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Proximal Methods with Invexity and Fractional Calculus

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Abstract

We present some proximal methods with invexity results involving fractional calculus.

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1 Introduction

We are concerned with the solution of the optimization problem defined by

$$\begin{aligned} &\min F(x^*) \\ &s.t,\ x^* \in D \end{aligned} \tag{1.1}$$

where $F : D \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$ is a convex mapping and D is an open and convex set. We shall study the convergence of the proximal point method for solving problem (1.1) defined by

$$x_{n+1} = \underset{x^* \in D}{argmin} \{F(x^*) + \frac{\gamma}{2} d^2(x_n, x^*)\} \tag{1.2}$$

where $x_0 \in X$ is an initial point, $\gamma > 0$ and d is the distance on D.

The rest of the paper is organized as follows. In Section 2 we present the convergence of method (1.2) and in Section 3 we present the application of the method using fractional derivatives.

2 Convergence of method (1.2)

We need an auxiliary result about convex functions.

LEMMA 2.1 *Let $D_0 \subseteq D$ be an open convex set, $F : D \longrightarrow \mathbb{R}$ and $x^* \in D$. Suppose that $F + \frac{\gamma}{2}d^2(., x^*) : D \longrightarrow \mathbb{R}$ is convex on D_0. Then, mapping F is locally Lipschitz on D_0.*

Proof By hypothesis $F + \frac{\gamma}{2}d^2(., x^*)$ is convex, so there exist $L_1, r_1 > 0$ such that for each $u, v \in U(x^*, r_1)$

$$|F(u) + \frac{\gamma}{2}d^2(u, x^*) - (F(v) + \frac{\gamma}{2}d^2(v, x^*))| \leq L_1 d(u, v). \tag{2.1}$$

It is well known that the mapping $\frac{d^2(.,x^*)}{2}$ is strongly convex. That is there exist $L_2, r_2 > 0$ such that for each $u, v \in U(x^*, r_2)$

$$|\frac{1}{2}d^2(u, x^*) - \frac{1}{2}d^2(v, x^*)| \leq L_2 d(u, v). \tag{2.2}$$

Let

$$r = \min\{r_1, r_2\} \text{ and } L_0 = L_1 + \gamma L_2. \tag{2.3}$$

Then, using (2.1)–(2.3), we get in turn that

$$\begin{aligned} |F(u) - F(v)| &\leq |F(u) + \frac{\gamma}{2}d^2(u, x^*) - (F(v) + \frac{\gamma}{2}d^2(v, x^*))| \\ &\quad +|\frac{\gamma}{2}d^2(u, x^*) - \frac{\gamma}{2}d^2(v, x^*)| \\ &\leq L_1 d(u, v) + L_2 \gamma d(u, v) = L_0 d(u, v). \end{aligned} \tag{2.4}$$

□

Next, we present the main convergence result for method (1.2).

THEOREM 2.2 *Under the hypotheses of Lemma 2.1, further suppose:*

$$-\infty < \inf_{x^* \in D} F(x^*), \tag{2.5}$$

$$S_y = \{x^* \in D : F(x^*) \leq F(y)\} \subseteq D. \inf_{x^* \in D} F(x^*) < F(y), \tag{2.6}$$

the minimizer set of F is non-empty, i.e.

$$T = \{x^* : F(x^*) = \inf_{x^* \in D} F(x^*)\} \neq \emptyset, \tag{2.7}$$

$$\|F(x^*) - x^*\| \leq L_3, \tag{2.8}$$

$$L := L_1 + 2\gamma L_2 < 1. \tag{2.9}$$

Then, the sequence $\{x_n\}$ generated for $x_0 \in S^* := S_y \cap U(x^*, r^*)$ is well defined, remains in S^* and converges to a point $x^{**} \in T$, where

$$r^* := \frac{L_3}{1 - L}. \tag{2.10}$$

Proof. Define the operator

$$G(x) := F(x) + \frac{\gamma}{2}\|x - x^*\|. \tag{2.11}$$

We shall show that operator G is a contraction on $U(x^*, r^*)$. Clearly sequence $\{x_n\}$ is well defined and since $x_0 \in S_y$ we get that $\{x_n\} \subseteq S_y$ for each $n = 0, 1, 2, \ldots$. In view of Lemma 2.1 and the definitions (2.8)–(2.11) we have in turn for $u, v \in U(x^*, r^*)$

$$\begin{aligned} |G(u) - G(v)| &\leq |F(u) - F(v)| + \gamma|\frac{1}{2}d^2(u, x^*) - \frac{1}{2}d^2(v, x^*)| \\ &\leq (L_0 + \gamma L_2)d(u, v) = Ld(u, v) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} |G(u) - x^*| &\leq |G(u) - G(x^*)| + |G(x^*) - x^*| \\ &\leq Ld(u, x^*) + |F(x^*) - x^*| \\ &\leq Ld(u, x^*) + L_3 \leq r^*. \end{aligned} \tag{2.13}$$

The result now follows from (2.9), (2.12), (2.13) and the contraction mapping principle[1, 3, 4, 5, 6].

□

3 Fractional derivatives with invexity

1. Let $0 < \alpha < 1$, we consider the left Caputo fractional partial derivatives of f of order α :

$$\frac{\partial^\alpha f(x)}{\partial x_i^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_{a_i}^{x_i} (x_i - t_i)^{-\alpha} \frac{\partial f(x_1, x_2, \ldots, t_i, \ldots, x_n))}{\partial x_i} dt_i, \tag{3.1}$$

where $x = (x_1, \ldots, x_n) \in X, i = 1, 2, \ldots n$ and $\frac{\partial f((x_1, \ldots, \ldots, \ldots, x_n))}{\partial x_i} \in L_\infty(a_i, b_i), i = 1, 2, \ldots n$. Here Γ stands for gamma function. Note that

$$\begin{aligned}\left|\frac{\partial^\alpha f(x)}{\partial x_i^\alpha}\right| &\leq \frac{1}{\Gamma(1-\alpha)}\left(\int_{a_i}^{x_i}(x_i-t_i)^{-\alpha}dt_i\right)\\ &\quad \| \frac{\partial f(x_1,x_2,\ldots,x_{i-1},.,x_{i+1},\ldots,x_n))}{\partial x_i}dt_i\|_{\infty,a_i,b_i}\\ &= \frac{(x_i-t_i)^{1-\alpha}}{\Gamma(2-\alpha)}\left\|\frac{\partial f(x_1,x_2,\ldots,x_{i-1},.,x_{i+1},\ldots,x_n))}{\partial x_i}dt_i\right\|_{\infty,a_i,b_i}\\ &< \infty, \end{aligned} \tag{3.2}$$

for all $i = 1, 2, \ldots, n$. Therefore, $\frac{\partial^\alpha f(x)}{\partial x_i^\alpha}$ exist for all $i = 1, 2, \ldots n$.

Now we consider the left fractional Gradient of F of order $\alpha, 0 < \alpha < 1$:

$$\nabla_\alpha^+ f(x^*) = \left(\frac{\partial f(x^*)}{\partial x_1^\alpha}, \ldots, \frac{\partial f(x^*)}{\partial x_n^\alpha}\right).$$

2. Let $0 < \alpha < 1$, we consider the right Caputo fractional partial derivatives of f of order α :

$$\frac{\bar{\partial}^\alpha f(x)}{\partial x_i^\alpha} = \frac{-1}{\Gamma(1-\alpha)}\int_{x_i}^{b_i}(t_i-x_i)^{-\alpha}\frac{\partial f(x_1,x_2,\ldots,t_i,\ldots,x_n))}{\partial x_i}dt_i, \tag{3.3}$$

where $x = (x_1, \ldots, x_n) \in X, i = 1, 2, \ldots n$ and $\frac{\partial f((x_1,\ldots,.,\ldots,x_n))}{\partial x_i} \in L_\infty(a_i, b_i), i = 1, 2, \ldots n$. Note that

$$\begin{aligned}\left|\frac{\bar{\partial}^\alpha f(x)}{\partial x_i^\alpha}\right| &\leq \frac{1}{\Gamma(1-\alpha)}\left(\int_{x_i}^{b_i}(x_i-t_i)^{-\alpha}dt_i\right)\\ &\quad \| \frac{\partial f(x_1,x_2,\ldots,x_{i-1},.,x_{i+1},\ldots,x_n))}{\partial x_i}dt_i\|_{\infty,a_i,b_i}\\ &= \frac{(b_i-x_i)^{1-\alpha}}{\Gamma(2-\alpha)}\left\|\frac{\partial f(x_1,x_2,\ldots,x_{i-1},.,x_{i+1},\ldots,x_n))}{\partial x_i}dt_i\right\|_{\infty,a_i,b_i}\\ &< \infty, \end{aligned} \tag{3.4}$$

for all $i = 1, 2, \ldots, n$. Therefore, $\frac{\bar{\partial}^\alpha f(x)}{\partial x_i^\alpha}$ exist for all $i = 1, 2, \ldots n$.

Now we consider the right fractional Gradient of F of order $\alpha, 0 < \alpha < 1$:

$$\bar{\nabla}_\alpha^+ f(x^*) = \left(\frac{\bar{\partial} f(x^*)}{\partial x_1^\alpha}, \ldots, \frac{\bar{\partial} f(x^*)}{\partial x_n^\alpha}\right).$$

3. Define for $k \in \mathbb{N}$: $\nabla_{k\alpha}^+ f = \nabla_\alpha^+ \ldots \nabla_\alpha^+ f, k-$ times composition of left fractional gradient, i.e.,

$$\nabla_{k\alpha}^+ f = \left(\frac{\partial^{k\alpha} f(x^*)}{\partial x_1^\alpha}, \ldots, \frac{\partial^{k\alpha} f(x^*)}{\partial x_n^\alpha}\right),$$

where $\frac{\partial^{k\alpha} f(x)}{\partial x_i^\alpha} = \frac{\partial^\alpha}{\partial x_i^\alpha} \dots \frac{\partial^\alpha}{\partial x_i^\alpha} f, k-$times composition of left partial fractional derivative, $i = 1, 2, \dots n$. We assume that $\frac{\partial^{k\alpha} f}{\partial x_i^\alpha}$ exist for all $i = 1, 2, \dots n$.

4. Define for $k \in \mathbb{N} : \bar{\nabla}^-_{k\alpha} f = \bar{\nabla}^-_\alpha \dots \bar{\nabla}^-_\alpha f, k-$ times composition of right fractional gradient, i.e.,

$$\bar{\nabla}^-_{k\alpha} f = \left(\frac{\bar{\partial}^{k\alpha} f(x^*)}{\partial x_1^\alpha}, \dots, \frac{\bar{\partial}^{k\alpha} f(x^*)}{\partial x_n^\alpha} \right),$$

where $\frac{\bar{\partial}^{k\alpha} f(x)}{\partial x_i^\alpha} = \frac{\bar{\partial}^\alpha}{\partial x_i^\alpha} \dots \frac{\bar{\partial}^\alpha}{\partial x_i^\alpha} f, k-$times composition of right partial fractional derivative, $i = 1, 2, \dots n$. We assume that $\frac{\bar{\partial}^{k\alpha} f}{\partial x_i^\alpha}$ exist for all $i = 1, 2, \dots n$.

5. Let $\alpha \geq 1$, we consider the left Caputo fractional partial derivatives of f of order α ($\lceil \alpha \rceil = m \in \mathbb{N}, \lceil . \rceil$ ceiling of the number [2]):

$$\frac{\partial^\alpha f(x)}{\partial x_i^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_{a_i}^{x_i} (x_i - t_i)^{m-\alpha-1} \frac{\partial^m f(x_1, x_2, \dots, t_i, \dots, x_n))}{\partial x_i^m} dt_i,$$

$i = 1, 2, \dots n$. We set $\frac{\partial^m f(x)}{\partial x_i^m}$ equal to the ordinary partial derivative $\frac{\partial^m f(x)}{\partial x_i^m}$. We assume that

$$\frac{\partial^m f}{\partial x_i^m}(x_1, \dots, ., \dots, x_n) \in L_\infty(a_i, b_i)$$

i.e.,

$$\left\| \frac{\partial^m f}{\partial x_i^m}(x_1, \dots, ., \dots, x_n) \right\|_{\infty,(a_i,b_i)} < \infty$$

for all $i = 1, 2, \dots, n$. Note that

$$|\frac{\partial^\alpha f(x)}{\partial x_i^\alpha}| \leq \frac{(x_i - a_i)^{m-\alpha}}{\Gamma(m-\alpha+1)} \| \frac{\partial^m f}{\partial x_i^m}(x_1, \dots, ., \dots, x_n) \|_{\infty,(a_i,b_i)} < \infty,$$

for all $i = 1, 2, \dots n$. Therefore, $\frac{\partial^\alpha f(x)}{\partial x_i^\alpha}$ exist for all $i = 1, 2, \dots n$. Now we consider the left fractional gradient of f of order $\alpha, \alpha \geq 1$:

$$\nabla^{++}_\alpha f(x^*) = \left(\frac{\partial^\alpha f(x^*)}{\partial x_1^\alpha}, \dots, \frac{\partial^\alpha f(x^*)}{\partial x_n^\alpha} \right).$$

6. Let $\alpha \geq 1$, we consider the right Caputo fractional partial derivatives of f of order α ($\lceil \alpha \rceil = m$):

$$\frac{\bar{\partial}^\alpha f(x)}{\partial x_i^\alpha} = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x_i}^{b_i} (x_i - t_i)^{m-\alpha-1} \frac{\partial^m f(x_1, x_2, \dots, t_i, \dots, x_n))}{\partial x_i^m} dt_i,$$

$i = 1, 2, \ldots n$. We set $\frac{\bar{\partial}^m f}{\partial x_i^m} = (-1)^m \frac{\partial^m f}{\partial x_i^m}$ (where $\frac{\partial^m f}{\partial x_i^m}$ is the ordinary partial). We assume that

$$\frac{\partial^m f}{\partial x_i^m}(x_1, \ldots, ., \ldots, x_n) \in L_\infty(a_i, b_i)$$

for all $i = 1, 2, \ldots, n$. Note that

$$|\frac{\bar{\partial}^\alpha f(x)}{\partial x_i^\alpha}| \leq \frac{(b_i - x_i)^{m-\alpha}}{\Gamma(m - \alpha + 1)} \| \frac{\partial^m f}{\partial x_i^m}(x_1, \ldots, ., \ldots, x_n) \|_{\infty,(a_i,b_i)} < \infty,$$

for all $i = 1, 2, \ldots n$. Therefore, $\frac{\bar{\partial}^\alpha f(x)}{\partial x_i^\alpha}$ exist for all $i = 1, 2, \ldots n$. Now we consider the right fractional gradient of f of order $\alpha, \alpha \geq 1$:

$$\nabla_\alpha^{--} f(x^*) = \left(\frac{\bar{\partial}^\alpha f(x^*)}{\partial x_1^\alpha}, \ldots, \frac{\bar{\partial}^\alpha f(x^*)}{\partial x_n^\alpha} \right).$$

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