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Hyers-Ulam Stability of Linear Operators in Frechet Spaces

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Abstract: Hyers-Ulam stability of a linear operator between Frechet spaces is defined. Necessary and sufficient conditions for the existence of Hyers-Ulam stability of a continuous linear operator from a Frechet space to another Frechet space are given.

Keywords: Hyers-Ulam stability, Frechet space.

1. Introduction

Ulam [6] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability problem of functional equations: "For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism?" In 1941, Hyers [4] gave an answer to the problem by considering approximately mappings as follows. Let E and E' be real Banach spaces. If there exists an $\varepsilon \geq 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in E$, then there exists the unique additive mappings $L : E \to E'$ satisfying

$$\|f(x) - L(x)\| \leq \varepsilon.$$

Rassias [3] provided a generalization of Hyers's theorem which allows the Cauchy difference to be unbounded. Since then several mathematicians were attracted to the result of Rassias and investigated a number of stability problems of functional equations. This stability phenomenon that was introduced and proved by Rassias in his 1978 paper is called *Hyers-Ulam-Rassias stability*.

The notion of the Hyers-Ulam stability of a mapping between two normed spaces was introduced in [2]. Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed spaces and T be a (not necessarily linear) mapping from X into Y. We say that T has the Hyers-Ulam stability if there exists $K > 0$ with the following property : For any $v \in R(T)$, $\varepsilon \geq 0$ and $u \in X$ with $\|Tu - v\|_Y \leq \varepsilon$, there exists a $u_0 \in X$ such that $Tu_0 = v$ and

$$\|u - u_0\|_X \leq K\varepsilon.$$

In other words, if T has the Hyers-Ulam stability, then to each "ε-approximate solution" u of the equation $Tx = v$ there corresponds an exact solution u_0 of the equation in the $K\varepsilon$-neighbourhood of u.

The linearity of T implies the following condition : For any $u \in X$ and $\varepsilon \geq 0$ with $\|Tu\|_Y \leq \varepsilon$, there exists a $u_0 \in X$ such that $Tu_0 = 0$ and $\|u - u_0\|_X \leq K\varepsilon$. The above condition is equivalent to : For given $u \in X$, there is a $u_0 \in X$ such that $Tu = Tu_0$ and $\|u_0\|_X \leq K\|Tu\|_Y$.

We call such $K > 0$ a HUS constant for T, and denote by K_T the infimum of all HUS constants for T. If, in addition, K_T becomes a HUS constant for T, then we call it the HUS constant for T. Miura at el. [2] have given a necessary and sufficient condition for the existence of the best HUS constant. The existence of the best HUS constants for the weighted composition operators and the first order linear differential operators are shown in [1].

We define the Hyers-Ulam stability of a linear operator between Frechet spaces. A Frechet space is a complete metrizable topological vector space [4].

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By an operator we shall mean a non-zero linear operator. $R(T)$ and $N(T)$ denote the range and null spaces of T respectively. The closure of A is denoted by $\overline{A}$.

2. Characterizations

Let X and Y be Frechet spaces and $T : X \to Y$ be a linear operator. T has the Hyers-Ulam stability if for a given open neighbourhood U of 0 in X there is an open neighbourhood V of 0 in Y such that for a given $x \in X$ with $Tx \in V$ there is a $y \in U$ satisfying $Tx = Ty$.

We first give a necessary and sufficient condition in order that T have the Hyers-Ulam stability.

Theorem 1.*Let X and Y be Frechet spaces and $T : X \to Y$ be a continuous linear operator. Then the following statements are equivalent:*

1.T has the Hyers-Ulam stability
2.T has closed range.

*Proof.*Suppose $R(T)$ is closed in Y. We denote $N = N(T)$ and $\widetilde{X} = X/N$. Define $\widetilde{T} : \widetilde{X} \to R(T)$ by

$$\widetilde{T}(x + N) = Tx.$$

Then $\widetilde{T}$ is a one-to-one continuous linear operator from $\widetilde{X}$ onto $R(T)$ and by the open mapping theorem $\widetilde{T}^{-1}$ is continuous. Let $\pi : X \to \widetilde{X}$ be the quotient mapping. Now fix an open neighborhood U of 0 in X. Then $\pi(U) = U + N = \widetilde{U}$ (say) is an open neighborhood of $0 + N$ in $\widetilde{X}$. Then there is an open neighborhood V of 0 in Y such that

$$R(T) \cap V \subseteq \widetilde{T}(\widetilde{U}) = \widetilde{T}(\pi(U)) = T(U).$$

Thus, for a given $x \in X$ with $Tx \in V$, there is a $y \in U$ such that $Tx = Ty$. This proves that T has the Hyers-Ulam stability.

Conversely assume that T has the Hyers-Ulam stability. Let $(U_n)_{n=1}^{\infty}$ be a sequence of balanced open neighborhoods of 0 which form a local base at 0 in X such that $U_{n+1} + U_{n+1} \subseteq U_n$, for every n. For each U_n, let us find an open neighborhood V_n of 0 in Y such that if $Tx \in V_n$ for some $x \in X$, then $Tx = Ty$ for some $y \in U_n$. Without loss of generality, we assume that $\{V_n : n = 1, 2, \cdots\}$ is a local base at 0 in Y such that $V_{n+1} + V_{n+1} \subseteq V_n$ for every n.

Let $y_0 \in \overline{R(T)}$. Find a sequence (x'_n) in X such that $Tx'_n \to y_0$ as $n \to \infty$, and $Tx'_{n+1} - Tx'_n \in V_n$ for every n. For every n, find $x_n \in U_n$ such that $Tx_n = Tx'_{n+1} - Tx'_n \in V_n$. Then, as $m \to \infty$,

$$\begin{aligned}\sum_{n=1}^{m} Tx_n &= (Tx'_2 - Tx'_1) + \cdots + (Tx'_{m+1} - Tx'_m) \\ &= Tx'_{m+1} - Tx'_1 \to y_0 - Tx'_1.\end{aligned}$$

Thus $\sum_{n=1}^{\infty} Tx_n$ converges to $y_0 - Tx'_1$. Also for $m < n$, we have

$$\begin{aligned}x_m + \cdots + x_n &\in U_m + U_{m+1} + \cdots + U_{n-1} + U_n \\ &\subseteq U_m + U_{m+1} + \cdots + U_{n-1} + U_{n-1} \\ &\subseteq U_m + U_{m+1} + \cdots + U_{n-2} + U_{n-2} \\ &\subseteq \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\subseteq U_m + U_m \\ &\subseteq U_{m-1}.\end{aligned}$$

This proves that $\sum_{n=1}^{\infty} x_n$ converges to x_0, say, in the Frechet space X, and hence $\sum_{n=1}^{\infty} Tx_n$ converges to Tx_0 in Y. Therefore

$$Tx_0 = y_0 - Tx'_1 = \sum_{n=1}^{\infty} Tx_n$$

so that $y_0 = Tx_0 + Tx'_1 \in R(T)$. This proves that $R(T)$ is closed in Y.

The following theorem gives a particular version of the Hyers-Ulam stability of a bounded linear operator between Banach spaces which was proved in [5]. However, our proof enjoys the standard technique for the proof of the open mapping theorem.

Theorem 2.*Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Then the following statements are equivalent:*

1.T has the Hyers-Ulam stability
2.T has closed range.

*Proof.*Suppose $R(T)$ is closed in Y. Define $\widetilde{T} : \widetilde{X} \to R(T)$ by $\widetilde{T}(x+N) = Tx$, for $x \in X$. Then $\widetilde{T}$ is a well defined one-to-one continuous linear operator from $\widetilde{X}$ onto $R(T)$. Therefore, by the open mapping theorem, there exists a constant $K' > 0$ such that

$$\|x + N\| \le K' \|\widetilde{T}(x + N)\| = \|Tx\|$$

for every $x \in X$. Take $K = K' + 1$. Then for given $x \in X$, if $Tx \neq 0$, then there is an element $z \in N$ such that

$$\begin{aligned}\|x + z\| &\le \|x + N\| + \|Tx\| \\ &\le K' \|Tx\| + \|Tx\| \\ &= K\|Tx\|.\end{aligned}$$

In this case, we take $y = x + z$ so that $\|y\| \le K\|Tx\|$. If $Tx = 0$, then we take $y = 0$ so that $\|y\| \le K\|Tx\|$. Thus T has the Hyers-Ulam stability with a HUS constant K.

Conversely assume that T has a HUS constant K. Fix $y_0 \in \overline{R(T)}$, the closure of $R(T)$ in Y. Then there is a sequence (x_n) in X such that $\|x_n\| \le K\|Tx_n\|$ and for every $n = 1, 2, 3, \ldots$,

$$\|(y_0 - Tx_1 - Tx_2 - \cdots - Tx_{n-1}) - Tx_n\| \le \frac{1}{2^{n+2}}.$$

Then $\frac{1}{K}\|x_n\| \le \|Tx_n\| \le \|y_0 - Tx_1 - Tx_2 - \cdots - Tx_n\| + \|y_0 - T_{x_1} - T_{x_2} - \cdots - T_{x_{n-1}}\| \le \frac{1}{2^{n+2}} + \frac{1}{2^{n+1}} \le \frac{1}{2^n}$.

Therefore, the series $\sum_{n=1}^{\infty} x_n$ converges to x_0, say, in X and the series $\sum_{n=1}^{\infty} Tx_n$ converges to y_0. Since T is continuous, $\sum_{n=1}^{\infty} Tx_n$ converges to $T(\sum_{n=1}^{\infty} x_n) = Tx_0$. Therefore $y_0 = Tx_0 \in R(T)$. This proves that $R(T)$ is closed in Y.

Corollary 1. *Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Then T has the Hyers-Ulam stability if and only if for a given bounded sequence (y_n) in $R(T)$ there is a bounded sequence (x_n) in X such that $Tx_n = y_n$ for every n.*

Proof. Suppose T has the Hyers-Ulam stability. Then there is a constant $K > 0$ such that for a given $y \in R(T)$, there is an element $x \in X$ such that $\|x\| \le K\|y\|$ and $Tx = y$. Consider a bounded sequence (y_n) in $R(T)$. To each y_n, there is an element $x_n \in X$ such that $Tx_n = y_n$ and $\|x_n\| \le K\|y_n\|$. Then (x_n) is a bounded sequence in X such that $Tx_n = y_n$, for every n.

To prove the converse part, assume that T does not have a Hyers-Ulam stability constant. Then, for every given n, there is an element y_n in X such that $n\|y_n\| < \|x\|$, for any $x \in X$ with $Tx = y_n$; and such that $\|y_n\| = 1$. Therefore, if there is an element x_n such that $Tx_n = y_n$, then $\|x_n\| > n$. Thus, there is no bounded sequence (x_n) in X such that $Tx_n = y_n$ for all n, when (y_n) is a bounded sequence in $R(T)$.

3. Compositions

Let X, Y and Z be Frechet spaces. Let $S : X \to Y$ and $T : Y \to Z$ be linear operators each of which having a HUS constant. It is not ture in general that the composition $TS : X \to Z$ has the Hyers-Ulam stability. The following example shows that even for continuous linear operators between Frechet spaces the composition of operators with HUS constants need not have the Hyers-Ulam stability.

Example 1. Suppose $X = Y = Z = \ell^2$ with the usual norm on this Hilbert space. Define $S : X \to Y$ and $T : Y \to Z$ by

$$S(x_1, x_2, x_3, x_4, \cdots) = (x_1, 0, x_2, 0, x_3, 0, x_4, 0, \cdots)$$

and

$$T(x_1, x_2, x_3, x_4, \cdots) = (x_1 + x_2, \frac{x_3}{3} + x_4, \frac{x_5}{5} + x_6, \cdots).$$

Then S and T have HUS constants. But TS does not have a HUS constant because $R(TS)$ is not closed in Z.

We provide a necessary and sufficient condition which gives that the composition of operators with HUS constants is again an operator with HUS constant.

Theorem 3. *Suppose $S : X \to Y$ and $T : Y \to Z$ are continuous linear operators between Frechet spaces such that S and T have the Hyers-Ulam stability. Then TS has the Hyers-Ulam stability if and only if $R(S) + N(T)$ is closed in Y.*

Proof. Let $Y' = R(S) + N(T)$. Then $R(S)$, $N(T)$ and Y' are Frechet spaces under the subspace topologies. Define $P : R(S) \times N(T) \to Y'$ by

$$P(x, y) = x + y,$$

for $x \in R(S)$ and $y \in N(T)$. Then P is a continuous linear mapping when the domain is endowed with the product topology and the coordinatewise algebraic operations. Then, by the open mapping theorem, for a given open neighborhood V_1 of 0 in $R(S)$ and V_2 of 0 in $N(T)$, there is an open neighborhood V_3 of 0 in Y' such that $V_1 + V_2 \supseteq V_3$. We shall use this observation in the following part.

Fix an open neighborhood U of 0 in X and find, by theorem 1, an open neighborhood V of 0 in Y such that if $x_1 \in X$ and $Sx_1 \in V$, then there is a $x_2 \in U$ such that $Sx_1 = Sx_2$. Find an open neighborhood V' of 0 in Y such that

$$V' \cap Y' \subseteq [V \cap R(S)] + [V \cap N(T)].$$

For this neighborhood V', we find, by theorem 1, an open neighborhood W of 0 in Z such that if $y_1 \in Y$ and $Ty_1 \in W$, then $Ty_1 = Ty_2$ for some $y_2 \in V'$.

Now if $x_1 \in X$ and $T(Sx_1) \in W$, then there is a $y_2 \in V'$ such that $T(Sx_1) = Ty_2$. Then $y_2 - Sx_1 \in N(T)$ and $y_2 = Sx_1 + (y_2 - Sx_1) \in R(S) + N(T) = Y'$ which implies that

$$y_2 \in V' \cap Y' \subseteq [V \cap R(S)] + [V \cap N(T)].$$

Therefore, there are y_3 and y_4 in Y such that $y_3 \in V \cap R(S)$ and $y_4 \in V \cap N(T)$ and $y_2 = y_3 + y_4$. Since $y_3 \in V \cap R(S)$, there is a $x_2 \in U$ such that $Sx_2 = y_3$. Therefore $T(Sx_1) = Ty_2 = T(y_3 + y_4) = T(Sx_2) + Ty_4 = T(Sx_2) + 0 = T(Sx_2)$. Thus, for a given $x_1 \in X$ with $(TS)x_1 \in W$, there is a $x_2 \in U$ such that

$$(TS)x_1 = (TS)x_2.$$

This proves that $R(TS)$ is closed in Z. The proof for the other way implication comes from

$$R(S) + N(T) = T^{-1}[T(S(X))]$$

and $T(S(X))$ is closed in Z.

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