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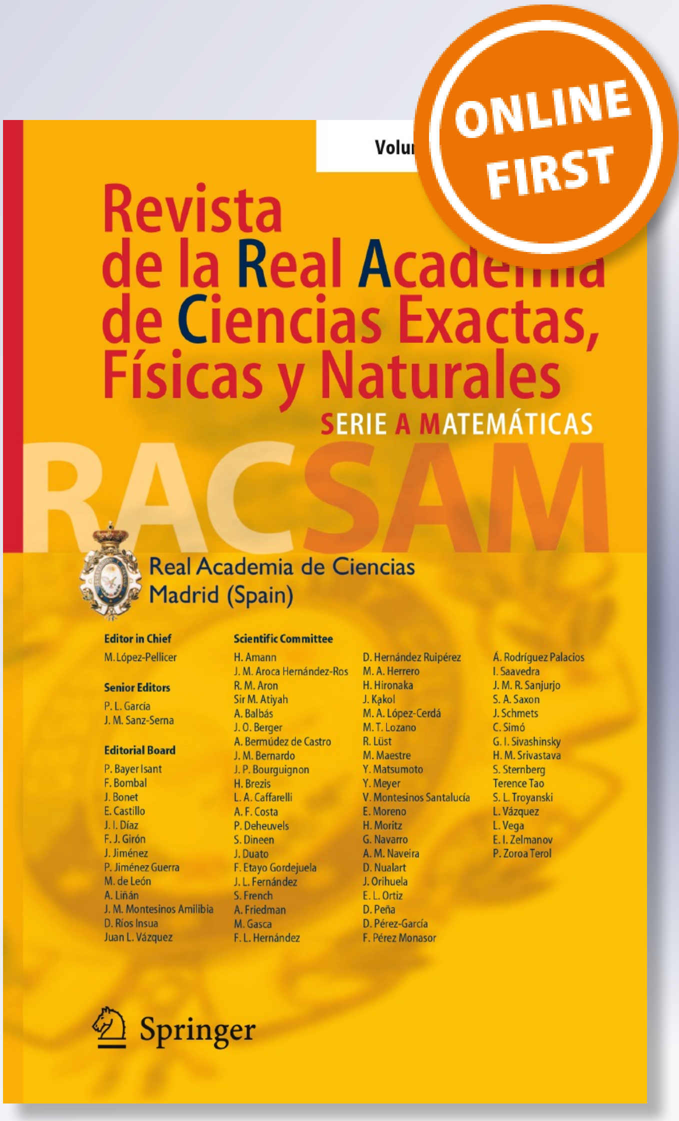

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ORIGINAL PAPER

On the complexity of choosing majorizing sequences for iterative procedures

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Abstract
The aim of this paper is to introduce general majorizing sequences for iterative procedures which may contain a non-differentiable operator in order to solve nonlinear equations involving Banach valued operators. A general semi-local convergence analysis is presented based on majorizing sequences. The convergence criteria, if specialized can be used to study the convergence of numerous procedures such as Picard's, Newton's, Newton-type, Stirling's, Secant, Secant-type, Steffensen's, Aitken's, Kurchatov's and other procedures. The convergence criteria are flexible enough, so if specialized are weaker than the criteria given by the aforementioned procedures. Moreover, the convergence analysis is at least as tight. Furthermore, these advantages are obtained using Lipschitz constants that are least as tight as the ones already used in the literature. Consequently, no additional hypotheses are needed, since the new constants are special cases of the old constants. These ideas can be used to study, the local convergence, multi-step multi-point procedures along the same lines. Some applications are also provided in this study.

Keywords Majorizing sequence · Banach space · Semi-local convergence · Iterative procedures

Mathematics subject classification 47H99 · 65H10 · 65J15

1 Introduction

Kantorovich proved in 1939 the semi-local convergence of Newton's method using first the contraction mapping principle of Banach and then using recurrence relations. Later he gave a proof based on the concept of a majorant function [9]. The novelty of the Newton-Kantorovich theorem, or related results is that the theorem is a convergence result for Newton's method

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and a theorem of existence of solution of equations in a Banach space setting. The theorem also provides information about where the solution is located, without finding the solution which is sometimes more important than the actual knowledge of the solution. A plethora of results has been published after the Newton–Kantorovich theorem was established concerning convergence and error bounds of Newton-type or secant-type methods under numerous assumptions. We refer the reader to [1–14] and the references therein for relevant publications.

Most of these results are based on finding a scalar majorizing sequence for the iterative procedure. The novelty of our paper lies in the fact that we introduce a generalized majorizing sequence that can be used in the case of Picard's method; Newton's method; Stirling's method; Steffensen's method; Secant method; Kurchatov's method; Aitken's method and many other methods [1–14]. This way, we unify the semi-local results for these methods. It turns out that when specialized the convergence criteria can be weaker than the existing ones for the preceding methods.

Many problems in computational sciences can be brought in the form

$$F(x) = 0, \tag{1.1}$$

where $F : D \subset X \longrightarrow Y$ is continuous, X, Y are Banach spaces and D is convex. We are looking for a continuous and nondecreasing function $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ and a nonnegative, nondecreasing scalar sequence $\{v_n\}$ such that for each $n = 0, 1, 2, \ldots$

$$\|x_{n+1} - x_n\| \leq \psi(v_n) = v_{n+1} - v_n, \tag{1.2}$$

where $\{x_n\}$ is an iterative process related to F, so that

$$\lim_{n \longrightarrow \infty} x_n = x^*$$

and $F(x^*) = 0$.

If there exists v^* such that

$$\lim_{n \longrightarrow \infty} v_n = v^* < \infty,$$

then the limit point x^* also exists. Additional hypotheses are needed so that $F(x^*) = 0$. The determination of function ψ and sequence $\{v_n\}$ is a complex task in general. In the present study, we introduce ψ and $\{v_n\}$ general enough so they can be used to study all aforementioned methods.

The rest of the paper is structured as follows: Sect. 2 contains the semi-local convergence, whereas in the concluding Sect. 3, we present applications and the special cases.

2 Majorizing sequences

We present sufficient convergence criteria for general majorizing sequences.

Lemma 2.1 [10,11] *Let $\{u_n\}$ be any sequence in X. Then, a sequence $\{v_n\} \subset [0, \infty)$ for which*

$$\|u_{n+1} - u_n\| \leq v_{n+1} - v_n \text{ for each } n = 0, 1, 2, \ldots \tag{2.1}$$

holds in a majorizing sequence for $\{u_n\}$. Suppose that

$$\lim_{n \longrightarrow \infty} v_n = v^* < \infty$$

exists. Then,

$$u^* = \lim_{n \longrightarrow \infty} u_n$$

exists and

$$\|u^* - u_n\| \leq v^* - v_n \quad \text{for each } n = 0, 1, 2, \ldots.$$

Note that any majorizing sequence is necessarily monotonically increasing.

Scalar iteration $\{s_n\}$ defined for each $n = 1, 2, \ldots$ by $s_0 = 0$,

$$s_{n+1} = s_n - \frac{\frac{1}{2}a_0(s_n - s_{n-1})^2 + (rs_{n-1} + s)(s_n - s_{n-1})}{sd_n + p}, \tag{2.2}$$

where $s_1 > 0$ is given and for some $a_0 > 0$, $r \geq 0$, $s \geq 0$, $d > 0$ and $p < 0$ is a majorizing sequence for many iterative processes (see Sect. 3). Therefore, finding sufficient convergence criteria for $\{s_n\}$ is very important. Next, we present two competing ideas for generating such criteria. Firstly, let us define sequence $\{t_n\}$ for each $n = 0, 1, 2, \ldots$ by

$$t_0 = 0,\ t_{n+1} = t_n - \frac{g(t_n)}{h(t_n)}, \tag{2.3}$$

where

$$g(t) = \frac{1}{2}at^2 - bt + c, \tag{2.4}$$

$$h(t) = dt + p, \tag{2.5}$$

$b > 0, c > 0$

$$a = \max\{d + r, a_0\} \tag{2.6}$$

and

$$t_1 = -\frac{c}{p}. \tag{2.7}$$

We can present the following convergence result for sequence $\{t_n\}$.

Theorem 2.2 *Suppose that*

$$ac \leq \frac{b^2}{2} \tag{2.8}$$

and

$$a > 0,\ b \leq -p. \tag{2.9}$$

Then, the following items hold:

(1) *Sequence $\{t_n\}$ generated by (2.3) is well defined nondecreasing and converges to the smallest root of quadratic function g given by*

$$t^* = \frac{b - \sqrt{b^2 - 4ac}}{a} > 0. \tag{2.10}$$

(2) *$\{t_n\}$ is a majorizing sequence for $\{s_n\}$ such that*

$$0 \leq s_n \leq t_n, \tag{2.11}$$
$$0 \leq s_{n+1} - s_n \leq t_{n+1} - t_n \tag{2.12}$$

and

$$s^* = \lim_{n \longrightarrow \infty} s_n \leq t^*. \tag{2.13}$$

Proof (1) It follows from (2.8) that function g has two positive roots t^* and t^{**} with $t^* \leq t^{**}$ and $t^{**} = \frac{b+\sqrt{b^2-4ac}}{a}$. We shall show using induction that

$$t_k \leq t_{k+1} \leq t^* \text{ for each } k = 0, 1, 2, \ldots. \tag{2.14}$$

We must show

$$t_1 \leq t^* \tag{2.15}$$

or equivalently by (2.7) and (2.10) that

$$-\frac{c}{p} \leq \frac{b - \sqrt{b^2 - 4ac}}{a}$$

or

$$ac + bp \leq \sqrt{b^2 - 4ac}. \tag{2.16}$$

Case $ac + bp \leq 0$. Then (2.15) hold by (2.16).
Case $ac + bp > 0$. We can show instead of (2.16) that

$$a^2c^2 + b^2p^2 + 2abcp \leq b^2 - 2ac. \tag{2.17}$$

By (2.8) and $-bp < ac$,

$$a^2c^2 + b^2p^2 + 2abcp < a^2c^2 + a^2c^2 - 2ac(ac) = 0.$$

Then, to show (2.17) it suffices $0 \leq b^2 - 2ac$, which is true by (2.8). Hence, (2.15) is true. We can write by (2.7), (2.8) and (2.9) in turn that

$$\begin{aligned} g(t_1) &= \frac{1}{2}at_1^2 - bt_1 + c \\ &= \frac{c}{2p^2}(ac + 2pb + 2p^2) \\ &\geq \frac{c}{2p^2}(ac + 2p\sqrt{2ac} + 2p^2) \\ &= \frac{c}{2p^2}[(\sqrt{2ac} + p)^2 + p^2 - 2ac] \\ &\geq \frac{c}{2p^2}[(\sqrt{2ac} + p)^2 + b^2 - 2ac] > 0, \end{aligned}$$

so $g(t_1) > 0 = g(t^*)$, and $g(t)$ is decreasing in $[0, \frac{b}{a}]$, leading to $t_1 < t^*$. That is (2.14) is true for $k = 0$. Suppose that $t_{k-1} \leq t_k < t^*$. Then, since $g(t)$ is decreasing and $g'(t)$ is increasing in $[0, \frac{b}{a}]$, we get $g(t_k) > 0$ and $g'(t_k) \leq 0$, respectively. Moreover by (2.6) and

(2.10), $h(t_k) \leq g'(t_k)$ which implies $-h(t_k) \geq 0$, so $t_{k+1} > t_k$. Furthermore, the function $t - \frac{g(t)}{g'(t)}$ is increasing in $[0, \frac{b}{a}]$. Then, by $-\frac{g(t_k)}{h(t_k)} \leq -\frac{g(t_k)}{g'(t_k)}$, we get

$$t_{k+1} \leq t^* - \frac{g(t^*)}{g'(t^*)} = t^*, \tag{2.18}$$

which completes the induction for (2.14). Sequence $\{t_k\}$ is increasing, bounded from above by t^* and as such it converges to its unique least upper bound t_0^*. By letting $k \longrightarrow \infty$ in (2.3) we deduce that $t_0^* = t^*$.
(2) Choose $s_1 = t_1$. Then, items (2.11)–(2.13) follow by a simple inductive argument, Lemma 2.1, (2.2) and (2.3). □

We need an auxiliary resul connecting sequence $\{x_n\}$ to sequence $\{t_n\}$.

Lemma 2.3 *Under the hypotheses of Theorem 2.2, further suppose that*

$$\|x_1 - x_0\| \leq t_1 - t_0 \tag{2.19}$$

$$b + p + s \leq 0 \tag{2.20}$$

and for each $n = 2, 3, \ldots$

$$\|x_n - x_{n-1}\| \leq -\frac{\frac{1}{2}a(t_n - t_{n-1})^2 + (rt_{n-1} + s)(t_n - t_{n-1})}{dt_n + p} \leq 0 \tag{2.21}$$

hold for some sequence $\{x_n\} \subset X$. *Then, sequence* $\{t_n\}$ *is majorizing for sequence* $\{x_n\}$ *and there exists* $x^* \in X$ *such that*

$$\lim_{n \longrightarrow \infty} x_n = x^*.$$

Proof It suffices to show the quantity in the above bracket is bounded above by $g(t_n)$. We have by the definition of sequence $\{t_n\}$, (2.4), (2.5), (2.6), (2.7), (2.19)–(2.21) in turn that

$$\begin{aligned}
&\frac{1}{2}a(t_n - t_{n-1})^2 + (rt_{n-1} + s)(t_n - t_{n-1})\\
&= g(t_n) - [g(t_n) - g(t_{n-1}) - \frac{1}{2}a(t_n - t_{n-1})^2 - (rt_{n-1} + s)(t_n - t_{n-1}) - h(t_{n-1})(t_n - t_{n-1})]\\
&= g(t_n) + (t_n - t_{n-1})[(d + r - a)t_n + (b + p + s)] \leq g(t_n),
\end{aligned} \tag{2.22}$$

since $t_{n-1} \leq t_n$, $d + r \leq a$ and $b + p + s < 0$. □

It is convenient for the convergence of sequence $\{s_n\}$ to introduce some scalar functions and parameters:

$$\varphi_0(t) = dt^2 + \frac{1}{2}a_0 t + r - \frac{1}{2}a_0 \tag{2.23}$$

$$\varphi(t) = -pt^2 + (ds_1 + p - s)t + rs_1, \tag{2.24}$$

$$\varphi_1(t) = d^2t^2 + 2((p - s)d + 2pr)t + (p - s)^2, \tag{2.25}$$

$$q_0 = -\frac{\frac{1}{2}a_0 s_1 + s}{ds_1 + p}, \tag{2.26}$$

Case 1: $2r_0 < a_0$.
Then, φ_0 has a unique positive root denoted by q. Suppose

$$(p - s)d + 2pr < 0, \; q_0 \geq 0, \tag{2.27}$$

$$\triangle_1 \geq 0 \tag{2.28}$$

and

$$s_1 \leq \bar{s}, \tag{2.29}$$

where $\triangle_1$ is the discriminant of φ_1 and $\bar{s}$ is the smallest root of φ_1. Then, $\triangle \geq 0$, where $\triangle$ is the discriminant of φ. It follows that function φ has two roots q_1 and q_2 such that $0 \leq q_1 \leq q_2$. Set $\bar{q} = \max\{q_0, q_1\}$. Moreover, suppose that

$$\bar{q} \leq q \leq q_2. \tag{2.30}$$

Denote the hypotheses of **Case 1** by (H_0).

Case 2: $a_0 \leq 2r$.

Then, for φ_0 has no positive roots. Suppose (2.28), (2.29), (2.30) and

$$\bar{q} \leq q_2 \tag{2.31}$$

hold. Then, there exists q such that

$$\bar{q} \leq q \leq q_2. \tag{2.32}$$

Denote the hypotheses of **Case 2** by (H). Then, we can show the convergence of sequence $\{s_n\}$ under hypotheses (H_0) or (H).

Theorem 2.4 *Suppose that hypotheses (H_0) or (H) hold. Then, sequence $\{s_n\}$ generated by (2.2) is well defined, nondecreasing, bounded above by*

$$s^{**} = \frac{s_1}{1-q} \tag{2.33}$$

and converges to its unique least upper bound s^* which satisfies

$$s_1 \leq s^* \leq s^{**}. \tag{2.34}$$

Proof We shall show using induction on k that

$$0 \leq -\frac{\frac{1}{2}a_0(s_k - s_{k-1}) + rs_{k-1} + s}{ds_k + p} \leq q. \tag{2.35}$$

Estimate (2.35) is true for $k = 1$ by (2.30) (or (2.32)). Then, we get by (2.2) and (2.35) $0 < s_2 - s_1 \leq q(s_1 - s_0)$ and

$$s_2 \leq \frac{1-q^2}{1-q}(s_1 - s_0) < s^{**}. \tag{2.36}$$

Suppose that

$$0 < s_{k+1} - s_k \leq q(s_k - s_{k-1}) \leq q^k(s_1 - s_0) \tag{2.37}$$

and

$$s_{k+1} \leq \frac{1-q^{k+1}}{1-q}(s_1 - s_0) < s^{**}. \tag{2.38}$$

We shall show that (2.35) holds with k replaced by $k+1$. Evidently (2.35) holds in this case, if

$$\frac{1}{2}a_0(s_{k+1} - s_k) + rs_k + s + (ds_{k+1} + p)q \leq 0$$

or using (2.37) and (2.38), if

$$\frac{1}{2}a_0 q^k s_1 + r(1+q+\cdots+q^{k-1})s_1 + s + dq(1+q+\cdots+q^k)s_1 + pq \leq 0. \tag{2.39}$$

Estimate (2.39) motivates us to define recurrent functions f_k defined on $[0, 1)$ by

$$f_k(t) = \frac{1}{2}a_0 s_1 t^k + rs_1(1+t+\cdots+t^{k-1}) + ds_1(1+t+\cdots+t^k)t + pt + s. \tag{2.40}$$

We need an estimate between two consecutive fucntions f_k :

$$\begin{aligned} f_{k+1}(t) &= \frac{1}{2}a_0 s_1 t^{k+1} + rs_1(1+t+\cdots+t^k) \\ &\quad + ds_1(1+t+\cdots+t^{k+1})t + pt + s - f_k(t) + f_k(t) \\ &= f_k(t) + \varphi_0(t)s_1 t^k. \end{aligned} \tag{2.41}$$

We can show instead of (2.39) that

$$f_k(q) \leq 0. \tag{2.42}$$

First by, under the hypotheses (H_0), we get

$$f_{k+1}(q) = f_k(q), \tag{2.43}$$

since $\varphi_0(q) = 0$. Define function f_∞ on $[0, 1)$ by

$$f_\infty(q) = \lim_{k\longrightarrow\infty} f_k(q) = \frac{rs_1}{1-q} + \frac{ds_1 q}{1-q} + pq + s. \tag{2.44}$$

Then, instead of (2.42), we can show

$$f_\infty(q) \leq 0, \tag{2.45}$$

(since (2.43) holds). But (2.45) is equivalent to

$$\varphi(q) \leq 0, \tag{2.46}$$

which is true by (2.31). Secondly, under the hypotheses (H), we get

$$f_k(q) \leq f_{k+1}(q) \leq f_\infty(q), \tag{2.47}$$

so again, we must show (2.45) or (2.46), which are true by (2.31) and (2.32). Hence, in either case, sequence $\{s_n\}$ is nondecreasing, bounded from above by s^{**} and as such it converges to s^* which satisfies (2.34). □

Lemma 2.5 *Under the hypotheses of Theorem 2.4, further suppose that*

$$\|x_1 - x_0\| \leq s_1 - s_0 \tag{2.48}$$

and

$$\|x_{n+1} - x_n\| \leq -\frac{[\frac{1}{2}a_0(s_{n+1}-s_n)^2 + (rs_n + s)(s_{n+1}-s_n)]}{ds_{n+1}+p} \tag{2.49}$$

hold for some sequence $\{x_n\} \subset X$. Then, sequence $\{s_n\}$ is majorizing for sequence $\{x_n\}$ and there exists $x^* \in X$ such that

$$x^* = \lim_{n\longrightarrow\infty} x_n.$$

Proof Simply notice that in view of (2.2), (2.48) and (2.49)

$$\|x_n - x_{n-1}\| \leq s_n - s_{n-1}. \tag{2.50}$$

The result now follows from Lemma 2.1 and (2.50). □

Next, we compare sequences $\{t_n\}$ and $\{s_n\}$.

Proposition 2.6 *Suppose that hypotheses of Theorem 2.2, Lemma 2.3, Theorem 2.4 and Lemma 2.5 hold. Then, the following assertions hold:*

$$s_n \leq t_n \tag{2.51}$$

and

$$\|x_{n+1} - x_n\| \leq s_{n+1} - s_n \leq t_{n+1} - t_n. \tag{2.52}$$

Proof Use a simple inductive argument and the definition of sequences $\{t_n\}$ and $\{s_n\}$. □

Remark 2.7 We have shown so far that $\{s_n\}$ is a tight sequence than $\{t_n\}$. However, we cannot directly compare the convergence criteria.Next we shall compare the convergence criteria in some popular cases.

3 Applications and special cases

Application 3.1 Let us consider the Newton-like method

$$x_{n+1} = x_n - A(x_n)^{-1} F(x_n), \tag{3.1}$$

where $A(x)^{-1} \in L(Y, X)$ the space of bounded linear operators from Y into X. Method (3.1) has been studied under the hypotheses (C) [13]:

Let $F : D \subseteq X \longrightarrow Y$ be Fréchet differentiable and let $A(x) \in L(X, Y)$ be an approximation to $F'(x)$.

(c_1) There exist an open convex subset D_0 of D, $x_0 \in D_0$, a bounded inverse $A(x_0)^{-1}$ of $A(x_0)$ and constants $\eta, K > 0, M, L, \mu, \ell \geq 0$ such that for each $x, y \in D_0$, the following conditions hold:

$$\begin{aligned}
&\|A(x_0)^{-1} F(x_0)\| \leq \eta,\\
&\|A(x_0)^{-1}(F'(x) - F'(y))\| \leq K\|x - y\|,\\
&\|A(x_0)^{-1}(F'(x) - A(x))\| \leq M\|x - x_0\| + \mu,\\
&\|A(x_0)^{-1}(A(x) - A(x_0))\| \leq L\|x - x_0\| + \ell,\\
&\bar{b} := \eta + \ell,\\
&\bar{h} = \sigma\eta \leq \frac{1}{2}(1 - \bar{b})^2,\ \sigma = \max\{K, M + L\}
\end{aligned}$$

and

$$\bar{U} = \bar{U}(x_0, t^*) \subseteq D_0,$$

where

$$t^* = \frac{1 - \bar{b} - \sqrt{(1 - \bar{b})^2 - 2\bar{h}}}{\sigma}.$$

Then, the following items hold:

(i) Sequence $\{x_n\}$ converges to a solution $x^* \in \bar{U}$ of equation $F(x) = 0$.
(ii) The equation $F(x) = 0$ has a unique solution in $\tilde{U}$, where

$$\tilde{U} = \begin{cases} \bar{U}(x_0, t^*) \cap D_0, & \text{if } \bar{h} = \frac{1}{2}(1-\bar{b})^2 \\ U(x_0, t^{**}) \cap D_0, & \text{if } \bar{h} < \frac{1}{2}(1-\bar{b})^2 \end{cases}$$

and

$$t^{**} = \frac{1 - \bar{b} + \sqrt{(1-\bar{b})^2 - 2\bar{h}}}{\sigma}.$$

(iii)

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \frac{1}{1 - Lt_n - \ell}\left(\frac{\sigma}{2}(t_n - t_{n-1})^2 + (Mt_{n-1} + \mu)(t_n - t_{n-1})\right) \\ &\leq \frac{g_1(t_n)}{h_1(t_n)},\end{aligned}$$

where $g_1(t) = \frac{\sigma}{2}t^2 - (1-\bar{b})t + \eta$ and $h_1(t) = 1 - Lt - \ell$.

Next, we shall show that these hypotheses are special cases of the results in Theorem 2.2 and Lemma 2.3. Indeed, let $d = L, r = M, a = \sigma, p = \ell - 1, c = \eta, b = 1 - \bar{b}$, $g_1 = g, h_1 = -h, s = \mu$ and notice that $b + p + s = 1 - \bar{b} + \ell - 1 + \mu = \mu + \ell - \bar{b} = 0$. Then, we also get

$$ac \leq \frac{b^2}{2} \Leftrightarrow \bar{h} \leq \frac{1}{2}(1-\bar{b})^2. \tag{3.2}$$

Hence, the new results reduce to the old ones for these special choices of the new parameters.

Application 3.2 Let us consider Newton's method [1–14]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{3.3}$$

i.e., set $A(x) = F'(x)$, $M = \mu = \ell = 0$ and $d = L$. Then, again we get

$$ac = K\eta \leq \frac{1}{2} \Leftrightarrow \bar{h} \leq \frac{1}{2}, \tag{3.4}$$

since $\bar{b} = 0$ and $\sigma = a = K$.

Application 3.3 Let us use Newton's method, sequence $\{s_n\}$ and Theorem 2.4 for $a_0 = K, d = L, r = s = 0, p = -1, c = \eta = s_1$. We get using (2.23)–(2.29) that

$$q_0 = \frac{K\eta}{2(1 - L\eta)},\; q = \frac{2K}{K + \sqrt{K^2 + 8KL}},$$

$q_1 = 0, q_2 = 1 - L\eta, \bar{q} = q_0, \bar{s} = \frac{1}{L}, 2r \leq K, (p - s)d + 2pr < 0, \triangle_1 = 0$ and t^* is replaced by s^* in Application 3.1. Then, (2.30) is satisfied provided that

$$\frac{K\eta}{2(1 - L\eta)} \leq \frac{2K}{K + \sqrt{K^2 + 8KL}} \leq 1 - L\eta, \tag{3.5}$$

which is true, if

$$\frac{1}{8}(K + 4L + \sqrt{K^2 + 8KL})\eta \leq \frac{1}{2}. \tag{3.6}$$

Notice however that

$$L \leq K \tag{3.7}$$

holds in general and $\frac{K}{L}$ can be arbitrarily large [5]. Moreover, we have

$$K\eta \leq \frac{1}{2} \Longrightarrow \frac{1}{8}(K + 4L + \sqrt{K^2 + 8KL})\eta \leq \frac{1}{2}, \tag{3.8}$$

but not necessarily vice versa, unless if $K = L$. Hence, if $L < K$ (3.6) is a weaker convergence criterion for Newton's method than $K\eta \leq \frac{1}{2}$.

Application 3.4 Let $F(x) = F_1(x) + F_2(x)$ for each $x \in D$, where F_1 is differentiable and F_2 is continuous. Let us also consider the iteration defined for each $n = 0, 1, 2, \ldots$ [2]

$$x_{n+1} = x_n - A(x_{n-1}, x_n)^{-1} F(x_n), \tag{3.9}$$

where $A(x, y)^{-1} \in L(Y, X)$ for each $x, y \in D$. Method like (3.9) have been studied extensively for different choice of A in connection to the solution of nonlinear equations containing a non differentiable term. Let us set $A(x, y) = F'(y) + [x, y; F_2]$. Suppose that $A(x_{-1}, x_0)^{-1} \in L(Y, X)$ and for each $u, x, y \in D$ there exist nonnegative constants $\eta_0, \eta, K, K_1, L_0, L_1, L_2$ such that

$$\begin{aligned}
&\|x_{-1} - x_0\| \leq \eta_0, \;\; \|x_1 - x_0\| \leq \eta, \\
&\|A(x_{-1}, x_0)^{-1}(F_1'(x) - F_1'(x_0))\| \leq L_0\|x - x_0\|, \\
&\|A(x_{-1}, x_0)^{-1}(F_1'(x) - F_1'(y))\| \leq K\|x - y\| \\
&\|A(x_{-1}, x_0)^{-1}([x, y; F_1] - [u, y; F_1])\| \leq K_1\|x - u\| \\
&\|A(x_{-1}, x_0)^{-1}([x, y; F_2] - [x_{-1}, x_0; F_2])\| \leq L_1\|x - x_{-1}\| + L_2\|y - x_0\|
\end{aligned}$$

and

$$U(x_0, \rho) \subseteq D,$$

where ρ is to be determined. Then, we can choose:$a_0 = K + 2K_1, r = 0, s = K_1\eta$, $d = L_0 + L_1 + L_2$, $p = L_1(\eta + \eta_0) - 1$ and $\rho = s^*$ (or t^*) depending on which sequence we use. Indeed, we have

$$\begin{aligned}
&\|A(x_{-1}, x_0)^{-1}(A(x_{k-1}, x_k) - A(x_{-1}, x_0))\| \\
&\quad \leq \|A(x_{-1}, x_0)^{-1}(F'(x_k) - F'(x_0))\| \\
&\qquad + \|A(x_{-1}, x_0)^{-1}([x_{k-1}, x_k; F_2] - [x_{-1}, x_0; F_2])\| \\
&\quad \leq L_0\|x_k - x_0\| + L_1\|x_{k-1} - x_{-1}\| + L_2\|x_k - x_0\| \\
&\quad \leq L_0\|x_k - x_0\| + L_1\|(x_{k-1} - x_k) + (x_k - x_0) + (x_0 - x_{-1})\| \\
&\qquad + L_2\|x_k - x_0\| \\
&\quad \leq (L_0 + L_1 + L_2)\|x_k - x_0\| \\
&\qquad + L_1\|x_k - x_{k-1}\| + L_1\|x_0 - x_{-1}\| \\
&\quad \leq (L_0 + L_1 + L_2)\|x_k - x_0\| + L_1\eta + L_1\eta_0
\end{aligned}$$

and

$$
\begin{aligned}
&\|A(x_{-1},x_0)^{-1}(F_1(x_k)+F_2(x_k))\| \\
&\quad \le \|A(x_{-1},x_0)^{-1}(F_1(x_k)-F_1(x_{k-1})-F_1'(x_k)(x_k-x_{k-1}))\| \\
&\qquad +\|A(x_{-1},x_0)^{-1}([x_k,x_{k-1};F_2]-[x_{k-2},x_{k-1};F_2])(x_k-x_{k-1})\| \\
&\quad \le \left(\frac{K}{2}\|x_k-x_{k-1}\|+K_1\|x_k-x_{k-2}\|\right)\|x_k-x_{k-1}\| \\
&\quad \le \left(\left(\frac{K}{2}+K_1\right)\|x_k-x_{k-1}\|+K_1\|x_{k-1}-x_{k-2}\|\right)\|x_k-x_{k-1}\|
\end{aligned}
$$

so

$$
\begin{aligned}
\|x_{k+1}-x_k\| &= \|A(x_{-1},x_0)^{-1}(F_1(x_k)+F_2(x_k))\| \\
&\le \|A(x_{-1},x_0)^{-1}A(x_{-1},x_0)\|\|A(x_{-1},x_0)^{-1}(F_1(x_k)+F_2(x_k))\| \\
&\le s_{k+1}-s_k.
\end{aligned}
$$

Then, the convergence of method (3.9) is guaranteed under the hypotheses of Theorem 2.4. In a future paper, we will extend these results even further using our ideas of the restricted convergence domain and the technique used in [4] for Newton's method. Examples where, the new Lipschitz constants are smaller than the old ones can be found in [1–4,6].

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