F}'(\beta_0))^{-1} \in L(\mathcal{Y}, \mathcal{X})$ and

$$\begin{aligned}\|(2(\mathcal{F}'(\alpha_0) + \mathcal{F}'(\beta_0)))^{-1}\mathcal{F}'(\alpha^*)\| &\le \frac{1}{1 - \frac{Ł_0}{2}(\|\alpha_0-\alpha^*\| + \|\beta_0-\alpha^*\|)}\\ &\le \frac{1}{1 - \frac{Ł_0}{2}(1 + g(\|\alpha_0-\alpha^*\|))\|\alpha_0-\alpha^*\|}\\ &\le \frac{1}{1 - \frac{Ł_0}{2}(1+g(r_2))r_2}. \end{aligned} \tag{2.25}$$

It also follows that γ_0 is well defined by the second step of method (1.2) for $n = 0$. Then, we have from the second step of method (1.2), (2.7)–(2.9), (2.20), (2.22) and (2.23) that

$$\begin{aligned}
\|\gamma_0 - \alpha^*\| &\le \|\beta_0 - \alpha^*\| + \|\mathcal{F}'(\alpha_0)^{-1}\mathcal{F}'(\alpha^*)\|(\|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(\beta_0) - \mathcal{F}'(\alpha^*))\| \\
&\quad + \|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(\alpha_0) - \mathcal{F}'(\alpha^*))\|) \\
&\quad \times \|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(\beta_0) + \mathcal{F}'(\alpha_0))^{-1}\|\|\mathcal{F}'(\alpha^*)^{-1}\mathcal{F}'(\alpha_0)\| \\
&\le \frac{Ł\|\alpha_0 - \alpha^*\|^2}{2(1 - Ł_0\|\alpha_0 - \alpha^*\|)} \\
&\quad + \frac{Ł_0\mathcal{M}(\|\alpha_0 - \alpha^*\| + \|\beta_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\|}{2(1 - Ł_0\|\alpha_0 - \alpha^*\|)\left(1 - \frac{Ł_0}{2}(\|\alpha_0 - \alpha^*\| + \|\beta_0 - \alpha^*\|)\right)} \\
&\le \frac{Ł\|\alpha_0 - \alpha^*\|^2}{2(1 - Ł_0\|\alpha_0 - \alpha^*\|)} \\
&\quad + \frac{Ł_0\mathcal{M}(1 + g(\|\alpha_0 - \alpha^*\|))\|\alpha_0 - \alpha^*\|^2}{2(1 - Ł_0\|\alpha_0 - \alpha^*\|)\left(1 - \frac{Ł_0}{2}(1 + g(\|\alpha_0 - \alpha^*\|))\|\alpha_0 - \alpha^*\|\right)} \\
&= g_1(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\| \\
&< \|\alpha_0 - \alpha^*\| < r_2, \qquad (2.26)
\end{aligned}$$

which shows (2.17) for $n = 0$. We also have by the third step of method (1.2) for $n = 0$ and (2.26) that α_1 is well defined. Then, using method (1.2) for $n = 0$, (2.8), (2.11), (2.21), (2.24)(for α_0 replaced by γ_0), (2.25) and (2.26) we obtain in turn that

$$\begin{aligned}
\|\alpha_1 - \alpha^*\| &\le \|\gamma_0 - \alpha^*\| \\
&\quad + \|(\mathcal{F}'(\beta_0) + \mathcal{F}'(\alpha_0))^{-1}\mathcal{F}'(\alpha^*)\|(\|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(\alpha_0) - \mathcal{F}'(\alpha^*))\| \\
&\quad + \|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(\beta_0) - \mathcal{F}'(\alpha^*))\| \\
&\quad + 2\|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(\alpha_0) - \mathcal{F}'(\alpha^*))\| + 2\|\mathcal{F}'(\alpha^*)^{-1}\mathcal{F}'(\alpha^*)\|) \\
&\quad \times \|\mathcal{F}'(\alpha_0)^{-1}\mathcal{F}'(\alpha^*)\|\left\|\int_0^1 \mathcal{F}'(\alpha^*)^{-1}\mathcal{F}'(\alpha^* + \theta(\gamma_0 - \alpha^*))(\gamma_0 - \alpha^*)d\theta\right\| \\
&\le \|\gamma_0 - \alpha^*\| \\
&\quad + \frac{[Ł_0(1 + g(\|\alpha_0 - \alpha^*\|))\|\alpha_0 - \alpha^*\| + 2Ł_0\|\alpha_0 - \alpha^*\| + 2]\mathcal{M}\|\gamma_0 - \alpha^*\|}{2\left(1 - \frac{Ł_0}{2}(1 + g(\|\alpha_0 - \alpha^*\|))\|\alpha_0 - \alpha^*\|\right)(1 - Ł_0\|\alpha_0 - \alpha^*\|)} \\
&\le \left[1 + \frac{Ł_0(3 + g(\|\alpha_0 - \alpha^*\|))\|\alpha_0 - \alpha^*\| + 2}{2\left(1 - \frac{Ł_0}{2}(1 + g(\|\alpha_0 - \alpha^*\|))\|\alpha_0 - \alpha^*\|\right)(1 - Ł_0\|\alpha_0 - \alpha^*\|)}\right] \\
&\quad \times \|\gamma_0 - \alpha^*\|^2 \\
&= g_2(\|\alpha_0 - \alpha^*\|)\|\alpha_0 - \alpha^*\| < \|\alpha_0 - \alpha^*\| < r_2, \qquad (2.27)
\end{aligned}$$

which shows (2.19) for $n = 0$. The estimates (2.16)–(2.18) are obtained by using $\alpha_k, \beta_k, \gamma_k, \alpha_{k+1}$ for $\alpha_0, \beta_0, \gamma_0, \alpha_1$ in the preceding estimates. By the estimate $\|\alpha_{k+1} - \alpha^*\| < \|\alpha_k - \alpha^*\| < r_2$, we deduce that $\alpha_{k+1} \in B(\alpha^*, r_2)$ and $\lim_{k\to\infty} \alpha_k = \alpha^*$. Let $T = \int_0^1 \mathcal{F}'(y^* + t(\alpha^* - y^*))dt$ for some $y^* \in \bar{B}(\alpha^*, R)$ with $\mathcal{F}(y^*) = 0$. Using (2.12) and the estimate

$$\|\mathcal{F}'(\alpha^*)^{-1}(T-\mathcal{F}'(\alpha^*))\| \leq \int_0^1 Ł_0 \|y^*+t(\alpha^*-y^*)-\alpha^*\|dt \leq \int_0^1 (1-t)\|\alpha^*-y^*\|dt \leq \frac{Ł_0}{2}R<1,$$

it follows that T^{-1} exists. Then, from the identity $0=\mathcal{F}(\alpha^*)-\mathcal{F}(y^*)=T(\alpha^*-y^*)$, we deduce that $\alpha^*=y^*$. □

Remark 2.2. 1. By (2.12) we get that

$$\|\mathcal{F}'(\alpha^*)^{-1}\mathcal{F}'(x)\| = \|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(x)-\mathcal{F}'(\alpha^*))+I\| \leq 1+\|\mathcal{F}'(\alpha^*)^{-1}(\mathcal{F}'(x)-\mathcal{F}'(\alpha^*))\| \leq 1+Ł_0\|x-\alpha^*\|.$$

Then, condition (2.14) can be eliminated and $\mathcal{M}$ can be defined by

$$\mathcal{M}(t)=1+Ł_0\, t$$

or by $\mathcal{M}(t)=\mathcal{M}=2$, since $t\in[0,\frac{1}{Ł_0})$.

2. If the operator $\mathcal{F}$ satisfies the equations of the form [2]

$$\mathcal{F}'(x)=P(\mathcal{F}(x)),$$

with a continuous operator P. Then, we have $\mathcal{F}'(\alpha^*)=P(\mathcal{F}(\alpha^*))=P(0)$. That is we can apply our result without knowing α^*. For example, let $\mathcal{F}(x)=e^x-1$. Then, we can choose: $P(x)=x+1$.

4. The convergence radius r given by (2.1) was shown to be the convergence radius of Newton's method [2,3]

$$\alpha_{n+1}=\alpha_n-\mathcal{F}'(\alpha_n)^{-1}\mathcal{F}(\alpha_n) \text{ for each } n=0,1,2,\ldots \tag{2.28}$$

under the conditions (2.12) and (2.13). By (2.1) and (2.7) the convergence radius r_2 of the method (1.2) is smaller than the convergence radius r of the second convergence order Newton's method (2.28). The radius r is at least as large as the convergence ball given by Rheinboldt [10]

$$r_R=\frac{2}{3Ł}. \tag{2.29}$$

If $Ł_0<Ł$ then, we have that

$$r_R<r$$

and

$$\frac{r_R}{r}\to\frac{1}{3} \quad as \quad \frac{Ł_0}{Ł}\to 0.$$

Then, the radius of convergence r is at most three times larger than Rheinboldt's. This value for r_R is also given by Traub in [11].

5. Method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger $(\mathcal{C})$ conditions used in [13]. We can find the computational order of convergence (COC) defined by

$$\xi=\ln\left(\frac{\|\alpha_{n+1}-\alpha^*\|}{\|\alpha_n-\alpha^*\|}\right)\Big/\ln\left(\frac{\|\alpha_n-\alpha^*\|}{\|\alpha_{n-1}-\alpha^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|\alpha_{n+1}-\alpha_n\|}{\|\alpha_n-\alpha_{n-1}\|}\right) \Big/ \ln\left(\frac{\|\alpha_n-\alpha_{n-1}\|}{\|\alpha_{n-1}-\alpha_{n-2}\|}\right).$$

Then, we obtain in practice the local order of convergence. However, we do not use the bounds requiring higher than the first Fréchet derivative of operator $\mathcal{F}$.

3. Numerical examples

We provide numerical examples in this section. Notice that by Remark 2.2, 1, we can set $\mathcal{M} = 2$ in both examples.

Example 3.1. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^3$, $\Omega = \bar{B}(0,1)$. Let $\mathcal{F}$ be defined on Ω for $v = (x, y, z)^T$ by

$$\mathcal{F}(v) = \left(e^x - 1, \frac{(e-1)y^2}{2} + y, z\right)^T. \tag{3.1}$$

The Fréchet derivative is defined by

$$\mathcal{F}'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $\alpha^* = (0,0,0)^T$, $\mathcal{F}'(\alpha^*) = \mathcal{F}'(\alpha^*)^{-1} = diag\{1,1,1\}$, $Ł_0 = e - 1 < Ł = 1.789572397$ and $\mathcal{M} = 2$. Then, we have

$$r_2 = 0.1149 < r_1 = 0.1532 < r = 0.3922$$

and

$$\|\alpha_1 - \alpha_0\| = 1.9667e^{-05}, \|\alpha_2 - \alpha_0\| = 4.7588e^{-17}.$$

The values of $Ł_0$ and $Ł$ are found in [2,3]. Hence, conditions (2.12)–(2.15) are satisfied. Moreover, condition (2.16) is also satisfied, since $\bar{B}(\alpha^*, r_2) \subset \Omega$. Hence, the conclusions of Theorem 2.1 apply in this case.

Example 3.2. In view of the example at the introduction, we have $Ł_0 = Ł = 96.662907$ and $\mathcal{M} = 2$. Then, we have by the definition of the radii that,

$$r_2 = 0.0024 < r_1 = 0.0028 < r = 0.0069$$

and

$$\|\alpha_1 - \alpha_0\| = 1.9666e^{-05}, \|\alpha_2 - \alpha_0\| = 4.7587e^{-17}.$$

The values of $Ł_0$ and $Ł$ are found using (2.13), (2.14) and by finding the maximum of the function $\mathcal{F}''(x) = 6x\ln x^2 + 20x^3 - 12x^2 + 10x$ using Matlab on the interval $\Omega = [-\frac{1}{2}, \frac{5}{2}]$ and dividing by three, since $\mathcal{F}'(\alpha^*) = 3$. That is conditions (2.12)–(2.15) are satisfied. Finally, condition (2.16) is also satisfied, since $\bar{B}(\alpha^*, r_2) \subset \Omega$. As already noted in the introduction the results in [13] cannot apply to solve this equation.

In the next example, $F^{(4)}$ is not bounded, so the results in [13] cannot apply.

Example 3.3. Let $\mathcal{X} = \mathcal{Y} = C[0,1], \Omega = \bar{B}(\alpha^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type defined by

$$x(s) = \int_0^1 Q(s,t)\left(x(t)^{\frac{5}{2}} + \frac{x(t)^2}{2}\right) dt, \tag{3.2}$$

where the kernel Q is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$Q(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases} \tag{3.3}$$

Define $\mathcal{F} : C[0,1] \longrightarrow C[0,1]$ by

$$\mathcal{F}(x)(s) := x(s) - \int_0^1 Q(s,t)\left(x(t)^{\frac{5}{2}} + \frac{x(t)^2}{2}\right) dt, \tag{3.4}$$

and

$$\mathcal{F}(x)(s) = 0.$$

Notice that $\alpha^*(s) = 0$ is one of the solutions of (1.1). Using (3.4), we obtain

$$\left\|\int_0^1 Q(s,t)dt\right\| \leq \frac{1}{8}. \tag{3.5}$$

Then, by (3.4)–(3.5), we have that

$$\|\mathcal{F}'(x) - \mathcal{F}'(y)\| \leq \frac{1}{8}\left(\frac{5}{2}\|x-y\|^{\frac{3}{2}} + \|x-y\|\right). \tag{3.6}$$

In view of (3.6), the earlier results requiring $\mathcal{F}^{(4)}$ to be bounded (such as [13]) cannot apply. However, our results can apply, if we choose $Ł_0 = Ł = \frac{1}{8}(\frac{5}{2}\sqrt{2}+1)$ and $\mathcal{M} = 2$. Then, we have by the definition of the radii that,

$$r_2 = 0.02808 < r_1 = 0.4298$$

and

$$\|\alpha_1 - \alpha_0\| = 1.9667e^{-05}, \|\alpha_2 - \alpha_0\| = 4.7588e^{-17}.$$

4. Conclusion

We provide a local convergence analysis of a fifth order method to compute a solution of an equation in a Banach space setting. Previous convergence analysis (on the real line) is based on conditions reaching up to the fourth Fréchet-derivative [1–13]. In this paper the local convergence analysis is based only on Lipschitz conditions of the first Fréchet-derivative. Hence, the usage of these methods is extended.

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