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ORIGINAL PAPER

On the convergence of Newton-like methods using restricted domains

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Abstract We present a new semi-local convergence analysis for Newton-like methods in order to approximate a locally unique solution of a nonlinear equation containing a non-differentiable term in a Banach space setting. The new idea uses more precise convergence domains. This way the new sufficient convergence conditions are weaker, and the error bounds are tighter than in earlier studies. Applications and numerical examples, involving a nonlinear integral equation of Chandrasekhar-type, are also provided in this study.

Keywords Newton's method · Banach space · Semi-local convergence · Kantorovich hypothesis · Restricted domains

Mathematics Subject Classification (2010) 65J15 · 65G99 · 47H17 · 45G10

1 Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) + G(x) = 0, \tag{1.1}$$

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where F is a Fréchet-differentiable operator defined on a convex subset Ω of a Banach space B_1 with values in a Banach space B_2 and $G : \Omega \longrightarrow B_2$ is a continuous operator.

A large number of problems in applied mathematics and also in engineering can be written like (1.1) using mathematical modelling [5, 7, 8, 11, 13, 15, 17, 29, 34]. The solution methods for solving (1.1) are iterative, i.e., starting from one or several initial approximations, a sequence is constructed that converges to a solution of the Eq. (1.1). Iteration methods, when applied for solving optimization problems, the iteration sequences converge to an optimal solution of the problem under consideration. Since all these methods have the same recursive structure, they can be introduced and studied in a general framework.

The Newton-like method(NLM)

$$\begin{aligned} x_{n+1} &= x_n - A(x_n)^{-1}P(x_n) \ \ (n \geq 0), \\ P(x) &= F(x) + G(x), \ \ (x \in \Omega) \end{aligned} \tag{1.2}$$

has been used by several authors to generate a sequence $\{x_n\}$ approximating x^* [1, 38]. Here, $A(x) \in L(B_1, B_2)$ is the space of all bounded linear operators from B_1 to B_2,and $F'(x)$ is the Fréchet derivative of operator $F(x)$ [8, 11, 34]. Note that at each step, the method requires one operator evaluation $P(x_n)$ and one inverse $A(x_n)^{-1}$.

- **Case** $G = 0$. Under Kantorovich-type assumptions (see Section 3), Rheinboldt [35] established a convergence theorem for NLM which includes the Kantorovich theorem for the Newton method($A(x) = F'(x)$) as a special case [8–12, 14, 16]. A further generalization was given by Dennis in [16], Deuflhard and Heindl in [17], Potra in [34].

  Miel [30, 31] improved the error bounds given by Rheinboldt in [35]. Using stronger condition than those of Rheinboldt, Moret in [32] not only obtained a convergence theorem and error bounds for NLM but also, using a numerical example showed that his bounds are sharper than those of Miel. However, no proof was given in [32]. Yamamoto in [36] presented a method for finding error bounds for NLM under Dennis assumptions and showed that the bounds obtained improve those of Rheinboldt, Dennis and Miel and reduce to Moret's bounds if we replace the assumptions by his strong assumptions. It was also shown that Moret's results can be derived from Rheinboldt's.
- **Case** $G \neq 0$. If $A(x) = F'(x)$, $(x \in \Omega)$,Zabrejko and Nguen [37] established a convergence theorem for the Krasnoselskii-Zincenko-type iteration [38]. Yamamoto and Chen [14] extended the results in [37, 38], when $A(x)$ is not necessarily equal to $F'(x)$. Under the same conditions, the results were specialized in [28] for Broyden-like methods. Related work can be found in the works by Amat et al. [1–7], Hernandez et al. [18–26] and Magrenan et al. [29] where the method of recurrent relations was used. Argyros [8–12] using his technique method of recurrent functions presented a unified convergence theory for even more general NLM with the following advantages over the above stated works under the same computational cost.

- **Semi-local Case.** weaker sufficient convergence conditions, tighter error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$, $(n \geq 0)$,and an at least as precise information on the location of the solution.

  The above advantages are obtained by using the needed center-Lipschitz instead of the Lipschitz condition commonly used for the derivation of the upper bounds on the norms $\|A(x_n)^{-1}A(x_0)\|$ $(n \geq 0)$. This modification leads to more precise majorizing sequences, which in turn result weaker sufficient convergence conditions in most interesting cases (see also Section 3).
- **Local Case.** A larger radius of convergence is also obtained.

In the present paper, we extend the applicability of NLM even further than in the preceding works using more precise domains containing the iterates x_n leading to smaller Lipschitz conditions which finally lead to a finer convergence analysis for this method.

The rest of the paper is organized as follows: Section 2 contains the semi-local convergence of NLM. Special case and applications are presented in Section 3 to show that our results can apply to solve equations, where earlier ones cannot.

2 Semi-local convergence analysis of NLM

We will be using the following auxiliary result on majorizing sequences for NLM.

Lemma 2.1 *Suppose:*

(i) *there exist constants* $K > 0$, $M > 0$, $\mu \geq 0$, $L > 0$, $\ell \geq 0$, $\eta > 0$ *such that*

$$2M < K \tag{2.1}$$

and

$$L\eta + \ell < 1. \tag{2.2}$$

Let

$$\delta_0 := \frac{K\eta + 2\mu}{1 - L\eta - \ell} \tag{2.3}$$

and

$$\delta := \frac{2(K - 2M)}{K + \sqrt{K^2 - 8L(2M - K)}}. \tag{2.4}$$

(ii) *Quadratic polynomial* $\bar{f}_\infty$ *defined by*

$$\bar{f}_\infty(s) = (1 - \ell)s^2 - (1 - \ell - L\eta + \mu)s + M\eta + \mu \tag{2.5}$$

has at least one root in the interval $(0, 1)$. *Denote such a root by* s_∞.

(iii)

$$\delta_0 \leq \delta \leq 2s_\infty. \tag{2.6}$$

Then, the scalar sequence $\{t_n\}$ $(n \geq 0)$ *given by*

$$t_0 = 0,\ t_1 = \eta,\ t_{n+2} = t_{n+1} + \frac{K(t_{n+1} - t_n) + 2(Mt_n + \mu)}{3(1 - Lt_{n+1} - \ell)}(t_{n+1} - t_n) \tag{2.7}$$

is increasing, bounded from above by

$$t^{**} = \frac{2\eta}{2-\delta}, \tag{2.8}$$

and converges to the unique least upper bound $t^* \in [\eta, t^{**}]$. *Moreover the following estimates hold for all* $n \geq 1$;

$$0 < t_{n+1} - t_n \leq \frac{\delta}{2}(t_n - t_{n-1}) \leq \left(\frac{\delta}{2}\right)^n \eta \tag{2.9}$$

and

$$t^* - t_n \leq \frac{2\eta}{2-\delta}\left(\frac{\delta}{2}\right)^n.$$

Proof We shall prove using induction on the integer m that:

$$0 < t_{m+2} - t_{m+1} = \frac{K(t_{m+1} - t_m) + 2(Mt_m + \mu)}{3(1 - Lt_{m+1} - \ell)}(t_{m+1} - t_m) \tag{2.10}$$

and

$$\ell + Lt_{m+1} < 1. \tag{2.11}$$

If (2.10) and (2.11) hold, we have that (2.9) holds, and

$$\begin{aligned} t_{m+2} &\leq t_{m+1} + \frac{\delta}{2}(t_{m+1} - t_m) \\ &\leq t_m + \frac{\delta}{2}(t_m - t_{m-1}) + \frac{\delta}{2}(t_{m+1} - t_m) \\ &\leq \eta + +\frac{\delta}{2}\eta + \cdots + \left(\frac{\delta}{2}\right)^{m+1}\eta \\ &= \frac{1 - \left(\frac{\delta}{2}\right)^{m+2}}{1 - \frac{\delta}{2}}\eta \\ &< \frac{2\eta}{2-\delta} = t^{**} \quad \text{(by (2.8))}. \end{aligned} \tag{2.12}$$

It will then also follow that sequence $\{t_m\}$ is increasing, bounded from above by t^{**} and converge to some $t^* \in [\eta, t^{**}]$.

Estimates (2.10) and (2.11) hold by the initial conditions for $m = 0$. Indeed, (2.10) and (2.11) become, respectively:

$$\begin{aligned} 0 < t_2 - t_1 &= \frac{K(t_1 - t_0) + 2(Mt_0 + \mu)}{2(1 - Lt_1 - \ell)}(t_1 - t_0) \\ &= \frac{K\eta + 2\mu}{2(1 - L\eta - \ell)}(t_1 - t_0) \\ &= \frac{\delta_0}{2}(t_1 - t_0) \leq \frac{\delta}{2}(t_1 - t_0) \end{aligned}$$

and

$$L\eta + \ell < 1,$$

which are true by the choice of δ_0, δ, (2.2), (2.7) and the initial conditions. Suppose that (2.10)–(2.11) hold for all $m \leq n+1$.

Estimate (2.11) can be re-written as

$$K(t_{m-1}-t_m)+2(Mt_m+\mu)\leq(1-Lt_{m+1}-\ell)\delta$$

or

$$K(t_{m-1}-t_m)+2(Mt_m+\mu)+Lt_{m+1}\delta+\ell\delta-\delta\leq 0$$

or

$$K\left(\frac{\delta}{2}\right)^m\eta+2\left(M\frac{1-\left(\frac{\delta}{2}\right)^m}{1-\frac{\delta}{2}}\eta+\mu\right)+\delta L\left(\frac{1-\left(\frac{\delta}{2}\right)^{m+1}}{1-\frac{\delta}{2}}\right)\eta+\delta(\ell-1)\leq 0. \quad (2.13)$$

Replace $\frac{\delta}{2}$ by s and define functions f_m on $[0,1)(m\geq 1)$:

$$\begin{aligned} f_m(s) = {} & Ks^m\eta+2(M(1+s+s^2+\cdots+s^{m-1})\eta+\mu) \\ & +2sL(1+s+s^2+\cdots+s^m)\eta+2s(\ell-1). \end{aligned} \quad (2.14)$$

In view of (2.13) and (2.14) estimates (2.10) and (2.11) certainly hold, if

$$f_m(s)\leq 0 \quad (m\geq 1). \quad (2.15)$$

We need to find a relationship between two consecutive functions f_m :

$$\begin{aligned} f_{m+1}(s) &= Ks^{m+1}\eta+2(M(1+s+s^2+\cdots+s^{m-1}+s^m)\eta+\mu) \\ &\quad +2sL(1+s+s^2+\cdots+s^{m-1}+s^m)\eta+2s(\ell-1) \\ &= Ks^{m+1}\eta-Ks^m\eta+Ks^{m-1}\eta+2(M(1+s+s^2+\cdots+s^{m-1})\eta+\mu) \\ &\quad +2Ms^m\eta+2sL(1+s+s^2+\cdots+s^{m-1}+s^m)\eta+2sLs^{m+1}\eta+2s(\ell-1) \\ &= f_m(s)+Ks^{m+1}\eta-Ks^m\eta+2Ms^m\eta+2sLs^{m+1}\eta \\ &= f_m(s)+g(s)s^m\eta, \end{aligned} \quad (2.16)$$

where

$$g(s)=2Ls^2+Ks+2M-K. \quad (2.17)$$

Quadratic polynomial g has a positive zero δ given by (2.4). Then, we have that

$$f_{m+1}(\delta)=f_m(\delta). \quad (2.18)$$

In view of (2.13)

$$f_\infty(s_\infty):=\lim_{m\longrightarrow\infty}f_m(s_\infty)=2\left(\frac{M}{1-s_\infty}\eta+\mu\right)+\frac{2s_\infty L}{1-s_\infty}\eta+2s_\infty(\ell-1), \quad (2.19)$$

by the choice of s_∞. Hence, (2.15) holds by (2.6) and (2.18). The induction is complete. That is, the sequence $\{t_n\}$ is increasing, bounded from above by t^{**} and as such it converges to its unique upper bound t^*. □

Let $x_0 \in \Omega$. Denote by $U(v,\rho)$, $\bar{U}(v,\rho)$ the open and closed balls in B_1 with center $v\in B_1$ and of radius $\rho>0$. Let $R>0$. Define

$$R_0:=\sup\{t\in[0,R]:U(x_0,R)\subset\Omega\}. \quad (2.20)$$

Set

$$S = \bar{U}(x_0, R_0). \tag{2.21}$$

We shall show the following semi-local convergence result for NLM.

Theorem 2.2 *Let $F : S \subseteq B_1 \longrightarrow B_2$ be a Fréchet differentiable operator, $G : S \longrightarrow B_2$ be a continuous operator and let $A(x) \in L(B_1, B_2)$. Suppose that there exist an open convex set S of B_1, $x_0 \in S$,a bounded inverse $A(x_0)^{-1}$ of $A(x_0)$ and constants $K > 0$, $M > 0$, $\mu_0 \geq 0$, $L > 0$, $\ell \geq 0$, $\eta > 0$,such that for all $x \in S :$*

$$\|A(x_0)^{-1}[F(x_0) + G(x_0)]\| \leq \eta \tag{2.22}$$

$$\|A(x_0)^{-1}[A(x) - A(x_0)]\| \leq L\|x - x_0\| + \ell \tag{2.23}$$

and for each $x, u, \in U(x_0, T) \cap S$

$$\|A(x_0)^{-1}[F'(x) - F'(y)]\| \leq K\|x - y\| \tag{2.24}$$

$$\|A(x_0)^{-1}[F'(x) - A(x)]\| \leq M\|x - x_0\| + \mu_0 \tag{2.25}$$

$$\|A(x_0)^{-1}[G(x) - G(y)]\| \leq \mu_1\|x - y\| \tag{2.26}$$

$$t^* \leq R_0 \text{ or } T \leq R_0, \tag{2.27}$$

the hypotheses of Lemma 2.1 hold with

$$\mu = \mu_0 + \mu_1, \tag{2.28}$$

where t^* is given in Lemma 2.1 and $T = \frac{1-\ell}{L}$. Then, the sequence $\{x_n\}(n \geq 0)$ generated by NLM is well defined, remains in $\bar{U}(x_0, t^*)$ for all $n \geq 0$ and converges to a solution x^* of (1.1) in $\bar{U}(x_0, t^*)$. Moreover, the following estimates hold for all $n \geq 0 :$

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{2.29}$$

and

$$\|x_n - x^*\| \leq t^* - t_n, \tag{2.30}$$

where, sequence $\{t_n\}(n \geq 0)$ is given in Lemma 2.1. Furthermore, the solution x^* of equation (1.1) is unique in $\bar{U}(x_0, t^*)$ provided that:

$$\left(\frac{K}{2} + M + L\right)t^* + \mu + \ell < 1.$$

Proof We shall show using induction on $m > 0 :$

$$\|x_{m+1} - x_m\| \leq t_{m+1} - t_m, \tag{2.31}$$

and

$$\bar{U}(x_{m+1}, t^* - t_{m+1}) \subseteq \bar{U}(x_m, t^* - t_m). \tag{2.32}$$

For every $x \in \bar{U}(x_1, t^* - t_1)$,

$$\|x - x_0\| \leq \|x - x_1\| + \|x_1 - x_0\| \leq t^* - t_1 + t_1 - t_0 = t^* - t_0,$$

implies $x \in \bar{U}(x_0, t^* - t_0)$. We also have

$$\|x_1 - x_0\| = \|A(x_0)^{-1}[F(x_0) + G(x_0)]\| \leq \eta = t_1 - t_0.$$

That is (2.31) and (2.32) hold for $m = 0$. Given they hold for $n \leq m$,then

$$\begin{aligned}\|x_{m+1} - x_m\| &\leq \sum_{i=1}^{m+1} \|x_i - x_{i-1}\| \\ &\leq \sum_{i=1}^{m+1} (t_i - t_{i-1}) = t_{m+1} - t_0 = t_{m+1},\end{aligned}$$

and

$$\|x_m + \theta(x_{m+1} - x_m) - x_0\| \leq t_m + \theta(t_{m+1} - t_m) \leq t^*,$$

for all $\theta \in (0, 1)$.

Using (2.11), (2.26) and the induction hypotheses, we get

$$\begin{aligned}\|A(x_0)^{-1}[A(x_{m+1}) - A(x_0)]\| &\leq L\|x_{m+1} - x_0\| + \ell \\ &\leq L(t_{m+1} - t_m) + \ell \\ &\leq Lt_{m+1} + \ell < 1. \end{aligned} \tag{2.33}$$

It follows from (2.33) and the Banach lemma on invertible operators [8, 11, 34] that $A(x_{m+1})$ is invertible and

$$\|A(x_{m+1})^{-1}A(x_0)\| \leq \frac{1}{1 - \ell - Lt_{m+1}}. \tag{2.34}$$

Using (1.2), we obtain the approximation:

$$\begin{aligned}x_{m+2} - x_{m+1} &= -(F(x_{m+1}) + G(x_{m+1})) \\ &= A(x_{m+1})^{-1}A(x_0)A(x_0)^{-1} \\ &\quad \bigg(\int_0^1 [F'(x_{m+1} + \theta(x_m - x_{m+1})) - F'(x_{m+1})](x_{m+1} - x_m)d\theta \\ &\quad +(F'(x_m) - A(x_m))(x_{m+1} - x_m) + G(x_{m+1}) - G(x_m))\bigg)\end{aligned} \tag{2.35}$$

Using (2.5), (2.6), (2.7), (2.34), (2.35) and the induction hypotheses, we obtain in turn that

$$\begin{aligned}\|x_{m+2} - x_{m+1}\| &\leq (1 - \ell - Lt_{m+1})^{-1}\left(\frac{K}{2}\|x_{m+1} - x_m\|^2\right. \\ &\quad \left.+(M\|x_m - x_0\| + \mu_0)\|x_{m+1} - x_m\| + \mu_1\|x_{m+1} - x_m\|\right) \\ &\leq (1 - \ell - Lt_{m+1})^{-1}\left(\frac{K}{2}(t_{m+1} - t_m) + Mt_m + \mu\right)(t_{m+1} - t_m) \\ &= t_{m+2} - t_{m+1},\end{aligned} \tag{2.36}$$

which shows (2.32) for all $m \geq 0$. Thus, for every $z \in \bar{U}(x_{m+2}, t^* - t_{m+2})$,we have

$$\begin{aligned}\|z - x_{m+1}\| &\leq \|z - x_{m+2}\| + \|x_{m+2} - x_{m+1}\| \\ &\leq t^* - t_{m+2} + t_{m+2} - t_{m+1} = t^* - t_{m+1},\end{aligned}$$

which shows (2.32) for all $m \geq 0$.

Lemma 2.1 implies that $\{t_n\}$ is a complete sequence. Moreover, it follows from (2.31) and (2.32) that $\{x_n\}(n \geq 0)$ is also a complete sequence in a Banach space B_1 and as such it converges to some $x^* \in \bar{U}(x_0, t^*)$ (since $\bar{U}(x_0, t^*)$ is a closed set).

In view of (2.36), we have that

$$\|A(x_0)^{-1}(F(x_{m+1}) + G(x_{m+1}))\| \leq \left(\frac{K}{2}(t_{m+1} - t_m) + Mt_m + \mu\right)(t_{m+1} - t_m). \tag{2.37}$$

By letting $m \longrightarrow \infty$ in (2.37), we obtain $F(x^*) + G(x^*) = 0$. Estimate (2.29) is obtained from (2.37) by using standard majorizing techniques [8, 11, 34]. To show the uniqueness part, let $y^* \in \bar{U}(x_0, t^*)$ with $F(y^*) + G(y^*) = 0$. Then, we have:

$$\begin{aligned}
\|y^* - x_{m+1}\| \leq\ & \|A(x_m)^{-1}A(x_0)\| \\
& \left\{\left(\int_0^1 \|A(x_0)^{-1}(F'(x_m + \theta(y^* - x_m)) - F'(x_m))\|d\theta\right.\right. \\
& \left. + \|A(x_0)^{-1}[F'(x_m) - A(x_m)]\|\right)\|y^* - x_m\| \\
& \left. + \|A(x_0)^{-1}[G(x_m) - G(y^*)]\|\right\} \\
\leq\ & (1 - Lt_{m+1})^{-1}\left(\frac{K}{2}\|y^* - x_m\|^2\right. \\
& \left. + (M\|x_m - x_0\| + \mu)\|y^* - x_m\|\right) \\
\leq\ & (1 - Lt_{m+1})^{-1}\left(\frac{K}{2}(t^* - t_m) + (Mt_m + \mu)\|y^* - x_m\|\right) \\
\leq\ & (1 - Lt_{m+1})^{-1}\left(\frac{K}{2}(t^* - t_0) + (Mt^* + \mu)\|x^* - x_m\|\right) \\
<\ & \|y^* - x_m\|,
\end{aligned} \tag{2.38}$$

by the uniqueness hypothesis. It follows by (2.38) that $\lim_{m \longrightarrow \infty} x_m = y^*$. But we showed $\lim_{m \longrightarrow \infty} x_m = x^*$. Hence, we deduce that $x^* = y^*$. □

Remark 2.3 Note that t^* can be replaced by t^{**} given in closed form by (2.8) in the uniqueness hypothesis provided that $t^* \leq R_0$ or in all the other hypotheses of the theorem.

3 Special cases and applications

The results in related studies can be improved, by simply noticing that the Lipschitz conditions in earlier studies are to be satisfied in $U(x_0, R_0)$ (i.e., in Ω). However, using our technique with the exception of the center Lipschitz condition (see (2.4)) the rest of the Lipschitz conditions are to be satisfied in $U(x_0, t^*)$ or $U(x_0, T)$. Then, since these balls are at least as small and inside $U(x_0, R_0)$. It follows that corresponding constants $\bar{K}, \bar{M}, \bar{\mu}_0, \bar{\mu}_1$ are at least as large. That is

$$K \leq \bar{K}, M \leq \bar{M}, \mu_0 \leq \bar{\mu}_0, \mu_1 \leq \bar{\mu}_1 \tag{3.1}$$

hold. Then, in case any of the inequalities in (3.1) is strict then our results in [8–11] using bar constants are improved. Moreover, other famous relevant results can be improved by repeating their proofs with the above changes.

Next, we list some examples.

- **Case 3.1** (**Newton-Like method**) Let $G = 0$. Using hypothesis

$$\bar{h} = \bar{\sigma}\eta \leq \frac{1}{2}(1-\bar{b})^2, \quad \mu + \ell < 1 \tag{3.2}$$

where $\bar{\sigma} := \max\{\bar{K}, \bar{M}+L\}$, with $\bar{b} := \bar{\mu}+\ell$, numerous semi-local convergence theorems were provided in [14, 16, 17, 30–38].

Then, following corresponding proofs using the new technique, the corresponding to (3.2) hypothesis is given by

$$h_A = \sigma\eta \leq \frac{1}{2}(1-b)^2, \tag{3.3}$$

where $\sigma := \max\{K, M+L\} \leq \bar{\sigma}$ and $b := \mu + \ell \leq \bar{b}$.

Then, we have that

$$h \leq \frac{1}{2}(1-\bar{b})^2 \Rightarrow h_A \leq \frac{1}{2}(1-b)^2, \tag{3.4}$$

but not necessarily vice versa unless if equality holds in all inequalities in (3.1). Clearly, corresponding majorizing sequences and the information on the location of the solution are more precise. Indeed, the corresponding to $\{t_n\}$ majorizing sequence given before is defined by

$$u_0 = 0, u_1 = \eta, \; u_{n+2} = u_{n+1} + \frac{\bar{K}(u_{n+1}-u_n) + 2(\bar{M}u_n + \bar{\mu})}{2(1-\bar{L}u_{n+1}-\ell)}(u_{n+1}-u_n). \tag{3.5}$$

Then, a simple inductive argument shows that

$$t_n \leq u_n, \quad t_{n+1} - t_n \leq u_{n+1} - u_n \tag{3.6}$$

and

$$t^* \leq u^* = \lim_{n\longrightarrow\infty} u_n. \tag{3.7}$$

Inequalities (3.6) are strict for $n > 1$, if any of the inequalities in (3.1) is strict.

- **Case 3.2** (**Newton's method.**) Let $G = 0$, $\bar{\sigma} = \bar{K}$ and $\bar{\mu}_0 = \bar{\mu}_1 = \ell = \bar{M} = 0$. Then, condition (3.2) reduces to the famous for its simplicity and clarity Newton-Kantorovich hypothesis for the semi-local convergence of Newton's method [3, 5, 8–12, 14, 25–27, 30–38].

$$\bar{h}_1 = \bar{K}\eta \leq \frac{1}{2}. \tag{3.8}$$

In this case, functions $f_m (m \geq 1)$ are defined by

$$f_m(s) = (Ks^{m-1} + 2L(1+s+s^2+\cdots+s^m))\eta - 2,$$

and

$$f_{m+1}(s) = f_m(s) + g(s)s^{m-1}\eta.$$

The conditions corresponding to Lemma 2.1 are for,

$$s_\infty = 1 - L\eta, \tag{3.9}$$

$$h_A^1 = \bar{L}_1\eta \leq \frac{1}{2}, \tag{3.10}$$

where,

$$\bar{L}_1 = \frac{1}{8}(K + 4L + \sqrt{K^2 + 8KL}). \tag{3.11}$$

Our earlier condition [12] is given by

$$h_A^2 = \bar{L}_2\eta \leq \frac{1}{2},$$

where

$$\bar{L}_2 = \frac{1}{8}(\bar{K} + 4L + \sqrt{\bar{K}^2 + 8\bar{K}L}). \tag{3.12}$$

Note also that

$$L \leq \bar{K} \tag{3.13}$$

holds in general, and $\frac{\bar{K}}{L}$ can be arbitrarily large [8, 11]. In view of (3.8), (3.10) and (3.12), we get

$$h_1 \leq \frac{1}{2} \Rightarrow h_A^2 \leq \frac{1}{2} \Rightarrow h_A^1 \leq \frac{1}{2} \tag{3.14}$$

but not necessarily vice versa unless, if $L = K = \bar{K}$.

In the next example, we show (3.8) is not satisfied but (3.10) or (3.12) hold.

Example 3.1 Let $B_1 = B_2 = \mathbb{R}, x_0 = 1, \Omega = \{x : |x - x_0| \leq 1 - \beta\}, \beta \in [0, \frac{1}{2}), R_0 = 1 - \beta, T = \frac{1}{L}$ and define function F on Ω by

$$F(x) = x^3 - \beta. \tag{3.15}$$

Using hypotheses of Theorem 2.2, we get:

$$\eta = \frac{1}{3}(1 - \beta),\ L = 3 - \beta,\ \bar{K} = 2(2 - \beta)\ \text{ and }\ K = 2\left(1 + \frac{1}{L}\right).$$

Notice that we have

$$L < \bar{K},\ K < \bar{K}\ \text{ and }\ T < R_0.$$

The Newton-Kantorovich condition (3.8) is violated, since

$$\frac{4}{3}(1 - \beta)(2 - \beta) > 1\ \text{ and }\ \beta \in [0, \frac{1}{2}). \tag{3.16}$$

Hence, there is no guarantee that Newton's method (1.2) converges to $x^* = \sqrt[3]{\beta}$, starting at $x_0 = 1$. However, our old condition (3.12) is true for all $\beta \in I = [0.450339002, \frac{1}{2})$ whereas new condition (3.10) holds for $I_N = [0.433124869, \frac{1}{2})$. Hence, the conclusions of our Theorem 2.2 can apply to solve equation (3.15) for all $\beta \in I_N$.

Example 3.2 Let $B_1 = B_2 = \mathcal{C}[0,1]$ be the space of continuous functions defined in [0, 1] equipped with the max-norm. Let $\Omega = \{x \in \mathcal{C}[0,1];\ \|x\| \leq R\}$, such that $R > 0$ and F is defined on Ω and given by [13]:

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 G(s,t)x(t)^3\,dt, \quad x \in C[0,1],\ s \in [0,1],$$

where $f \in \mathcal{C}[0,1]$ is a given function, λ is a real constant and the kernel G is the Green's function

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$$

In this case, for each $x \in \Omega$, $F'(x)$ is a linear operator defined on Ω by the following expression:

$$[F'(x)(v)](s) = v(s) - 3\lambda \int_0^1 G(s,t)x(t)^2 v(t)\,dt, \quad v \in C[0,1],\ s \in [0,1].$$

If we choose $x_0(s) = f(s) = 1$, it follows that $\|I - F'(x_0)\| \leq 3|\lambda|/8$. Thus, if $|\lambda| < 8/3$, $F'(x_0)^{-1}$ is defined and

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8 - 3|\lambda|}.$$

Moreover,

$$\|F(x_0)\| \leq \frac{|\lambda|}{8},$$

so

$$\eta = \|F'(x_0)^{-1}F(x_0)\| \leq \frac{|\lambda|}{8 - 3|\lambda|}.$$

On the other hand, for $x, y \in \Omega$ we have

$$\|F'(x) - F'(y)\| \leq \|x - y\|\frac{1 + 3|\lambda|(\|x + y\|)}{8} \leq \|x - y\|\frac{1 + 6R|\lambda|}{8}.$$

and

$$\|F'(x) - F'(1)\| \leq \|x - 1\|\frac{1 + 3|\lambda|(\|x\| + 1)}{8} \leq \|x - 1\|\frac{1 + 3(1 + R)|\lambda|}{8}.$$

Choosing $\lambda = 1.175$ and $R = 2$, we have $\eta = 0.26257\ldots$, $\bar{K} = 2.76875...$, $L = 1.8875\ldots$, $\frac{1}{L} = 0.529801\ldots$, $K = 1.47314\ldots$.

Using these values, we obtain that condition (3.8) is not satisfied, since:

$$\bar{h}^1 = 1.02688\ldots > 1,$$

but condition (3.10) is satisfied:

$$h_A^1 = 0.986217\ldots < 1,$$

so we can ensure the convergence of the Newton's method by Theorem 2.2.

Application 3.3 Let

$$A(y_n) = F'(y_n) + [y_{n-1}, y_n; G], \quad (n \geq 0)$$

and consider NLM in the form

$$y_{n+1} = y_n - (F'(y_n) + [y_{n-1}, y_n; G])^{-1}(F(y_n) + G(y_n)) \;\; (n \geq 0). \qquad (3.17)$$

This method has order $\frac{1+\sqrt{5}}{2}$ (see [8, 11, 25–27])(same as the method of Chord), but higher than the order of

$$z_{n+1} = z_n - F'(z_n)^{-1}(F(z_n) + G(z_n)) \;\; (n \geq 0) \qquad (3.18)$$

considered in [36, 37] and the method of Chord

$$w_{n+1} = w_n - [w_{n-1}, w_n; G]^{-1}(F(w_n) + G(w_n)) \;\; (n \geq 0), \qquad (3.19)$$

where $[x, y; G]$ denotes the divided difference of G at the points x and y considered in [8, 11].

Now, we shall provide an example for this case.

Example 3.4 Let $B_1 = B_2 = (\mathbb{R}^2, \|.\|_\infty)$. Consider the system

$$3x^2y + y^2 - 1 + |x - 1| = 0$$
$$x^4 + xy^3 - 1 + |y| = 0.$$

Table 1 Comparison Table 1

n	$z_n^{(1)}$	$z_n^{(2)}$	$\|z_n - z_{n-1}\|$
0	1	0	
1	1	0.333333333333333	3.333E-1
2	0.906550218340611	0.354002911208151	9.344E-2
3	0.885328400663412	0.338027276361322	2.122E-2
4	0.891329556832800	0.326613976593566	1.141E-2
5	0.895238815463844	0.326406852843625	3.909E-3
6	0.895154671372635	0.327730334045043	1.323E-3
7	0.894673743471137	0.327979154372032	4.809E-4
8	0.894598908977448	0.327865059348755	1.140E-4
9	0.894643228355865	0.327815039208286	5.002E-5
10	0.894659993615645	0.327819889264891	1.676E-5
11	0.894657640195329	0.327826728208560	6.838E-6
12	0.894655219565091	0.327827351826856	2.420E-6
13	0.894655074977661	0.327826643198819	7.086E-7
...			
39	0.894655373334687	0.327826521746298	5.149E-19

Table 2 Comparison Table 2

n	$w_n^{(1)}$	$w_n^{(2)}$	$\|w_n - w_{n-1}\|$
−1	5	5	
0	1	0	5.000E+00
1	0.989800874210782	0.012627489072365	1.262E-02
2	0.921814765493287	0.307939916152262	2.953E-01
3	0.900073765669214	0.325927010697792	2.174E-02
4	0.894939851625105	0.327725437396226	5.133E-03
5	0.894658420586013	0.327825363500783	2.814E-04
6	0.894655375077418	0.327826521051833	3.045E-04
7	0.894655373334698	0.327826521746293	1.742E-09
8	0.894655373334687	0.327826521746298	1.076E-14
9	0.894655373334687	0.327826521746298	5.421E-20

Set $\|x\|_\infty = \|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$, $F = (F_1, F_2)$, $G = (G_1, G_2)$. For $x = (x_1, x_2) \in \mathbb{R}^2$,we choose $F_1(x_1, x_2) = 3x_1^2 x_2 + x_2^2 - 1$, $F_2(x_1, x_2) = x_1^4 + x_1 x_2^3 - 1$, $G_1(x_1, x_2) = |x_1 - 1|$, $G_2(x_1, x_2) = |x_2|$. We shall take $[x, y; G] \in M_{2\times 2}(\mathbb{R})$ as

$$[x, y; G]_{i,1} = \frac{G_1(y_1, y_2) - G_i(x_1, y_2)}{y_1 - x_1},$$

$$[x, y; G]_{i,2} = \frac{G_1(x_1, y_2) - G_i(x_1, x_2)}{y_2 - x_2}, \quad i = 1, 2,$$

provided that $y_1 \neq x_1$ and $y_2 \neq x_2$. Otherwise define $[x, y; G]$ to be the zero matrix in $M_{2\times 2}(\mathbb{R})$. Moreover, using method (3.18) with $z_0 = (1, 0)$ we obtain Comparison Table 1. Furthermore, using the method of Chord (3.19) with $w_{-1} = (1, 0)$ and $w_0 = (5, 5)$,we obtain Comparison Table 2.

Finally, using our method (3.17) with $y_{-1} = (1, 0)$, $y_0 = (5, 5)$,we obtain Comparison Table 3.

The solution is

$$x^* = (0.894655373334687, 0.327826521746298)$$

Table 3 Comparison Table 3

n	$y_n^{(1)}$	$y_n^{(2)}$	$\|y_n - y_{n-1}\|$
−1	5	5	
0	1	0	5
1	0.909090909090909	0.363636363636364	3.636E-01
2	0.894886945874111	0.329098638203090	3.453E-02
3	0.894655531991499	0.327827544745569	1.271E-03
4	0.894655373334793	0.327826521746906	1.022E-06
5	0.894655373334687	0.327826521746298	6.089E-13
6	0.894655373334687	0.327826521746298	2.710E-20

chosen from the lists of the tables displayed above. Hence, method (3.17) converges faster than (3.18), suggested in Chen and Yamamoto [36], Zabrejko and Nguen [37] in this case, and the method of chord [25–27].

4 Conclusion

We presented a semi-local convergence analysis for NLM in order to approximate a locally unique solution of an equation in a Banach space. Using our new idea of restricted convergence domains, recurrent functions, a combination of Lipschitz and centerLipschitz conditions, instead of only Lipschitz conditions, we provided an analysis with the following advantages over the works in [3, 8–12, 14, 16, 17, 28–38]: weaker sufficient convergence conditions, tighter error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$,at least as precise information on the location of the solution x^* and a larger convergence domain. Note that these advantages are obtained under the same computational cost, since, in practice, the computation of the Lipschitz constants $\bar{K}$ requires the computation of L and K. Numerical examples further validating the results are also provided in this study.

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