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LOCAL RESULTS FOR AN ITERATIVE METHOD OF CONVERGENCE ORDER SIX AND EFFICIENCY INDEX 1.8171

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Abstract. We present a local convergence analysis of an iterative method of convergence order six and efficiency index 1.8171 in order to approximate a locally unique solution of a nonlinear equation. In earlier studies such as [16] the convergence order of these methods was given under hypotheses reaching up to the fourth derivative of the function although only the first derivative appears in these methods. In this paper, we expand the applicability of these methods by showing convergence using only the first and second derivatives. Moreover, we compare the convergence radii and provide computable error estimates for these methods using Lipschitz constants.

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1. Introduction

The problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq \mathbb{R} \to \mathbb{R}$ is a nonlinear function, D is a convex subset of $\mathbb{R}$ has many applications in mathematics and engineering. Newton-like methods are famous for finding solution of (1.1). These methods are usually studied based on: semi-local (that is based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure) and local convergence (that is based on the information around a solution, to find estimates of the radii of convergence balls [1–25]).

Many authors (see [1–25]) have used higher order multi-point methods for approximating a locally unique solution x^* of (1.2). Higher order methods such as Euler's, Halley's, super Halley's, Chebyshev's [1–25] require the evaluation of the higher order derivative of F at each step, which in general is very expensive.

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In this paper we present the local convergence analysis of method defined for each $n = 0, 1, 2, \cdots$ by

$$
\begin{aligned}
y_0 &= x_0, \\
x_1 &= x_0 - A_0^{-1}F(x_0), \\
y_n &= x_n - A_{n-1}^{-1}F(x_n), \\
x_{n+1} &= x_n - A_n^{-1}F(x_n), \qquad (1.2)
\end{aligned}
$$

where x_0 is an initial point and

$$A_n = F'(\frac{1}{2}(x_n + y_n)) - \frac{1}{2}\frac{F(x_n)F''(\frac{1}{2}(x_n + y_n))}{F'(\frac{1}{2}(x_n + y_n))}.$$

Method (1.2) was introduced and studied in [15]. The motivation and favorable comparisons were also given in [15]. The sixth order of convergence was shown in [16] using Taylor expansions, Maple software and hypotheses reaching up to the fourth derivative. The efficiency index is $6^{\frac{1}{3}} = 1.8171$ which is larger than the efficiency indices of other methods (see Table 1).

Method	Number of function or derivative evaluations	Efficiency index
Newton, quadratic	2	$2^{\frac{1}{2}} \approx 1.4142$
Cubic methods	3	$3^{\frac{1}{3}} \approx 1.4422$
Kou's 5^{th} order [25]	4	$5^{\frac{1}{4}} \approx 1.4953$
Kou's 6^{th} order [25]	4	$6^{\frac{1}{4}} \approx 1.5651$
Jarratt's 4^{th} order	3	$4^{\frac{1}{3}} \approx 1.5874$
Secant	1	$0.5(1+\sqrt{5}) \approx 1.6180$
Modified Halley's method	43	$6^{\frac{1}{3}} \approx 1.8171$

Table 1: table 1. Comparison of efficiencies of various methods

However, the hypotheses up to the fourth derivative of function F limit the applicability of these methods. As a motivational example, let us define the function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$
\begin{aligned}
f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \; f'(1) = 3, \\
f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\
f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22.
\end{aligned}
$$

Then, obviously, function f''' is unbounded on D. Notice that, in particular there is a plethora of iterative methods for approximating solutions of nonlinear equations defined on $\mathbb{R}$ [1–25]. These results show that if the initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^* should the initial guess x_0 be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.2) in Section 2. The same technique can be used to other methods [1–25].

In the present paper we only use hypotheses up to the second derivative. This way we expand the applicability of these methods.

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of the method. The numerical examples are presented in the concluding Section 3.

2. Local convergence analysis

We present the local convergence analysis of method (1.2) in this section. Let $L_0 > 0, L > 0, N > 0$ and $M \geq 1$ be parameters. It is convenient for the local convergence analysis of method (1.2) that follows to introduce some scalar functions and parameters. Define functions p, q, h_p and h_q on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned} p(t) &= (L_0 + \frac{MN}{2(1-L_0t)})t, \\ q(t) &= \frac{1}{2}(4L_0 + \frac{MN}{1-L_0t})t, \\ h_p(t) &= p(t) - 1, \end{aligned}$$

and

$$h_q(t) = q(t) - 1.$$

Notice that the functions p, q and h_q are increasing on the interval $[0, \frac{1}{L_0})$. We have that $h_p(0) = -1 < 0$ and $h_p(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. It follows from the intermediate value theorem that the function h_p has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_p the smallest such zero. Similarly, denote the smallest zero of the function h_q on the interval $(0, \frac{1}{L_0})$ by r_q. Notice that $h_q(t) = h_p(t) + \frac{L_0}{2}t$. In particular $h_q(r_p) = h_p(r_p) + \frac{L_0}{2}r_p = \frac{L_0}{2}r_p > 0$, since $h_p(r_p) = 0$ and $r_p > 0$. Hence, we deduce that $r_q < r_p$. Moreover, define functions g_1 and h_1 on the interval $[0, r_p)$ by

$$g_1(t) = \frac{1}{2(1-L_0t)}(Lt + \frac{2Mq(t)}{1-p(t)})$$

and

$$h_1(t) = g_1(t) - 1.$$

The functions g_1 and h_1 are increasing on $[0, r_p)$. We have that $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to r_p^-$. Denote by r_1 the smallest zero of the function h_1 in the interval $(0, r_p)$. Set

$$r = \min\{r_q, r_1\}. \tag{2.1}$$

Then, we have for each $t \in [0, r)$ that

$$0 \le p(t) < 1 \tag{2.2}$$

$$0 \le q(t) < 1, \tag{2.3}$$

and

$$0 \le g_1(t) < 1. \tag{2.4}$$

Let $U(v, \rho)$ denote an interval in $\mathbb{R}$, with center $v \in \mathbb{R}$ and of radius $\rho > 0$. Then, by $\bar{U}(v, \rho)$ we denote the closure of the interval $U(v, \rho)$. Next, we present the local convergence analysis of method (1.2) using the preceding notation.

Theorem 2.1. *Let $F : D \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose that there exist $x^* \in D$, $L_0 > 0, L > 0, N > 0$ and $M \ge 1$ such that for each x, $y \in D$*

$$F(x^*) = 0, \;\; F'(x^*) \neq 0, \tag{2.5}$$

$$|F'(x^*)^{-1}(F'(x) - F'(x^*))| \le L_0|x - x^*|, \tag{2.6}$$

$$|F'(x^*)^{-1}(F'(x) - F'(y))| \le L|x - y|, \tag{2.7}$$

$$|F'(x^*)^{-1}F'(x)| \le M, \tag{2.8}$$

$$|F'(x^*)^{-1}F''(x)| \le N, \tag{2.9}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.10}$$

hold, where the radius r is given by (2.1). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.2) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges linearly to x^*. Moreover, the following estimates hold

$$|y_n - x^*| \le c|x_n - x^*| \le |x_n - x^*| < r \text{ for each } n = 1, 2, \ldots \tag{2.11}$$

and

$$|x_{n+1} - x^*| \le c|x_n - x^*| \le |x_n - x^*| \text{ for each } n = 1, 2, \ldots, \tag{2.12}$$

where,

$$c = g_1(|x_0 - x^*|) \in [0, 1) \tag{2.13}$$

and the function g_1 is as defined previously. Furthermore, for $T \in [r, \frac{2}{L_0})$ the solution x^* is unique in $D_0 := \bar{U}(x^*, T) \cap D$.

Proof. We shall show estimates (2.11) and (2.12) using mathematical induction. By hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.1) and (2.6), we get since $|\frac{1}{2}(x_0 + y_0) - x^*| \le \frac{1}{2}(|x_0 - x^*| + |y_0 - x^*|) < r$ that

$$\begin{aligned} |F'(x^*)^{-1}(F'(\frac{x_0+y_0}{2}) - F'(x^*))| &\le L_0|\frac{x_0+y_0}{2} - x^*| \\ &\le \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|) \\ &= L_0|x_0 - x^*| < L_0 r < 1. \end{aligned} \tag{2.14}$$

It follows from (2.14) that $F'(\frac{x_0+y_0}{2}) \neq 0$ and by the Banach Lemma on invertible functions [3, 4, 24]

$$|F'(\frac{x_0+y_0}{2})^{-1}F'(x^*)| \le \frac{1}{1 - \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|)}. \tag{2.15}$$

We can write by (2.5) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.16}$$

Notice that $|x^* + \theta(y_0 - x^*) - x^*| = \theta|y_0 - x^*| < r$. That is $x^* + \theta(y_0 - x^*) \in U(x^*, r)$. Then, by (2.8) and (2.16), we get that

$$|F'(x^*)^{-1}F(x_0)| \le M|x_0 - x^*|. \tag{2.17}$$

Next, we show that $A_0 \neq 0$. We have by (2.1), (2.2), (2.9), (2.15) and (2.17) that

(2.18)

$$\begin{aligned} &|F'(x^*)^{-1}(A_0 - F'(x^*))| \\ &\le |F'(x^*)^{-1}(F'(\frac{1}{2}(x_0 + y_0)) - F'(x^*))| \\ &\quad + \frac{1}{2}|F'(x^*)^{-1}F(x_0)||F'(x^*)^{-1}F'(\frac{1}{2}(x_0 + y_0))||F'(\frac{1}{2}(x_0 + y_0))^{-1}F'(x^*)| \\ &\le L_0|\frac{1}{2}(x_0 + y_0) - x^*| + \frac{MN|x_0 - x^*|}{2(1 - \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|))} \\ &\le \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|) + \frac{MN|x_0 - x^*|}{2(1 - \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|)} \\ &\le [L_0 + \frac{MN}{2(1 - \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|))}]|x_0 - x^*| \\ &\le p(|x_0 - x^*|) < 1. \end{aligned}$$

That is, $A_0 \neq 0$ and

$$|A_0^{-1}F'(x^*)| \leq \frac{1}{1-p(|x_0-x^*|)}. \tag{2.19}$$

Hence, x_1 is well defined. Notice that we can write in turn that

$$\begin{aligned}
x_1 - x^* &\\
= \;& (x_0 - x^* - F'(x_0)^{-1}F(x_0)) + (F'(x_0)^{-1} - A_0^{-1})F(x_0)\\
= \;& (x_0 - x^* - F'(x_0)^{-1}F(x_0)) + (F'(x_0)^{-1}F'(x^*))(F'(x^*)^{-1}(A_0 - F'(x_0))\\
& \times(A_0^{-1}F'(x^*))(F'(x^*)^{-1}F(x_0)),
\end{aligned}$$

so

$$\begin{aligned}
&|x_1 - x^*|\\
&\quad\leq |x_0 - x^* - F'(x_0)^{-1}F(x_0)| + |F'(x_0)^{-1}F'(x^*)|\\
&\qquad\times|F'(x^*)^{-1}(A_0 - F'(x_0))|\\
&\qquad\times|A_0^{-1}F'(x^*)||F'(x^*)^{-1}F(x_0)|. \qquad (2.20)
\end{aligned}$$

Moreover, we also have that

$$\begin{aligned}
|F'(x^*)^{-1}(A_0 - F'(x_0))| &\leq L_0\left|\frac{x_0+y_0}{2} - x^*\right| + L_0|x_0 - x^*|\\
&\quad+\frac{1}{2}\frac{|F'(x^*)^{-1}F(x_0)||F'(x^*)^{-1}F''(\frac{1}{2}(x_0+y_0))|}{|F'(x^*)^{-1}F'(\frac{1}{2}(x_0+y_0))|}\\
&\leq 2L_0|x_0 - x^*|\\
&\quad+\frac{1}{2}\frac{MN|x_0 - x^*|}{1-\frac{L_0}{2}(|x_0-x^*| + |y_0 - x^*|)}\\
&\leq q(|x_0 - x^*|). \qquad (2.21)
\end{aligned}$$

Then, in view of (2.1), (2.4), (2.7), (2.19), (2.20) and (2.21) we get in turn that

$$\begin{aligned}
&(2.22)\\
&|x_1 - x^*|\\
&\leq |F'(x_0)^{-1}F'(x^*)||\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta|\\
&\quad+|F'(x_0)^{-1}F'(x^*)||F'(x^*)^{-1}(A_0 - F'(x_0))||A_0^{-1}F'(x^*)||F'(x^*)^{-1}F(x_0)|\\
&\leq \frac{L|x_0 - x^*|^2}{2(1-L_0|x_0 - x^*|)} + \frac{Mq(|x_0 - x^*|)|x_0 - x^*|}{(1-L_0|x_0-x^*|)(1-p(|x_0-x^*|))}\\
&\leq g_1(|x_0 - x^*|)|x_0 - x^*| \leq |x_0 - x^*| < r,
\end{aligned}$$

which shows (2.12) for $n = 0$ and $x_1 \in U(x^*, r)$, where we also used the estimates $|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \leq L_0|x_0 - x^*| < L_0 r < 1$, so as in (2.15)

$|F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{1-L_0|x_0-x^*|}$. Furthermore, we also get by (2.3), (2.6), (2.9), (2.15) and (2.17) that

$$\begin{aligned}
&|F'(x^*)^{-1}(A_0 - F'(x_1))| \\
&\quad \leq |F'(x^*)^{-1}(F'(\frac{1}{2}(x_0+y_0)) - F'(x^*))| \\
&\qquad +|F'(x^*)^{-1}(F'(x_1) - F'(x^*))| \\
&\qquad +\frac{1}{2}\frac{|F'(x^*)^{-1}F(x_0)||F'(x^*)^{-1}F''(\frac{1}{2}(x_0+y_0))|}{|F'(x^*)^{-1}F'(\frac{1}{2}(x_0+y_0))|} \\
&\quad \leq \frac{L_0}{2}(|x_0-x^*|+|y_0-x^*|) \\
&\qquad +L_0|x_1-x^*| + \frac{MN|x_0-x^*|}{2(1-\frac{L_0}{2}(|x_0-x^*|+|y_0-x^*|))} \\
&\quad \leq 2L_0|x_0-x^*| + \frac{MN|x_0-x^*|}{2(1-L_0|x_0-x^*|)} \\
&\quad = q(|x_0-x^*|). \qquad (2.23)
\end{aligned}$$

We have by the third sub-step of method (1.2) for $n=0$ that

$$\begin{aligned}
|y_1 - x^*| &\leq |x_1 - x^* - F'(x_1)^{-1}F(x_1)| \\
&\quad +|F'(x_1)^{-1}F'(x^*)||F'(x^*)^{-1}(A_0 - F'(x_1))| \\
&\quad \times|A_0^{-1}F'(x^*)||F'(x^*)^{-1}F(x_1)|. \qquad (2.24)
\end{aligned}$$

Then, we also get by (2.1), (2.20) and (2.22) that

$$\begin{aligned}
|y_1 - x^*| &\leq \frac{L|x_1-x^*|^2}{2(1-L_0|x_1-x^*|)} + \frac{Mq(|x_0-x^*|)|x_1-x^*|}{(1-L_0|x_1-x^*|)(1-p(|x_0-x^*|))} \\
&\leq g_1(|x_0-x^*|)|x_1-x^*| \leq |x_1-x^*| < r, \qquad (2.25)
\end{aligned}$$

which shows (2.11) for $n=1$ and $y_1 \in U(x^*,r)$ where we also used the estimates $|F'(x^*)^{-1}(F'(x_1) - F'(x^*))| \leq L_0|x_1-x^*| < L_0 r < 1$, so $|F'(x_1)^{-1}F'(x^*)| \leq \frac{1}{1-L_0|x_0-x^*|}$. Then, as in (2.19) and (2.21), respectively, we get that

$$|A_1^{-1}F'(x^*)| \leq \frac{1}{1-p(|x_0-x^*|)} \qquad (2.26)$$

and

$$|F'(x^*)^{-1}(A_1 - F'(x_1))| \leq q(|x_1-x^*|). \qquad (2.27)$$

It then follows from (2.1), (2.4), (2.7), (2.20), (2.26) and (2.27) and the last substep of method (1.2) for $n=1$ since

$$x_2 - x^* = x_1 - x^* - F'(x_1)^{-1}F(x_1) + F'(x_1)^{-1}(A_1 - F'(x_1))A_1^{-1}F(x_1), \qquad (2.28)$$

that

$$
\begin{aligned}
(2.29)\quad & |x_2 - x^*| \\
\leq\ & |F'(x_1)^{-1}F(x^*)|| \int_0^1 F'(x^* + \theta(x_1 - x^*))(x_1 - x^*)d\theta| \\
& +|F'(x_1)^{-1}F'(x^*)||F'(x^*)^{-1}(A_1 - F'(x_1))||A_1^{-1}F'(x^*)||F'(x^*)^{-1}F(x_1)| \\
\leq\ & \frac{L|x_1 - x^*|^2}{2(1 - L_0|x_1 - x^*|)} + \frac{Mq(|x_1 - x^*|)|x_1 - x^*|}{(1 - L_0|x_1 - x^*|)(1 - p(|x_1 - x^*|))} \\
\leq\ & g_1(|x_1 - x^*|)|x_1 - x^*| \\
\leq\ & g_1(|x_0 - x^*|)|x_1 - x^*| \leq |x_1 - x^*| < r,
\end{aligned}
$$

which shows (2.12) for $n = 1$ and $x_2 \in U(x^*, r)$, since function g_1 is increasing on $[0, r)$ and $|x_1 - x^*| \leq |x_0 - x^*|$. By simply replacing x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding estimates we arrive at estimates (2.11) and (2.12). Using the estimate

$$
(2.30)\qquad |x_{k+1} - x^*| \leq g_1(|x_0 - x^*|)|x_k - x^*| \leq c|x_k - x^*| < r,
$$

we deduce that $x_{k+1} \in U(x^*, r)$, $\lim_{k\to\infty} x_k = x^*$. To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$ for some $y^* \in D_0$ with $F(y^*) = 0$. Using (2.6) we get that

$$
\begin{aligned}
|F'(x^*)^{-1}(Q - F'(x^*))| &\leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\
(2.31)\qquad &\leq \int_0^1 (1 - \theta)|x^* - y^*|d\theta \leq \frac{L_0}{2}T < 1.
\end{aligned}
$$

It follows from (2.31) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we conclude that $x^* = y^*$. □

REMARK 2.2. 1. In view of (2.6) and the estimate

$$
\begin{aligned}
|F'(x^*)^{-1}F'(x)| &= |F'(x^*)^{-1}(F'(x) - F'(x^*)) + I| \\
&\leq 1 + |F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq 1 + L_0|x - x^*|
\end{aligned}
$$

condition (2.8) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t$$

or by

$$M = M(t) = 2,$$

since $t \in [0, \frac{1}{L_0})$. In view of (2.7) and (2.9), we can also choose $L = N$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then we can choose $P(x) = x + 1$.

3. The radius $r_A = \frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton's method [3], [4]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots$$

under the conditions (2.6)– (2.7). It follows from the definition that the convergence radius r of the method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method. As already noted in [3, 4] r_A is at least as large as the convergence ball given by Rheinboldt [22]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \quad as \quad \frac{L_0}{L} \to 0.$$

That is, our convergence ball r_A is at most three times larger than Rheinboldt's [22]. The same value for r_R was given by Traub [24].

4. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [16]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{|x_{n+1} - x^*|}{|x_n - x^*|}\right) / \ln\left(\frac{|x_n - x^*|}{|x_{n-1} - x^*|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}\right) / \ln\left(\frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the second derivative of operator F.

3. Numerical Examples

We present numerical examples in this section.

EXAMPLE 3.1. Let $D = (-\infty, +\infty)$. Define the function f of D by

$$f(x) = \sin(x). \tag{3.1}$$

Then we have for $x^* = 0$ that $L_0 = L = M = N = 1$. The parameters are

$$r_p = 0.5000,\ r_q = 0.5000 \text{ and } r_1 = 0.1896 = r.$$

EXAMPLE 3.2. Let $D = [-1, 1]$. Define the function f of D by

$$f(x) = e^x - 1. \tag{3.2}$$

Using (3.2) and $x^* = 0$ we get that $L_0 = e - 1 < L = N = e, M = 2$. The parameters are

$$r_p = 0.1776,\ r_q = 0.1657 \text{ and } r_1 = 0.0503 = r.$$

EXAMPLE 3.3. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = N = 146.6629073$, $M = 2$. The parameters are

$$r_p = 0.0026,\ r_q = 0.0025 \text{ and } r_1 = 0.0007 = r.$$

Conflict of interest: The author(s) declare(s) that there is no conflict of interest regarding the publication of this manuscript.

References

[1] Amat, S., Hernández, M.A., Romero, N., A modified Chebyshev's iterative method with at least sixth order of convergence. Appl. Math. Comput. 206(1) (2008), 164-174.

[2] Amat, S., Busquier, S., Plaza, S., Dynamics of the King's and Jarratt iterations. Aequationes. Math. 69 (2005), 212-213.

[3] Argyros, I.K., Convergence and Application of Newton-type Iterations. Springer, 2008.

[4] Argyros, I.K., Hilout, S., Computational methods in nonlinear Analysis. New Jersey, USA: World Scientific Publ. Co., 2013.

[5] Argyros, I.K., Ren, R., On the convergence of efficient King-Werner type methods of order $1 + \sqrt{2}$. J. Comput. Appl. Math. 285 (2015), 169-180.

[6] Argyros, I.K., Ren, R., On the convergence of King-Werner-Secant type methods free of derivatives. Appl. Math. Comput. 256 (2015), 148–159.

[7] Candela, V., Marquina. A, Recurrence relations for rational cubic methods I: The Halley method. Computing 44 (1990), 169-184.

[8] Chen, J., Some new iterative methods with three-order convergence. Appl. Math. Comput. 181 (2006), 1519-1522.

[9] Chun, C., Neta, B., Scott, M., Basins of attraction for optimal eighth order methods to find simple roots of nonlinear equations. Appl. Math. Comput. 227 (2014), 567-592.

[10] Cordero, A., Torregrosa, J., Variants of Newton's method using fifth order quadrature formulas. Appl. Math. Comput. 190 (2007), 686-698.

[11] Cordero, A., Maimo, J., Torregrosa, J., Vassileva, M.P., Vindel, P., Chaos in King's iterative family. Appl. Math. Lett. 26 (2013), 842-848.

[12] Cordero, A., Magrenan, A.A., Quemada, C., Torregrosa, J.R., Stability study of eight-order iterative methods for solving nonlinear equations. J. Comput. Appl. Math 283 (2015), 75–91.

[13] Ezquerro, J.A., Hernández, M.A., A uniparametric Halley-type iteration with free second derivative. Int. J.Pure and Appl. Math. 6(1) (2003), 99-110.

[14] Gutiérrez, J.M., Hernández, M.A., Recurrence relations for the super-Halley method. Computers Math. Applic. 36(7) (1998), 1-8.

[15] Mc Dougall, T.J., Wotherspoon, S.T., A simple modification of Newton's method to achieve convergence order $1 + \sqrt{2}$. Appl. Math. Letters 2 (2013), 183–187.

[16] Nazeer, W., Modified Halley's method for solving nonlinear equations with convergence order six and efficiency index 1.8171, Appl. Math. Comput. 283 (2015), 57–69.

[17] Noor, M.A, Noor, K.I., Some iterative schemes for nonlinear equations. Appl. Math. Comput. 183 (2006), 774–779.

[18] Parhi, S.K., Gupta, D.K., Semilocal convergence of a Stirling-like method in Banach spaces. Int. J. Comput. Methods 7(02) (2010), 215-228.

[19] Petković, M.S., Neta, B, Petković, Lj., Žunić, J., Multipoint methods for solving nonlinear equations. Elsevier, 2013.

[20] Potra, F.A., Ptak, V., Nondiscrete induction and iterative processes. Research Notes in Mathematics 103, Boston, MA: Pitman Publ., 1984.

[21] Ren, H., Wu, Q., Bi, W., New variants of Jarratt's method with sixth-order convergence. Numer. Algorithms 52(4), (2009), 585-603.

[22] Rheinboldt, W.C., An adaptive continuation process for solving systems of nonlinear equations. In: Mathematical models and numerical methods (A.N.Tikhonov et al. eds.) pub. 3(19), pp. 129-142. Banach Center, Warsaw Poland.

[23] Sharma, J.R., Improved Chebyshev-Halley-methods with sixth and eighth order of convergence. Appl. Math. Comput. 283 (2015), 107–121.

[24] Traub, J.F., Iterative methods for the solution of equations. New Jersey, USA: Prentice Hall Englewood Cliffs, 1964.

[25] Wang, X., Kou, J., Convergence for modified Halley-like methods with less computation of inversion. J. Diff. Eq. and Appl. 19(9) (2013), 1483-1500.

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