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On strong (weak) independent sets and vertex coverings of a graph

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Abstract

A vertex v in a graph $G=(V,E)$ is *strong* (*weak*) if $\deg(v) \geqslant \deg(u)$ ($\deg(v) \leqslant \deg(u)$) for every u adjacent to v in G. A set $S \subseteq V$ is said to be *strong* (*weak*) if every vertex in S is a strong (weak) vertex in G. A strong (weak) set which is independent is called a *strong independent set* [*SIS*] (*weak independent set* [*WIS*]). The *strong* (*weak*) *independence number* $s\alpha = s\alpha(G)(w\alpha = w\alpha(G))$ is the maximum cardinality of an SIS (WIS). For an edge $x=uv$, v *strongly covers* the edge x if $\deg(v) \geqslant \deg(u)$ in G. Then u *weakly covers* x. A set $S \subseteq V$ is a *strong vertex cover* [*SVC*] (*weak vertex cover* [*WVC*]) if every edge in G is strongly (weakly) covered by some vertex in S. The *strong* (*weak*) *vertex covering number* $s\beta = s\beta(G)$ $(w\beta = w\beta(G))$ is the minimum cardinality of an SVC (WVC).

In this paper, we investigate some relationships among these four new parameters. For any graph G without isolated vertices, we show that the following inequality chains hold: $s\alpha \leqslant \beta \leqslant s\beta \leqslant w\beta$ and $s\alpha \leqslant w\alpha \leqslant \alpha \leqslant w\beta$. Analogous to Gallai's theorem, we prove $s\beta + w\alpha = p$ and $w\beta + s\alpha = p$. Further, we show that $s\alpha \leqslant p - \Delta$ and $w\alpha \leqslant p - \delta$ and find a necessary and sufficient condition to attain the upper bound, characterizing the graphs which attain these bounds. Several Nordhaus–Gaddum-type results and a Vizing-type result are also established.

Keywords: Strong (weak) vertices; Strong (weak) vertex cover; Strong (weak) independent sets

1. Introduction

The terminologies and notations used here are as in [1]. Given a graph $G=(V,E)$, for any $v \in V$ the set $N(v)=\{u \in V \mid uv \in E\}$ is called the *open neighbourhood* of v and $N[v] = N(v) \cup \{v\}$ is called the *closed neighbourhood* of v. For any set $S \subseteq V$, $N(S) = \bigcup_{v\in S} N(v)$ is called the *open neighbourhood* of the set S.

A set $D \subseteq V$ is a *dominating set* of a graph $G=(V,E)$, if every $v \in V - D$ is adjacent to some $u \in D$. The *domination number* $\gamma = \gamma(G)$ is the minimum cardinality of a dominating set of G. This concept is well studied in [6,12]. A set $D \subseteq V$ is said to be *independent* if no two vertices in D are adjacent. The *independence number* $\alpha = \alpha(G)$ is the maximum cardinality of an independent set of G. We say that an edge x and a vertex v *cover* each other if x is incident on v. A set $D \subseteq V$ is said to be a *vertex cover* if every edge in G is covered by some vertex in D. The *vertex covering number* $\beta = \beta(G)$ is the minimum cardinality of a vertex cover of G. Sampathkumar and Pushpalatha [10]

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defined the strong (weak) domination parameters as follows. For any two adjacent vertices u and v, u *strongly* (*weakly*) *dominates* v if $\deg(u) \geqslant \deg(v)$ ($\deg(u) \leqslant \deg(v)$). A set $D \subseteq V$ is a *strong dominating set* [sd-set] (*weak dominating set* [wd-set]) if every $v \in V - D$ is strongly (weakly) dominated by some $u \in D$. The *strong domination number*, $s\gamma = s\gamma(G)$ (*weak domination number* $w\gamma = w\gamma(G)$) is the minimum cardinality of a sd-set (wd-set) of G. The *strong independent domination number* $si = si(G)$ (*weak independent domination number* $wi = wi(G)$) is the minimum cardinality of an independent sd-set (independent wd-set). These concepts are further studied in [2,4,5]. Routenbach [9] made a detailed study on strong (weak) domination.

1.1. Strong number, weak number, balance number and regular number of a graph

A vertex $v \in V$ is *strong* (*weak*) if $\deg(v) \geqslant \deg(u)$ ($\deg(v) \leqslant \deg(u)$) for every u adjacent to v in G. A vertex v is *balanced* if it is neither strong nor weak. A vertex v is *regular* if $\deg(v) = \deg(u)$ for every vertex u adjacent to v in G. A set $S \subseteq V$ is said to be *strong*, *weak*, *balanced*, and *regular* if every vertex in S is, respectively, a strong, weak, balanced, and regular vertex in G. The *strong number* $s = s(G)$, *weak number* $w = w(G)$, *balance number* $b = b(G)$, *regular number* $r = r(G)$ are, respectively, the maximum cardinalities of a strong, weak, balanced, regular set of G. One can easily note that a vertex $v \in V$ is balanced if, and only if, $\deg(v) < \deg(u)$ and $\deg(v) > \deg(w)$ for some u, w adjacent v. If all the vertices in G are regular then G itself is regular. Let S, W, B and R, respectively, denote the maximum strong, weak, balanced and regular sets of G. If G is a regular graph then $S = W = R$ and $B = \emptyset$. For any graph G, since regular vertices are both strong and weak, $S \cap W = R$ and $V = S \cup W \cup B$. Let $C = S - R$ and $D = W - R$. Then one can observe that V can be partitioned in to four mutually disjoint sets B, C, D and R such that $V = B \cup C \cup D \cup R$.

We now obtain a relationship between the above four parameters.

Proposition 1. *Let $G=(V, E)$ be a graph with p vertices. Let $s=s(G)$, $w=w(G)$, $b=b(G)$ and $r=r(G)$, respectively, denote the strong number, weak number, balance number and regular number of the graph G. Then $p=s+w+b-r$.*

Proof. Let S, W, B and R, respectively, denote the maximum strong, weak, balanced and regular sets of G.

Case 1: Let G be a regular graph. Then $r = p$ and $S = W = R$ and $B = \emptyset$. Therefore $V = S \cup W$ and $p = |V| = |S \cup W| = |S| + |W| - |S \cap W| = r$. Hence the result.

Case 2: Let G be a non-regular graph. Then we have $S \cap W = R$. Since any balanced vertex is neither strong nor weak it cannot be either in S or in W. Therefore $(S \cup W) \cap B = \emptyset$. Also we have $V = S \cup W \cup B$. Therefore, $|V| = |S \cup W| + |B| = |S| + |W| - |S \cap W| + |B| = |S| + |W| + |B| - |R| = s + w + b - r$. $\square$

1.2. Strong (weak) independent sets and vertex coverings

In view of the fact that for any graph G strong (weak) sets always exist, it is interesting to study the independence concepts in these sets. In this direction, we define the following.

A strong (weak) set which is independent is called a *strong independent set* [SIS] (*weak independent set* [WIS]). The *strong* (*weak*) *independence number* $s\alpha = s\alpha(G)$ ($w\alpha = w\alpha(G)$) is the maximum cardinality of an SIS (WIS).

For an edge $x = uv$, v *strongly covers* the edge x if $\deg(v) \geqslant \deg(u)$ in G. Then u *weakly covers* x. A set $S \subseteq V$ is a strong vertex cover [SVC] (weak vertex cover [WVC]) if every edge in G is strongly (weakly) covered by some vertex in S. The *strong* (*weak*) *vertex covering number* $s\beta = s\beta(G)$ ($w\beta = w\beta(G)$) is the minimum cardinality of an SVC (WVC). A minimum vertex covering is written as β-set and a minimum SVC (WVC) is written as *$s\beta$-set* (*$w\beta$-set*).

Besides investigating some relationships among these new parameters the following results are established in this paper.

- For any graph G without isolated vertices, we show that the following inequality chains hold: $s\alpha \leqslant \beta \leqslant s\beta \leqslant w\beta$ and $s\alpha \leqslant w\alpha \leqslant \alpha \leqslant w\beta$.
- Analogous to Gallai's theorem, we prove $s\beta + w\alpha = p$ and $w\beta + s\alpha = p$.
- Further, we show that $s\alpha \leqslant p - \Delta$ and $w\alpha \leqslant p - \delta$ and find a necessary and sufficient condition to attain the upper bound, characterizing the graphs which attain these bounds.
- Several Nordhaus–Gaddum-type results are obtained and few Vizing-type theorems are also established.

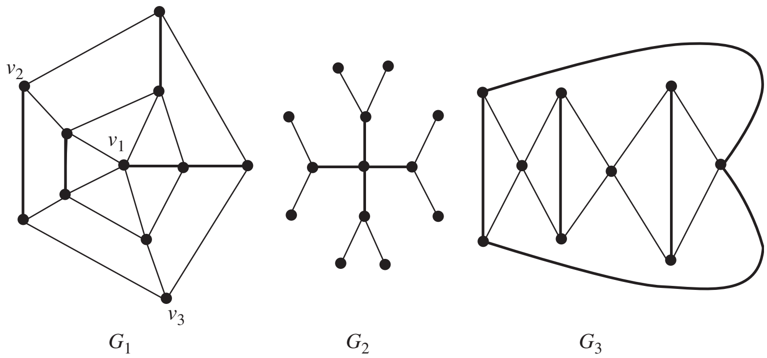

Fig. 1.

1.3. Motivation

In a road network, if every road is guarded at each junction by a policeman, for security purpose or so, then the minimum number of policemen required is nothing but the vertex covering number of the road network. But guarding every road at any junction may not serve the purpose and it is essential that every road must be guarded at a heavily crowded junction. In such a situation, we look for a strong covering than just a covering.

1.4. Illustrations

In G_1, an $s\alpha$-set is $S_1 = \{v_1\}$, and a $w\alpha$-set is $S_2 = \{v_2, v_3\}$. Here $s\alpha = 1$, $w\alpha = 2$ and $\alpha = 4$. Further, $V - S_1$ is a $w\beta$-set and $V - S_2$ is a $s\beta$-set. Hence $w\beta = 10$, $s\beta = 9$, $\beta = 7$. This is an example of a graph where the strict inequalities $\beta < s\beta < w\beta$ and $s\alpha < w\alpha < \alpha$ hold.

A vertex adjacent to a pendant vertex is called a support. In G_2, all the supports form a β-set and all non-pendant vertices form an $s\beta$-set where as all pendant vertices together with supports form a $w\beta$-set. All the pendant vertices form a $w\alpha$-set while the maximum degree vertex forms an $s\alpha$-set. All the pendant vertices together with maximum degree vertex forms an α-set.

G_3 is an example of a graph where $\alpha = s\alpha = w\alpha = 3$ and $\beta = s\beta = w\beta = 6$.

We observe the following facts:

Fact 1. *If G is a regular graph all the three covering numbers coincide and we have $\beta = s\beta = w\beta$. So also for the three independence numbers, and we have $\alpha = s\alpha = w\alpha$. But the converse is not true as we can see from the graph G_3 in Fig. 1.*

Fact 2. *Since every SIS (WIS) is a subset of a maximum strong (weak) set, it is immediate that $s\alpha \leqslant s(G)$ and $w\alpha \leqslant w(G)$.*

Fact 3. *Ore [8] established that every maximal independent set is a minimal dominating set. But this result is not true in the case of a maximal SIS (WIS). For example in the graph G_2 in Fig. 1. the singleton set with vertex of maximum degree is a maximal SIS and not a dominating set, and the set of all pendant vertices is a maximal WIS and not a dominating set.*

2. Gallai-type results

From the Gallai's theorem [3], it is well known that for any graph G with p vertices, $\alpha + \beta = p$. We obtain similar results for the new parameters defined. A set $S \subseteq V$ is a vertex cover if, and only if, $V - S$ is an independent set. Similarly, for the strong (weak) coverings we have the following result.

Proposition 2. *Let $G = (V, E)$ be any graph. Any set $S \subseteq V$, is an SIS (WIS) if, and only if, $V - S$ is a WVC (SVC).*

Proof. Let S be an SIS. Then S is an independent set and every vertex in S is a strong vertex. Since S is independent, $V-S$ is a vertex cover. It remains to show that $V-S$ is a WVC. Partition the edge set E as follows. $E_1=\{x\in E|x=uv, u\in S, v\in V-S$ with $\deg(u)\geqslant\deg(v)\}$ and $E_2=E-E_1$. Since S contains strong vertices, every edge in E_1 is strongly covered by some $u\in S$ but then every edge in E_1 is weakly covered by some $v\in V-S$. Moreover, E_2 contains all those edges whose end vertices are in $V-S$, and hence all the edges in E_2 are weakly covered by some vertex in $V-S$. Thus $V-S$ is a WVC. Conversely, let $V-S$ be a WVC. Since it is a covering, S is an independent set. It remains to show that S is an SIS. That is, we have to show that S consists of only strong vertices. Suppose there exists a vertex $u\in S$ such that $\deg(u)<\deg(v)$ for some $v\in V-S$. But then the line $x=uv$ is weakly covered by the vertex $u\in S$ and hence $V-S$ cannot be a WVC, a contradiction.

The following algorithm generates an SIS in any graph G:

$S=\emptyset$
$D=\emptyset$
While $(D\neq V)$
begin
  Let $v\in V-D$ such that $\deg(v)$ is maximum
  if $(\deg(v)\geqslant\deg(u))$ for every $u\in N(v)$
    then
      $S=S\cup\{v\}$
      $D=D\cup N[v]$
    else
      $D=D\cup N[v]$
    endif
  endwhile

Note that the set S in the algorithm will yield a maximal SIS and then from Proposition 2, $V-S$ is a minimal WVC. With a similar algorithm, we can get a maximal WIS and a minimal SVC of any graph. □

Analogous to Gallai's theorem, we now prove the following:

Proposition 3. *For any graph $G=(V,E)$ with p vertices*,

$$s\beta+w\alpha=p, \tag{1}$$

$$w\beta+s\alpha=p. \tag{2}$$

Proof. Let S be a minimum SVC. Then $V-S$ is a WIS by Proposition 2. Hence $w\alpha\geqslant|V-S|\geqslant p-s\beta$. Again, if D is a maximum WIS, then $V-D$ is an SVC by Proposition 2. Hence $s\beta\leqslant|V-D|\leqslant p-w\alpha$. Then (1) follows from the above inequalities. The proof of (2) is similar. □

The following proposition depicts the relationship among the various newly defined parameters.

Proposition 4. *For any graph G without isolated vertices*, *the following inequality chains hold*:

$$s\alpha\leqslant\beta\leqslant s\beta\leqslant w\beta, \tag{3}$$

$$s\alpha\leqslant w\alpha\leqslant\alpha\leqslant w\beta. \tag{4}$$

Proof. To prove $s\alpha\leqslant\beta$: we observe that if $A\subseteq V$ is an SIS then $|A|\leqslant|N(A)|$, since G is isolate free. Let S be an $s\alpha$-set and D be a β-set. There are two possibilities for S and D.

Case 1: $S\cap D=\emptyset$. Then since D is a covering, $N(S)\subseteq D$. Since S is an SIS, we have, $|S|\leqslant|N(S)|$. Then we have $s\alpha=|S|\leqslant|N(S)|\leqslant|D|=\beta$.

Case 2: $S\cap D\neq\emptyset$.

Subcase (i). Let $S \subseteq D$. Then the result is immediate.

Subcase (ii). Let $S \not\subset D$. We partition the set S as follows. Let $S_1 = \{v | v \in S \text{ and } v \notin D\}$. By the choice of S and D we have $S_1 \neq \emptyset$. Let $S_2 = S \cap D$. We now partition the set D as follows. Let $D_1 = N(S_1)$ and $D_2 = D - D_1$. Then, since S is an SIS, we have $|D_1| = |N(S_1)| \geqslant |S_1|$ and $S_2 \subseteq D_2$. Now $s\alpha = |S_1 \cup S_2| \leqslant |D_1 \cup D_2| = |D| = \beta$ as desired.

$\beta \leqslant s\beta$ follows from the fact that every strong covering is a covering.

To prove $s\beta \leqslant w\beta$: let $S = \{v_1, v_2, \ldots, v_k\}$ be an $s\beta$-set and without loss of generality assume that $\deg(v_1) \leqslant \deg(v_2) \leqslant \cdots \leqslant \deg(v_k)$. Choose $u_1 \in N(v_1)$ such that $\deg(u_1)$ is as small as possible. Then the edge $u_1 v_1$ is weakly covered by u_1. Let $D_1 = \{u_1\}$. Choose $u_i \in N(v_i) - D_i$, $2 \leqslant i \leqslant k$, such that $\deg(u_i)$ is as small as possible, where $D_i = D_{i-1} \cup \{u_i\}$, $2 \leqslant i \leqslant k$. Then $\{u_i v_i : 1 \leqslant i \leqslant k\}$ is a matching and each u_i weakly covers the edge $u_i v_i$. Therefore $D_k \subseteq W$ for some $w\beta$-set W. Thus $s\beta = |S| = |D_k| \leqslant |W| = w\beta$ as required. Then (4) follows immediately from (1), (2) and Gallai's theorem. □

Corollary 4.1. *For any (p, q) graph G without isolated vertices*,

$$s\beta + s\alpha \leqslant p,$$

$$w\beta + w\alpha \geqslant p.$$

Proposition 5. *For any (p, q) graph G without isolated vertices*,

$$s\alpha + w\alpha \leqslant p, \tag{5}$$

$$w\beta + s\beta \geqslant p. \tag{6}$$

Further, equality holds in (5) *and* (6) *if, and only if, there exists an SVC which is also an SIS.*

Proof. Since $s\alpha \leqslant s\beta$, (5) follows from (1). Likewise (6) follows. To show that equality holds in (6) it is enough to show that the reverse inequality holds in this case. Let S be an SVC which is an SIS. Since S is an SVC we have $s\beta \leqslant |S|$. Also since S is an SIS, $V - S$ is a WVC from Proposition 2. Therefore, $w\beta \leqslant |V - S| = p - |S|$. Hence $w\beta + s\beta \leqslant p$. Thus $w\beta + s\beta = p$. Then $p - s\alpha + p - w\alpha = p$ which yields equality in (5). Conversely, suppose that (i) $w\beta + s\beta = p$ and (ii) $s\alpha + w\alpha = p$. From Proposition 3, for any graph G, Eqs. (1) and (2) hold. Comparing (1) and (ii), $s\beta = s\alpha$ follows. Hence there must exist an SVC which is an SIS. □

We now obtain a necessary and sufficient condition for $\beta = s\beta = s\alpha$ and $\alpha = w\alpha = w\beta$.

Proposition 6. *For any graph G without isolated vertices, $\beta = s\beta = s\alpha$ and $\alpha = w\alpha = w\beta$ if, and only if, there exists an SVC which is also an SIS.*

Proof. Let S be an SVC which is also an SIS. Then $s\beta = s\alpha$ follows from the proof of converse of Proposition 5. Therefore, it remains to show that $\beta = s\beta$. We already have $\beta \leqslant s\beta$. Also from Proposition 4, we have $s\alpha \leqslant \beta$ and $s\beta = s\alpha$ implies $s\beta \leqslant \beta$. Thus $\beta = s\beta$. Hence, $\beta = s\beta = s\alpha$. Then $p - \alpha = p - w\alpha = p - w\beta$ which yields $\alpha = w\alpha = w\beta$. Converse is trivial. □

3. Bounds on strong (weak) independence and covering numbers

The complementary aspect of the domination number is defined by Sampathkumar and Pushpalatha [10]. A set $D \subseteq V$ is *full* (*s-full*, *w-full*, respectively) if every $u \in D$ dominates (strongly dominates, weakly dominates, respectively) some $v \in V - D$. The *full number* (*s-full number*, *w-full number*, respectively) $f = f(G)(sf = sf(G), wf = wf(G)$, respectively) is the maximum cardinality of a full set (s-full set, w-full set, respectively) of G. The following result is used in sequel.

Proposition 7 (*Sampathkumar and Pushpalatha [10]*). *For any graph G with p vertices,*

$$\gamma + f = p,$$

$$s\gamma + wf = p,$$

$$w\gamma + sf = p.$$

Proposition 8. *For any graph* $G = (V, E)$

$$\gamma \leqslant s\gamma \leqslant s\beta, \tag{7}$$

$$\gamma \leqslant w\gamma \leqslant w\beta, \tag{8}$$

$$s\alpha \leqslant sf \leqslant f, \tag{9}$$

$$w\alpha \leqslant wf \leqslant f. \tag{10}$$

Proof. Since every SVC is an sd-set and every sd-set is a dominating set we have (7). Likewise (8) follows. Using Propositions 7 and 3, Eq. (7) can be written as $p - f \leqslant p - f_w \leqslant p - w\alpha$ which yields (10). Likewise (9) follows. □

Let $I_s = \{v \in V \mid \deg(v) > \deg(u)$ for every $u \in N(v)\}$. It is clear that $I_s \subseteq D$ where D is an $s\alpha$-set. Now we give a bound in terms of $N(I_s)$.

Proposition 9. *For any graph G with p vertices,* $s\alpha \leqslant p - |N(I_s)|$, *and equality holds if, and only if,* $N(I_s)$ *is a minimum weak covering.*

Proof. Let D be an $s\alpha$-set. Then $I_s \subseteq D$, and since D is independent, $D \cap N(I_s) = \emptyset$. Therefore $D \subseteq V - N(I_s)$ and the result follows. If equality holds, then (i) $s\alpha + |N(I_s)| = p$. But from Proposition 3, we have (ii) $w\beta + s\alpha = p$. By comparing (i) and (ii), we have $N(I_s)$ is a minimum weak covering. Converse is trivial. □

We state the corresponding result on weak independence number without proof.

Proposition 10. *Let G be a graph with p vertices and let* $I_w = \{v \in V \mid \deg(v) < \deg(u)$ *for every* $u \in N(v)\}$. *Then* $w\alpha \leqslant p - |N(I_w)|$, *and equality holds if, and only if,* $N(I_w)$ *is a minimum strong covering.*

Even though the above bounds are very sharp, it is not simple to determine the cardinality of I_s (I_w). For the graphs in which $I_s = \emptyset$ $(I_w = \emptyset)$, the bound is trivial. Therefore, we try to give a bound in terms of some graph parameters.

Proposition 11. *For any* (p, q) *graph G without isolated vertices,*

$$s\alpha \leqslant \frac{p}{2} \quad \text{and} \quad w\beta \geqslant \frac{p}{2}.$$

Proof. We have $s\alpha \leqslant \alpha$. Also we have $s\alpha \leqslant \beta = p - \alpha$. Adding the two, we get the desired inequality. Then the second inequality is straight forward from Proposition 3. □

The bounds are attained for C_{2n} (cycle with $2n$ vertices) and $K_{n,n}$ (complete bipartite graph with n vertices in each part).

Domke et al. [2] proved that $si \leqslant p - \Delta$ and $wi \leqslant p - \delta$ and characterized the triangle free graphs that attain the upper bound. But si and $s\alpha$ are not comparable. For the graph G_2 in Fig. 1, $si = 9$, $s\alpha = 1$. For a path, $s\alpha(P_n) = \lfloor (n-1)/2 \rfloor$, $n \geqslant 3$; but $si(P_n) = \lceil n/3 \rceil$. Similarly, for the same graphs one may note that wi and $w\alpha$ are not comparable. Here we prove that $s\alpha \leqslant p - \Delta$ and $w\alpha \leqslant p - \delta$ and characterize the graphs which attain these upper bounds. We first prove the following result.

Proposition 12. *Let G be any graph and* V_Δ *be the set of all maximum degree vertices in G. Let S be any maximum independent set of vertices in* V_Δ. *Then there exists an* $s\alpha$*-set D such that* $V_\Delta \cap D = S \neq \emptyset$.

Proof. Clearly, $S \neq \emptyset$. Let S_1 be an SIS in G such that $\deg(x) < \Delta$ for every $x \in S_1$ and $|S_1|$ is as large as possible. Then $D = S \cup S_1$ is an $s\alpha$-set of G. Since $S_1 \cap V_\Delta = \emptyset$ and $S \subseteq D$, $S \subseteq V_\Delta$ we have $V_\Delta \cap D = S$. $\square$

Proposition 13. *For any connected graph G with p vertices, $s\alpha \leqslant p - \Delta$. Further, this bound is sharp.*

Proof. Let V_Δ be the set of all maximum degree vertices in G. Then by Proposition 12, there exists an $s\alpha$-set D such that $V_\Delta \cap D \neq \emptyset$. Let $v \in V_\Delta \cap D$. Since D is independent $D \cap N(v) = \emptyset$. Therefore $D \subseteq V - N(v)$. Hence $s\alpha = |D| \leqslant |V - N(v)| = p - \Delta$ as desired. This bound is sharp is evident as the complete bipartite graph $K_{m,n}$ attains the upper bound. $\square$

Corollary 13.1. *For any graph G with p vertices $\Delta \leqslant w\beta$.*

We are now ready to answer the question—when is $s\alpha = p - \Delta$? In the following proposition we get a necessary and sufficient condition for $s\alpha = p - \Delta$.

Proposition 14. *Let G be any connected graph with p vertices, and S be a maximum independent set of vertices in V_Δ. Then $s\alpha = p - \Delta$ if, and only if, $V - N(v)$ is an SIS for every $v \in S$.*

Proof. Assume $s\alpha = p - \Delta$ and let $v \in S$. Suppose $V - N(v)$ is not an SIS. Then either $V - N(v)$ is not independent or there exists some vertex in $V - N(v)$ which is not strong. If $V - N(v)$ is not independent then there exist at least two vertices which are adjacent in $V - N(v)$. Let x and y be adjacent in $V - N(v)$. Then only one of x or y will be in an $s\alpha$-set. Therefore $s\alpha = |V - N(v)| - 1 = p - \Delta - 1 < p - \Delta$, a contradiction. On the other hand, if there exists some vertex $x \in V - N(v)$ which is not strong then x will not be in an $s\alpha$-set and again we have $s\alpha = |V - N(v)| - 1 = p - \Delta - 1 < p - \Delta$, a contradiction. Conversely, assume that $V - N(v)$ is an SIS for every $v \in S$ and let D be a maximum SIS. Then $s\alpha = |D| \geqslant |V - N(v)|$. Since $v \in S$ then as in Proposition 13, $D \subseteq V - N(v)$. Hence $s\alpha = |D| \leqslant |V - N(v)|$. Thus we have $s\alpha = p - \Delta$. $\square$

In the next theorem, we characterize the graphs for which $s\alpha = p - \Delta$.

Theorem 15. *For any connected graph G with $p \geqslant 2$ vertices, $s\alpha = p - \Delta$ if, and only if, the vertex set of G can be partitioned into two sets V_1 and V_2 satisfying the following conditions.*

(i) *V_1 is an SIS and*
(ii) *There exists a vertex $v \in V_1$ such that $N(v) = V_2$.*

Proof. Suppose $s\alpha = p - \Delta$. Then by Proposition 14, $D = V - N(v)$ is an SIS for every $v \in S$ where S is the maximum independent set of vertices in V_Δ. Then $V_1 = D$ and $V_2 = N(v)$ is a partition of V such that V_1 is an SIS. Hence condition (i) is satisfied. Since $V_2 = N(v)$ we have $v \in S$. Hence the condition (ii) is satisfied. Conversely, let G be a graph such that its vertex set V can be partitioned into two sets V_1 and V_2 satisfying the conditions (i) and (ii) of the theorem. Since there exists a vertex $v \in S$ such that $N(v) = V_2$ we have $\deg(v) = \Delta$ for otherwise V_1 is not an SIS. Therefore $\Delta = |N(v)| = |V_2|$. Then $|V_1| = p - \Delta$ which implies V_1 is a maximum SIS. Therefore $s\alpha = p - \Delta$. $\square$

Corollary 15.1. *For any connected graph G with $p \geqslant 2$, the equalities $\beta = s\beta = s\alpha = p - \Delta$ hold if, and only if, G is a bipartite graph with bipartition $V = V_1 \cup V_2$ satisfying the following conditions*:

(i) *V_1 is an SIS and*
(ii) *There exists a vertex $v \in V_1$ such that $N(v) = V_2$.*

Proof. Suppose that G is a connected graph with $\beta = s\beta = s\alpha = p - \Delta$. Then by Proposition 6, there exists an SIS which is also an SVC. Let S be such a set. Then by Proposition 2, $V - S$ is a WIS which is also a WVC. Thus S and $V - S$ are independent sets. Since G is a connected graph, it must be a connected bipartite graph. Since $s\alpha = p - \Delta$, from Theorem 15, G must satisfy conditions (i) and (ii). Converse is trivial. $\square$

Corollary 15.2. *Let T be a tree with* $p \geqslant 2$. *Then* $\beta = s\beta = s\alpha = p - \Delta$ *if, and only if, T is the star graph* $K_{1,p-1}$.

Next we obtain a bound on the weak independence number involving the minimum degree δ. The following results are stated without proof as they can be proved analogously to those above proved for the strong independence number.

Proposition 16. *Let G be any graph. Let* V_δ *be the set of all minimum degree vertices in G. Let S be any maximum independent set of vertices in* V_δ. *Then there exists a* $w\alpha$*-set D such that* $V_\delta \cap D = S \neq \emptyset$.

Proposition 17. *For any connected graph G with p vertices,*

$$w\alpha \leqslant p - \delta \quad \text{and} \quad \delta \leqslant s\beta.$$

Proposition 18. *Let G be a connected graph with p vertices. Then* $w\alpha = p - \delta$ *if, and only if,* $V - N(v)$ *is a WIS for every* $v \in S$ *where S is any maximum independent set of vertices in* V_δ.

We now characterize the class of graphs for which $w\alpha = p - \delta$.

Theorem 19. *Let G be a connected graph with* $p \geqslant 2$ *vertices. Then* $w\alpha = p - \delta$ *if, and only if, the vertex set of G can be partitioned into two sets* V_1 *and* V_2 *satisfying the following conditions.*

(i) V_1 *is a WIS and*
(ii) *Every vertex in* V_1 *is adjacent to every vertex in* V_2.

Proof. Assume that the vertex set of G can be partitioned in to V_1 and V_2, as in the statement of the theorem. Then from condition (ii) every vertex in V_1 has degree δ, for otherwise V_1 is not a WIS. This implies $|V_2| = \delta$ and $|V_1| = p - \delta$. Therefore V_1 is a maximum WIS, and $w\alpha = |V_1| = p - \delta$. Conversely, assume $w\alpha = p - \delta$. Then, by Proposition 18, $V - N(v)$ is a WIS for every $v \in S$, where S is any maximum independent set in V_δ. Then $V_1 = V - N(v)$ and $V_2 = N(v)$ is a partition of V such that V_1 is a WIS. Hence condition (i) is satisfied. Since $v \in S$, $\deg(v) = \delta$, and thus $|V_2| = \delta$. Then $|V_1| = p - \delta$. Since $|V_2| = \delta$, and V_1 is a WIS, every vertex in V_1 has degree δ. This implies that every vertex in V_1 is adjacent to every vertex in V_2. That is, condition (ii) is also satisfied. □

Corollary 19.1. *Let G be a connected graph with* $p \geqslant 2$. *Then* $w\beta = \alpha = w\alpha = p - \delta$ *if, and only if, G is a complete bipartite graph.*

Corollary 19.2. *Let T be a tree with* $p \geqslant 2$. *Then* $w\beta = \alpha = w\alpha = p - \delta$ *if, and only if, T is the star graph* $K_{1,p-1}$.

We now obtain some bounds for the strong (weak) independence number when G is a tree.

Proposition 20. *Let T be a tree with* $p \geqslant 3$ *vertices. Let m be the number of pendant vertices, and b the number of supports. Then*:

$$\Delta \leqslant m \leqslant w\alpha \leqslant p - b, \tag{11}$$

$$\Delta \leqslant m \leqslant w\beta, \tag{12}$$

$$b \leqslant s\beta \leqslant p - m \leqslant p - \Delta, \tag{13}$$

$$s\alpha \leqslant p - m. \tag{14}$$

Proof. Let v be a vertex of maximum degree Δ in T. Then there exists at least Δ pendant vertices. Clearly, all the pendant vertices are in a $w\alpha$-set D. Hence $\Delta \leqslant m \leqslant w\alpha$. Let B be the set of all supports. Since no support is a weak vertex, no support can be contained in a weak independent set. Therefore $D \subseteq V - B$. Hence the upper bound in (11) follows. The result (12) follows from (11) as $w\alpha \leqslant w\beta$. The upper bound in (13) follows from (11) and Proposition 3. As pendant edges are strongly covered only by supports, any minimum strong covering must contain all supports. Hence the lower bound in (13) follows. Since $s\alpha \leqslant s\beta$, (14) follows from (13). □

4. Nordhaus–Gaddum-type results

Here we obtain Nordhaus–Gaddum-type results [7] for strong (weak) independence numbers. For any graph G, we denote $w\alpha(\overline{G}) = \overline{w\alpha}$ and $s\alpha(\overline{G}) = \overline{s\alpha}$.

Proposition 21. *For any graph G with p vertices*

$$s\alpha + \overline{w\alpha} \leqslant p + 1, \tag{15}$$

$$\overline{s\alpha} + w\alpha \leqslant p + 1, \tag{16}$$

$$s\alpha + \overline{s\alpha} \leqslant p + 1 + \delta - \Delta, \tag{17}$$

$$w\alpha + \overline{w\alpha} \leqslant p + 1 + \Delta - \delta, \tag{18}$$

$$w\beta + \overline{s\beta} \geqslant p - 1, \tag{19}$$

$$\overline{w\beta} + s\beta \geqslant p - 1, \tag{20}$$

$$\overline{s\beta} + s\beta \geqslant p - 1 + \delta - \Delta, \tag{21}$$

$$w\beta + \overline{w\beta} \geqslant p - 1 + \Delta - \delta. \tag{22}$$

Proof. First we note that $\Delta+\overline{\delta}=\overline{\Delta}+\delta=p-1$. From Propositions 13 and 17 we have $s\alpha+\overline{w\alpha}\leqslant 2p-(\Delta+\overline{\delta})=2p-(p-1)=p+1$. Thus (15) follows. Likewise (16) follows. Again $s\alpha+\overline{s\alpha}\leqslant p-\Delta+p-\overline{\Delta}=p-\Delta+p-(p-1-\delta)=p+1+\delta-\Delta$. Thus (17) holds. Likewise (18) follows. Using Proposition 3 we get (19)–(22). The bounds in (15)–(22) are sharp. The graphs K_p and $\overline{K}_p$ attain these bounds. □

Proposition 22. *If both G and $\overline{G}$ are isolate free graphs, then*

$$(s\alpha) + (\overline{s\alpha}) \leqslant p, \tag{23}$$

$$(s\alpha)(\overline{s\alpha}) \leqslant \frac{p^2}{4}, \tag{24}$$

$$(w\beta) + (\overline{w\beta}) \geqslant p. \tag{25}$$

Proof. (23)and (24) follow from Proposition 11. Then (25) follows from (23) and Proposition 3. □

5. Vizing-type results

In this section, we obtain a bound on the number of edges in a simple graph when the strong (weak) independence number is given. This result corresponds to the well known theorem of Vizing [11] which gives a bound on the number of edges when the domination number is given. We also give an upper bound for $s\alpha$ and $w\alpha$ in terms of number of vertices and edges of the graph.

Theorem 23. *Let G be a connected (p, q) graph with strong independence number $s\alpha = k$. Then $q \leqslant p(p - k)/2$. Further, this bound is sharp.*

Proof. Let G be a connected graph with $s\alpha=k$. Let S be an $s\alpha$-set. Since S is independent, a vertex in S can be adjacent to at most $p-k$ vertices in $V-S$. Thus, $\deg(v)\leqslant p-k$, for every $v \in S$. Further, since S is strongly independent, we have $\deg(y)\leqslant\deg(x)\leqslant p-k$ for every $x \in S$ and $y \in N(x) = V - S$. Then by the handshaking lemma we have $2q\leqslant k(p-k)+(p-k)(p-k)=p(p-k)$. Then the result follows. □

The following example shows that the bound is sharp.

Example 1. A $(p-k)$-regular graph G with $V = V_1 \cup V_2$, where V_1 is independent and $|V_1| = k = s\alpha \leqslant p-k = |V_2|$ satisfies $q = p(p-k)/2$.

An upper bound for $s\alpha$ in terms of number of vertices and edges of the graph is immediate from the above theorem.

Corollary 23.1. *If G is a connected* (p, q) *graph*, *then* $s\alpha \leqslant (p^2 - 2q)/p$

Theorem 24. *Let G be a connected* (p, q) *graph and weak independence number* $w\alpha = k$. *Then* $q \leqslant (p-k)(p+k-1)/2$. *Further*, *this bound is sharp*.

Proof. Let W be $w\alpha$-set. Since W is independent, a vertex in W can be adjacent to at most $p-k$ vertices in $V - W$. Thus $\deg(v) \leqslant p-k$, for every $v \in W$. Further since W is weakly independent, we have $\deg(x) \leqslant \deg(y)$ for every $x \in W$ and $y \in N(x) = V - W$. But $\deg(y)$ can be at most $p-1$ for every $y \in V - W$. Then by the handshaking lemma $2q \leqslant k(p-k) + (p-k)(p-1) = (p-k)(p+k-1)$. Then the result follows. □

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Their *join* $G_1 + G_2$ as defined by Zykov [13] has $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ together with all the edges joining V_1 and V_2.

The following example shows that the bound in Theorem 24 is sharp.

Example 2. The graph $G = \overline{K}_k + K_{p-k}$ satisfies $w\alpha = k$ and $q = (p-k)(p+k-1)/2$.

As earlier, this theorem suggests an upper bound for $w\alpha$ in terms of the number of vertices and edges of the graph.

Corollary 24.1. *Let G be a connected* (p, q) *graph. Then* $w\alpha \leqslant \frac{1}{2} + \sqrt{p(p-1) - 2q + \frac{1}{4}}$

Proof. From the above theorem we have $2q \leqslant (p-k)(p+k-1)$. This gives $k^2 - k + 2q - p(p-1) \leqslant 0$ and solving for k we get the required bound. □

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