and, by (14)

$$\hat{U}(s)=\begin{pmatrix} se^{-sh} & -1 \\ 1 & 0\end{pmatrix}, \quad \hat{X}(s)=\begin{pmatrix} 0 & 0 \\ e^{-sh}-1 & 0\end{pmatrix}.$$

Starting, e.g., from zero initial conditions except for $x(0)=(x_{10}, x_{20})^T$ we get the Laplace transforms of the distributional control and trajectory

$$\hat{u}(s)=\left((se^{-sh}-s)x_{10}-x_{20}, x_{10}\right)^T, \quad \hat{x}(s)=\left(0,(e^{-sh}-1)x_{10}\right)^T.$$

To get regular controls we use Procedure 2. Solving (16) we obtain (nonuniquely)

$$\overline{U}_1=\begin{pmatrix} c^2(d-1) & -2cd(d-1) \\ -c^2 & 2cd\end{pmatrix},$$

$$\overline{U}_2=\begin{pmatrix} c^3(d-1) & -2c^2d(d-1)-c^2 \\ -c^2(c+1) & 2c(c+1)d\end{pmatrix}.$$

Choose $c_1=0$, $c_2=1$. Hence, $U^1=0$, $X^1=\theta_0(1-d)I+\theta_0^2A$ and

$$U^2=\theta_1\begin{pmatrix} \left(e^{-h}-d+\theta_1\right)(d-1) & -2d(d-1)\left(e^{-h}-d+\theta_1\right)-\theta_1 \\ -e^{-h}+d-2\theta_1 & 2d\left(e^{-h}-d+2\theta_1\right)\end{pmatrix},$$

$$X^2=\theta_1\begin{pmatrix} e^{-h}-d-\theta_1(d^2+1) & \theta_1(2d^3+d+1) \\ \theta_1(d-1) & \theta_1(-2d^2+2d-1)+e^{-h}-d\end{pmatrix}.$$

Next, we solve (21), which takes the form

$$q_1(1-d)^2+q_2(e^{-h}-d)^2=1$$

obtaining the following solution (of minimal degree)

$$q_1=(e^{-h}-1)^{-3}(-2d+3e^{-h}-1), \quad q_2=(e^{-h}-1)^{-3}(2d+e^{-h}-3).$$

Performing Steps 5–7 we get formulas for Laplace transforms of regular control and trajectory corresponding to given initial conditions. For instance, if $\Phi=0$, $\Psi=0$, and $x(0)=[1,1]^T$, the control is given by $\hat{u}(s)=[\hat{u}_1(s), \hat{u}_2(s)]^T$

$$\begin{aligned}\hat{u}_1(s)&=(e^{-h}-1)^{-3}(2e^{-sh}+e^{-h}-3)\hat{\theta}_1(s)\\ &\cdot\left[(e^{-sh}-1)(e^{-h}-e^{-sh})(1-2e^{-2sh})\right.\\ &\left.+\hat{\theta}_1(s)(-2+3e^{-sh}-2e^{-2sh})\right]\\ \hat{u}_2(s)&=(e^{-h}-1)^{-3}(2e^{-sh}+e^{-h}-3)\hat{\theta}_1(s)\\ &\cdot\left(e^{-h}-e^{-sh}+2\hat{\theta}_1(s)(-1+2e^{-sh}\right)\end{aligned}$$

where

$$\hat{\theta}_1(s)=(e^{-h}-e^{-sh})(s-1)^{-1}.$$

The Laplace transform $\hat{x}(s)=[\hat{x}_1(s), \hat{x}_2(s)]^T$ of the state trajectory is

$$\begin{aligned}\hat{x}_1(s)&=\hat{q}_1(s)\hat{\theta}_0(s)\left(1-e^{-sh}+\hat{\theta}_0(s)+\hat{\theta}_0(s)e^{-sh}\right)\\ &+\hat{q}_2(s)\hat{\theta}_1(s)\left(e^{-h}-e^{-sh}+\hat{\theta}_1(s)(e^{-sh}-e^{-2sh}+2e^{-3sh})\right)\\ \hat{x}_2(s)&=\hat{q}_1(s)\hat{\theta}_0(s)(1-e^{-sh})\\ &+\hat{q}_2(s)\hat{\theta}_1(s)\left(e^{-h}-e^{-sh}+\hat{\theta}_1(s)(-2+3e^{-sh}-2e^{-2sh})\right)\end{aligned}$$

where

$$q_1(s)=(e^{-h}-1)^{-3}(-2e^{-sh}+3e^{-h}-1),$$

$$q_2(s)=(e^{-h}-1)^{-3}(2e^{-sh}+e^{-h}-3),$$

and

$$\hat{\theta}_0(s)=(1-e^{-sh})s^{-1}.$$

IV. Conclusions

Under the assumption of reachability over the polynomial ring $\mathbb{R}[d]$ of the matrix pair characterizing a linear system with commensurate delays it has been shown that the system can be controlled to the origin and stay there while the control also vanishes identically after some time. This means that the full state of the system becomes zero identically after some finite time. Two constructive procedures have been presented which allow us to calculate easily Laplace transforms of the control and state trajectory.

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On Model Reduction by Modified Cauer Form

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***Abstract*—A simple algorithm for obtaining the continued fraction quotients in the modified Cauer form (MCF) from the given system matrices in companion form is presented. In the sequel, the triple of all lower order models in companion form is directly obtained. A matrix method of obtaining the time-moments and Markov parameters from the MCF quotients is also outlined. Finally, it is shown that system reduction by matching a set of MCF quotients is equivalent to system reduction by matching a set of time-moments and Markov parameters.**

I. Introduction

The problem considered by Khatwani *et al.* [1] is how to obtain the scalar quotients $h_1, k_1, h_2, k_2, \cdots$ in the following modified Cauer form (MCF) representation:

$$g(s)=\cfrac{1}{h_1+\cfrac{s}{k_1+\cfrac{1}{h_2+\cfrac{s}{k_2+\cfrac{1}{\vdots}}}}} \tag{1}$$

given the system matrices (A, B, C) in companion form. Chuang [9] modified the continued fraction technique for model reduction into this form to overcome instability.

Manuscript received February 22, 1982; revised June 14, 1982.

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0018-9286/83/0400-0523$01.00 ©1983 IEEE

The procedure proposed in [1] requires the construction of a large array, the computation of whose elements involves first the evaluation of higher powers of A and A^{-1} and then the matrix products CA^kB, $k=0,1,2,\cdots$, and $CA^{-k}B$, $k=1,2,\cdots$.

In this paper, we first formulate a Routh-type array which is simple to construct. We then evaluate the scalar quotients of MCF and use them to develop an inversion table from which the triple of all lower order models in companion form is directly read off. A matrix method to obtain the time-moments and Markov parameters of the system is also presented. Finally, we prove that the method of model simplification via the MCF is equivalent to model reduction by matching the first k time-moments and first k Markov parameters corresponding to the kth order of the reduced model. In this respect, the result for the second Cauer form is available in [7], [8].

II. Development of the Algorithm

Given the system matrices in companion form, viz.

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{1,1} & -a_{1,2} & -a_{1,3} & \cdots & -a_{1,n} \end{bmatrix}$$

$$B=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\quad C=[b_{1,1}b_{1,2}b_{1,3}\cdots b_{1,n}]. \tag{2}$$

The transfer function can straightaway be written as

$$g(s)=\frac{b_{1,1}+b_{1,2}s+b_{1,3}s^2+\cdots+b_{1,n}s^{n-1}}{a_{1,1}+a_{1,2}+a_{1,3}s^2+\cdots+a_{1,n}s^{n-1}+s^n} \tag{3}$$

from which, using algorithm [3], we evaluate the MCF quotients by formulating the modified Routh array as follows:

$$\begin{array}{lcccccc} & a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} & 1 \\ h_1=\dfrac{a_{1,1}}{b_{1,1}} & & & & & & \\ & b_{1,1} & b_{1,2} & \cdots & b_{1,n-1} & b_{1,n} & k_1=b_{1,n} \\ & a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & 1 & \\ & \vdots & \vdots & & & & \\ & a_{n,1} & 1 & & & & \\ h_n=\dfrac{a_{n,1}}{b_{n,1}} & & & & & & \\ & b_{n,1} & k_n=b_{n,1}. & & & & \\ & 1 & & & & & \end{array} \tag{4}$$

Once the MCF quotients are evaluated, we formulate the inversion table [4]:

$$\begin{array}{lcccc|ccccc} & p_{1,1} & & & & l_{1,1} & & & & \\ & q_{1,1} & 1 & & & r_{1,1} & & & & \\ & p_{2,1} & p_{2,2} & & & l_{2,1} & & & & \\ k_2 & & & & & & & & & \\ & q_{2,1} & q_{2,2} & 1 & & r_{2,1} & r_{2,2} & h_3 & & \\ & \vdots & \vdots & & & \vdots & \vdots & & & h_n \\ & p_{n,1} & p_{n,2}\cdots & p_{n,n} & & l_{n,1} & l_{n,2}\cdots & l_{n,n-1} & & \\ k_n & & & & & & & & & \\ & q_{n,1} & q_{n,2}\cdots & q_{n,n} & 1 & r_{n,1} & r_{n,2}\cdots & r_{n,n-1} & r_{n,n}. & \end{array} \tag{5}$$

The first two rows are built out of the elements $p_{1,1}=h_1$, $q_{1,1}=h_1k_1$, $q_{1,2}=1$, $l_{1,1}=1$, and $r_{1,1}=k_1$. The subsequent rows are evaluated for $i=2,3,\cdots,n$ by the recursive relations

$$\begin{aligned} p_{i,j}&=p_{i-1,j}+q_{i-1,j}h_i, & j&=1,2,\cdots,i \\ q_{i,j}&=q_{i-1,j-1}+p_{i,j}k_i, & j&=1,2,\cdots,i+1 \\ l_{i,j}&=l_{i-1,j}+r_{i-1,j}h_i, & j&=1,2,\cdots,i-1 \\ r_{i,j}&=r_{i-1,j-1}+l_{i,j}k_i, & j&=1,2,\cdots,i \end{aligned} \tag{6}$$

with $q_{i,0}=0$ and $r_{i,0}=0$.

Lower Order Canonical Realizations

These are obtained in the form defined in (2), by reading off from (5) the values $a_{1,1}, a_{1,2},\cdots$ and $b_{1,1}, b_{1,2},\cdots$ as per the relations

$$\begin{aligned} a_{1,i}&=q_{k,i} \\ b_{1,i}&=r_{k,i} \\ k&=2,3,\cdots,n-1,\ i=1,2,\cdots,k. \end{aligned} \tag{7}$$

Evaluation of Time-Moments and Markov Parameters

From the entries of the inversion table in (5), the weighted time-moments C_i (viz. $C_i=\{(-1)^i/(i!)\}m_i$, where m_i is the ith time-moment of the system) and Markov parameters D_i are determined by the following matrix relations [2]:

$$\begin{bmatrix} q_{1,1} & & & & \\ q_{2,2} & q_{2,1} & & \bigcirc & \\ q_{3,3} & q_{3,2} & q_{3,1} & & \\ \vdots & \vdots & \vdots & \ddots & \\ q_{n,n} & q_{n,n-1} & q_{n,n-2} & \cdots & q_{n,1} \end{bmatrix}\begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_{n-1} \end{bmatrix}=\begin{bmatrix} r_{1,1} \\ r_{2,2} \\ r_{3,3} \\ \vdots \\ r_{n,n} \end{bmatrix} \tag{8}$$

and

$$\begin{bmatrix} 1 & & & & \\ q_{2,2} & 1 & & \bigcirc & \\ q_{3,2} & q_{3,3} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ q_{n,2} & q_{n,3} & q_{n,4} & \cdots & 1 \end{bmatrix}\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_n \end{bmatrix}=\begin{bmatrix} r_{1,1} \\ r_{2,1} \\ r_{3,1} \\ \vdots \\ r_{n,1} \end{bmatrix}. \tag{9}$$

The Q-matrices composed of $q_{i,j}$ elements as shown in (8) and (9) are formed, respectively, out of the elements in (5) as defined by

$$\begin{aligned} &q_{i,i-j}, & &i=1,2,\cdots,n,\ j=0,1,\cdots,i-1 \\ &q_{i,j}, & &i=1,2,\cdots,n,\ j=2,3,\cdots,i+1. \end{aligned} \tag{10}$$

III. Equivalence of the MCF Technique with Mixed Method

In the MCF approach to system reduction, the model parameters for the lower order model are obtained by solving the set of equations

$$\begin{aligned} h_i'&=h_i \\ k_i'&=k_i \end{aligned}\qquad i=1,2,\cdots \tag{11}$$

while in the mixed method [5], the model parameters are obtained by solving the equations

$$\begin{aligned} C_i'&=C_i, & i&=0,1,2,\cdots \\ D_i'&=D_i, & i&=1,2,\cdots. \end{aligned} \tag{12}$$

We now prove that matching in (11) is equivalent to matching in (12).

Proof: From the recursive relations in (6), we observe that

$$r_{1,1}=r_{2,2}=r_{3,3}=\cdots=r_{n,n}=k_1. \tag{13}$$

From (6), (8), and (9), $h_1=q_1$, $1/k_1$, $C_0=r_{1,1}/q_{1,1}=k_1/q_{1,1}$, and $D_1=r_{1,1}$. Thus, $h_1=1/C_0$ and $k_1=D_1$. Matching $h_1'=h_1$ and $k_1'=k_1$, we obtain

$$C_0' = C_0 \quad \text{and} \quad D_1' = D_1. \tag{14}$$

From (8), $q_{2,2}C_0 + q_{2,1}C_1 = r_{2,2}$. Substituting for $q_{2,2}, q_{2,1}, r_{2,2}$, and C_0 from (6) and (13) we get

$$(h_1k_1 + h_2k_2)\cdot(1/h_1) + (h_1k_2 + h_1k_1h_2k_2)C_1 = k_1$$

which gives

$$h_2 = -\left(h_1^2C_1k_2\right)/\left(k_2 + h_1^2k_1k_2C_1\right) = -(C_1)/\left(C_0^2 + C_1D_1\right). \tag{15}$$

Similarly, from $D_1q_{2,2} + D_2 = r_{2,1}$ [see (9)] we obtain

$$k_2 = \left(D_1^2 + C_0D_2\right)/C_0. \tag{16}$$

Matching $h_2' = h_2$ and $k_2' = k_2$ leads to

$$\frac{C_1'}{C_0'^2 + C_1'D_1'} = \frac{C_1}{C_0^2 + C_1D_1}$$

and

$$\frac{D_1'^2 + C_0'D_2'}{C_0'} = \frac{D_1^2 + C_0D_2}{C_0}.$$

Since $C_0' = C_0$ and $D_1' = D_1$, it follows that

$$C_1' = C_1, \quad D_2' = D_2 \tag{17}$$

and so on.

IV. Example

Consider the same example treated earlier in [1] for which lower order models are required:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -90 & -60 & -24 & -5 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [15 \quad 5 \quad 1 \quad 1/6]. \tag{18}$$

The corresponding transfer function is

$$g(s) = \frac{15 + 5s + s^2 + (1/6)s^3}{90 + 60s + 24s^2 + 5s^3 + s^4}. \tag{19}$$

MCF Quotients: By forming the table in (4), we obtain

$$h_1 = 6, k_1 = 1/6, h_2 = 3, k_2 = 1/3, h_3 = 2, k_3 = 1,$$
$$h_4 = 1/5, \text{and } k_4 = 5. \tag{20}$$

The inversion table is as follows

	6					1					
	1	**1**				**1/6**					
	9	3				3/2					
$k_2 = 1/3$											
	3	2	**1**			1/2	**1/6**			$h_3 = 2$	
	15	7	2			5/2	1/3				
$k_3 = 1$											
	15	10	4	**1**		5/2	5/6	**1/6**		$h_4 = 1/5$	
	18	9	14/5	1/5		3	1/2	1/30			
$k_4 = 5$											
	90	60	24	5	**1**	15	5	1	**1/6.**		(21)

Lower Order Models: From (7),

$$a_3 = (-a_{1,1} \; -a_{1,2} \; -a_{1,3}) = (-15 \; -10 \; -4)$$
$$c_3 = [b_{1,1} \; b_{1,2} \; b_{1,3}] = [5/2 \; 5/6 \; 1/6] \tag{22}$$
$$a_2 = (-a_{1,1} \; -a_{1,2}) = (-3, \; -2),$$
$$c_2 = [b_{1,1} \; b_{1,2}] = [1/2 \; 1/6].$$

Weighted Time-Moments and Markov Parameters: From (8) and (9),

$$C_0 = 1/6, C_1 = -1/18, C_2 = 1/270, C_3 = 2/405$$
$$D_1 = 1/6, D_2 = 1/6, D_3 = 1/6, D_4 = 1/6. \tag{23}$$

V. Conclusion

A computationally efficient procedure which involves constructing only a simple modified Routh array is presented to evaluate the continued fraction quotients from the given system matrices in companion form. The saving in computation over the earlier method of Khatwani *et al.* [1] is obvious: the proposed method does not require any matrix inversion or multiplication of the system matrices. The canonical realizations of all lower order models are directly read off from the inversion table.

A matrix method is presented to determine the time moments and Markov parameters from the knowledge of the MCF quotients. Furthermore, it is shown that system reduction by MCF is equivalent to system reduction by matching a set of time-moments and Markov parameters. These results find application in many practical problems, as the canonical realization possesses distinct advantages for simulation studies and for system design [6].

The steps in the algorithm are oriented for easy and direct programming. The relationship between the original state vector and the state vector of the model obtained through MCF quotients has been investigated and has been reported elsewhere [3].

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A Note on the Model Reduction Problem

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***Abstract*—A mixed method of model reduction is proposed; it is based on the differentiation method suggested by Gutman *et al.* [1] and on the**

Manuscript received June 10, 1982.

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0018-9286/83/0400-0525$01.00 ©1983 IEEE