European Journal of Combinatorics 30 (2009) 986–995

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Set colorings of graphs

S.M. Hegde

Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, Srinivasanagar-575025, India

ARTICLE INFO

Article history:
Received 22 September 2007
Accepted 24 June 2008
Available online 2 September 2008

ABSTRACT

A ***set coloring*** of the graph G is an assignment (function) of distinct subsets of a finite set X of ***colors*** to the vertices of the graph, where the colors of the edges are obtained as the symmetric differences of the sets assigned to their end vertices which are also distinct. A set coloring is called a ***strong set coloring*** if sets on the vertices and edges are distinct and together form the set of all nonempty subsets of X. A set coloring is called a ***proper set coloring*** if all the nonempty subsets of X are obtained on the edges. A graph is called ***strongly set colorable (properly set colorable)*** if it admits a strong set coloring (proper set coloring).

In this paper we give some necessary conditions for a graph to admit a strong set coloring (a proper set coloring), characterize strongly set colorable complete bipartite graphs and strongly (properly) set colorable complete graphs, etc. Also, we give a construction of a planar strongly set colorable graph from a planar graph, a strongly set colorable tree from a tree and a properly set colorable tree from a tree, etc., thereby showing their embeddings.

1. Introduction

In this paper we consider only finite simple graphs. For all notation in graph theory we follow Harary [3] and West [5].

Colorings of the vertices and edges of a graph G which are required to obey certain conditions have often been motivated by their utility in various applied fields and their intrinsic mathematical interest (logico-mathematical). An enormous amount of literature has built up on several kinds of colorings of graphs.

Motivated by the papers of Hopkroft and Krishnamurty [4], Balister et al. [2], and Acharya [1], we introduce set colorings of graphs: Let X be a nonempty set of colors, 2^X denote the set of all possible

E-mail address: smhegde@nitk.ac.in.

0195-6698/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2008.06.005

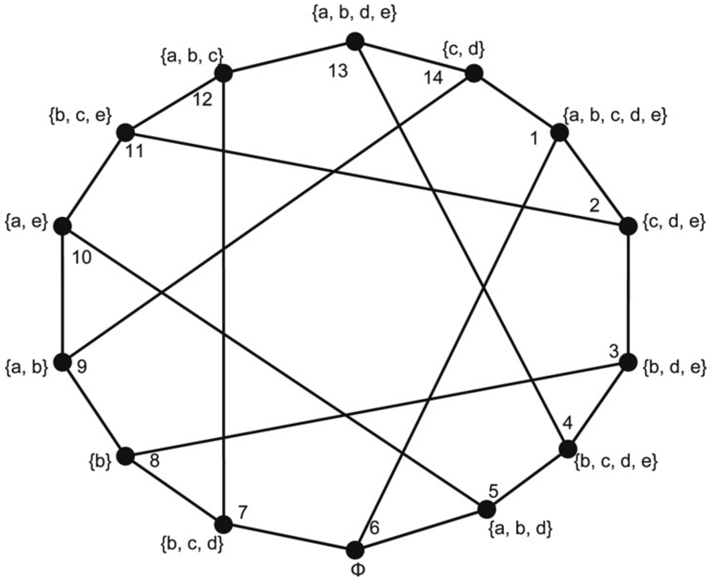

Fig. 1. Set coloring of a Heawood graph.

combinations of colors (or power set) of X and $Y(X) = 2^X \setminus \{\emptyset\}$. For any two subsets A and B of X let $A \oplus B$ denote the symmetric difference of A, B and be given by $A \oplus B = (A \cup B) - (A \cap B)$.

Given a (p, q)-graph $G = (V, E)$ and a nonempty set X of colors, we define a function f on the vertex set V of G as an assignment of subsets of X to the vertices of G, and given such a function f on the vertex set V we define $f^{\oplus}$ on the set of edges E as an assignment of the colors $f^{\oplus}(e) = f(u) \oplus f(v)$ to the edge $e = uv$ of G.

Let $f(G) = \{f(u) : u \in V\}$ and $f^{\oplus}(G) = \{f^{\oplus}(e) : e \in E\}$.

We call f a ***set coloring*** of G if both $f(G)$ and $f^{\oplus}(G)$ are injective functions. A graph is called ***set colorable*** if it admits a set coloring. A set coloring f of G is called a ***strong set coloring*** if $f(G)$ and $f^{\oplus}(G)$ are disjoint subsets of X and, further, they form a partition of $Y(X)$. If G admits such a coloring then G is a called a ***strongly set colorable graph.***

A set coloring f is called a ***proper set coloring*** if $f^{\oplus}(G) = Y(X)$. If a graph G admits such a set coloring then it is called a ***properly set colorable graph***.

The ***set coloring number*** $\sigma(G)$ of a graph G is the least cardinality of a set X with respect to which G has a set coloring. Further, if $f : V \to 2^X$ is a set coloring of G with $|X| = \sigma(G)$ we call f an optimal set coloring of G.

Theorem 1. *For any graph G,*

$$\lceil \log_2(q + 1) \rceil \le \sigma\ (G) \le p - 1,$$

where $\lceil x \rceil$ denotes the least integer not less than the real number x, and the bounds are best possible.

Fig. 1 gives an optimal set coloring of a Heawood graph.

2. Strongly (properly) set colorable graphs

Since all the nonempty subsets have to appear in any strong set coloring of a (p, q)-graph G, a necessary condition for G to be strongly set colorable is that $p + q + 1 = 2^m$, for the positive integer $m = |X|$. This necessary condition immediately yields that no cycle is strongly set colorable. Also, we observe that the above condition is not sufficient for saying that a graph G is strongly set colorable

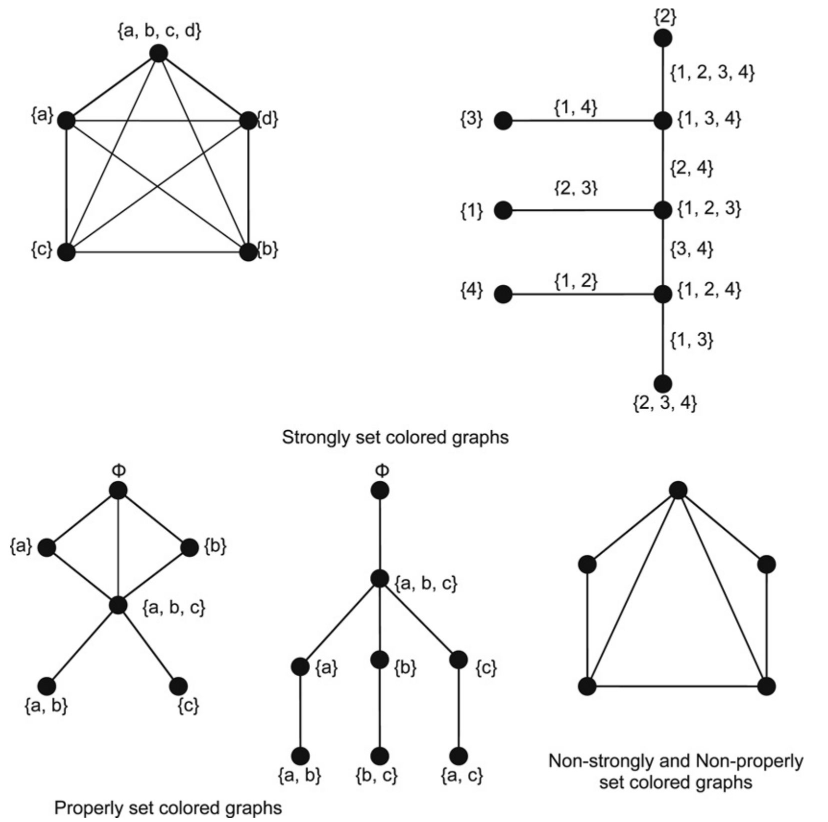

Fig. 2.

as the path of length 3 satisfies the condition for $m = 3$ but one can verify that it is not strongly set colorable. We propose:

Conjecture 1. *No path of length greater than* 2 *is strongly set colorable.*

Similarly, a necessary condition for G to be properly set colorable is that $q + 1 = 2^m$. From the above necessary condition it follows that the cycles of lengths not equal to $2^m - 1$ are not properly set colorable.

Fig. 2 gives examples of properly, strongly, non-strongly and non-properly colorable graphs.

The following result gives a natural link between strongly set colorable and properly set colorable graphs.

Theorem 2. *A graph G is strongly set colorable if and only if $G + K_1$ with $V(K_1) = \{v\}$ has a proper set coloring F such that $F(v) = \emptyset$.*

Proof. Let f be a strong set coloring of G. Then, extend f to the vertices of $G + K_1$ to a set assignment F so that the restriction map $F|V(G)$ of F to $V(G)$ is f and $F(v) = \emptyset$. Since f is a strong set coloring of G the edges of $G + K_1$ having the form uv where $u \in V(G)$ will receive $f(u)$. So F turns out to be a required proper set coloring of $G + K_1$. □

Conversely, if $H = G + K_1$ has a proper set coloring F with $F(v) = \emptyset$ then the removal of v from H obviously results in a strong set coloring of G.

The following two results give stronger necessary conditions for a strong set coloring of graphs.

Theorem 3. *If a graph G $(p > 2)$ has:*

(i) *exactly one or two vertices of even degree* ***or***
(ii) *exactly three vertices of even degree, say,* v_1, v_2, v_3*, and any two of these vertices are adjacent* ***or***
(iii) *exactly four vertices of even degree, say,* v_1, v_2, v_3, v_4 *such that* v_1v_2 *and* v_3v_4 *are edges in G, then G is not strongly set colorable.*

Proof. Let G be a graph with a strong set coloring f with respect to a set X having m colors. Let $\{v_1, v_2, v_3, \ldots, v_p\}$ be the vertices of G such that $f(v_i) = A_i$, $1 \leq i \leq p$, and $A_i \in Y(X)$. Then we get $f(G) \cup f^{\oplus}(G) = \{A_1, A_2, \ldots, A_p, \{A_i \oplus A_j : v_iv_j \in E\}\} = Y(X)$. As the symmetric difference of all the nonempty subsets of any set is the empty set, we get

$$f(G) \cup f^{\oplus}(G) = \{A_1, A_2, \ldots, A_p\} \cup \{A_i \oplus A_j : v_iv_j \in E\} = \emptyset. \tag{1}$$

One can see that if the degree of a vertex v is even then the set A assigned to v appears an odd number of times and if the degree of a vertex u is odd then the set B assigned to u appears an even number of times in (1).

(i) Suppose that G has exactly one vertex of even degree, say v_1. Then A_1 will appear an odd number of times and all the other sets will appear an even number of times in (1). So, as the binary operation $\oplus$ is commutative, when the symmetric differences of the sets in (1) are taken, all the subsets which are assigned to the vertices of odd degree will vanish and hence we get that $A_1 = \emptyset$, a contradiction to the definition of the strong set coloring of G. Hence, if G has exactly one vertex of even degree then G is not strongly set colorable. Suppose that G has exactly two vertices of even degree, say v_1, v_2. Then using arguments similar to those above and from (1) we obtain that $A_l \oplus A_2 = \emptyset$, which implies that $A_l = A_2$, a contradiction to the injectivity of f. Hence, if G has exactly two vertices of even degree then G is not strongly set colorable.
(ii) Suppose G has three vertices of even degree, say, v_1, v_2, v_3 such that v_1v_2 is an edge in G. Then by arguments similar to those for (i) and from (1) we obtain that $A_3 = A_1 \oplus A_2$, or $A_2 \oplus A_3 = A_1$, or $A_l \oplus A_3 = A_2$, a contradiction to the definition of a strong set coloring of G. Hence, if G has exactly three vertices of even degree as mentioned in Theorem 1, then G is not strongly set colorable.
(iii) Suppose that G has exactly four vertices of even degree as given in the statement of Theorem 3. Then, by similar arguments and using (1), we get $A_1 \oplus A_2 = A_3 \oplus A_4$ or $A_1 \oplus A_3 = A_2 \oplus A_4$ or $A_1 \oplus A_4 = A_2 \oplus A_3$ respectively, a contradiction to the injectivity of $f^{\oplus}$. Hence, if G has exactly four vertices of even degree then G is not strongly set colorable. □

Theorem 4. *If a graph G has a strong set coloring f with respect to a set X of cardinality m, there exists a partition of the vertex set V into two sets* V_1 *and* V_2 *such that the number of edges joining the vertices of* V_1 *with those of* V_2 *is exactly* $2^{m-1} - |V_2|$.

Proof. Suppose that G is strongly set colorable with respect to a set X of cardinality $m \geq 2$. Consider a partition of V into two sets V_1 and V_2 such that $V_1 = \{u \in V : |f(u)| \text{ is even}\}$ and $V_2 = \{v \in V : |f(v)| \text{ is odd}\}$. One can obtain other odd subsets of X which are not there on the vertices only by taking the symmetric differences between the vertices of V_1 with those of V_2 and hence the result follows, as there are exactly 2^{m-l} subsets of each parity for a set $X$of cardinality m. □

The following two results give stronger necessary conditions for a proper set coloring of graphs.

Theorem 5. ***If*** *a graph G $(p > 2)$ has:*

(i.) *exactly two vertices of odd degree* ***or***
(ii.) *exactly four vertices of odd degree, say,* v_1v_2, v_3, v_4 *with* v_1v_2 *and* v_3v_4 *being edges in G, then G is not properly set colorable.*

Proof. Let G have a proper set coloring f with respect to a set X of cardinality m. Let $\{v_1, v_2, v_3, \ldots, v_p\}$ be the vertices of G such that $f(v_i) = A_i$, $1 \leq i \leq p$, and $A_i \in Y(X)$. As G is properly set colorable, we have

$$f^{\oplus}(G) = \{A_i \oplus A_j : v_iv_j \in E\} = Y(X). \tag{2}$$

(i) Suppose that G has exactly two vertices of odd degree, say, v_1, v_2. From (1) we obtain $A_l \oplus A_2 = \emptyset$, which implies that $A_l = A_2$, a contradiction to the injectivity of f. Hence if G has exactly two vertices of odd degree then it is not properly set colorable.

(ii) Suppose that G has exactly four vertices of odd degree as mentioned in the theorem. Then by similar arguments we obtain
$A_l \oplus A_2 = A_3 \oplus A_4$ or $A_l \oplus A_3 = A_2 \oplus A_4$ or $A_l \oplus A_4 = A_3 \oplus A_2$ respectively, a contradiction to the injectivity of $f^{\oplus}$. Hence, if G has exactly four vertices of odd degree as mentioned in the theorem then G is not properly set colorable. □

Corollary 5.1. *No path of length greater than* 2 *is properly set colorable.*

Theorem 6. *If a graph G has a proper set coloring f with respect to a set X of cardinality m, then there exists a partition of the vertex set V into two sets V_1 and V_2 such that the number of edges joining the vertices of V_1 with those of V_2 is exactly 2^{m-1}.*

Proof. Suppose that G is properly set colorable with respect to a set X of cardinality $m \geq 2$. Let V_1 and V_2 be a partition of V as mentioned in the proof of Theorem 4. As one can obtain all the odd subsets of X by taking the symmetric differences between the vertices of V_1 with those of V_2, the result follows. □

The next result characterizes the strongly colorable complete graphs.

Consider the complete graph K_n. Suppose that it is strongly set colorable with respect to a set X of cardinality m. Then it follows that the sum of the number of vertices and the edges of K_n must be equal to $2^m - 1$, i.e., $n + n(n-1)/2 = 2^m - 1$, which yields the quadratic equation

$$n^2 + n - (2^{m+1} - 2) = 0. \tag{3}$$

Solving (3) for positive integer values of n, we get

$$n = (1/2)(\sqrt{(2^{m+3} - 7)} - 1). \tag{4}$$

The first four values of n for which K_n may possibly be strongly set colorable are 1, 2, 5 and 90, where the values of m are 1, 2, 4 and 12, respectively. The following result characterizes the strongly set colorable complete graphs.

Theorem 7. *The nontrivial complete graph K_n is strongly set colorable if and only if $n = 2, 5$.*

Proof. Suppose that G is *strongly set colorable* with respect to a set X of cardinality $m \geq 2$. Consider a partition of V into two sets V_1 *and* V_2 as mentioned in the proof of Theorem 4. Then one can obtain all the other odd subsets of X which are not covered by V_2 by taking the symmetric differences between the vertices of V_1 and those of V_2. Thus, $|V_1||V_2| = 2^{(m-1)} - |V_2|$,

$$\text{i.e., } (|V_1| + 1)\,|V_2| = 2^{(m-1)}. \tag{5}$$

Hence, $|V_1| + 1$ is a power of 2 and $|V_2|$ is a power of 2. By the symmetric differences among the vertices of V_1 as well as among the vertices of V_2, we obtain all the other even nonempty subsets of X which are not covered by V_1. Thus, we get

$$|V_1|(|V_1| - 1)/2 + |V_2|(|V_2| - 1)/2 = 2^{(m-1)} - |V_1| - 1 \tag{6}$$

$$\text{i.e., } |V_1|(|V_1| + 1)/2 + |V_2|(|V_2| - 1)/2 = 2^{(m-1)} - 1. \tag{7}$$

Eq. (7) implies that one of the terms in (7) is odd, say $|V_1|(|V_1|+1)/2$ is odd. We know that $(|V_1|+1)$ is a power of 2. Suppose that $(|V_1| + 1) = 2^t,\ t \geq 2$; then we obtain that $|V_1|(|V_1| + 1)/2$ is even, a contradiction. Hence, $(|V_1| + 1) = 2$. Thus, we obtain $|V_1| = 1$ and hence, $|V_2| = n - 1$. Then, from (5) we get

$$2(n - 1) = 2^{(m-1)}. \tag{8}$$

Also, from (7), we obtain

$$2 + (n-1)(n-2)/2 = 2^{(m-1)}. \tag{9}$$

Similarly, if $|V_2|(|V_2|-1)/2$ is odd, then (8) and (9) are again obtained.

Equating (8) and (9), we obtain
$2(n-1) = 2 + (n-1)(n-2)/2$ or $n^2 - 7n + 10 = 0$, which implies that $n = 2, 5$.

Conversely, suppose that $n = 2, 5$; then one can easily verify that K_2 and K_5 are strongly set colorable. □

Theorem 8. *The complete graph K_n is properly set colorable with respect to a set X of cardinality m if and only if $n = 2, 3$ and 6.*

Proof. The proof follows from Theorems 7 and 2. □

Theorem 9. *The nontrivial complete n-ary tree T_n^t is strongly set colorable if and only if $n = 2^\alpha - 1$ and $t = 1$, where t is the number of levels of T_n^t.*

Proof. Suppose that $G = T_n^t$ is strongly set colorable with respect to a set X of cardinality m. Then we obtain $|V(G)| + |E(G)| = 2^m - 1$.

The case when n is even follows from Theorem 3. Thus, no complete n-ary tree G is strongly set colorable when n is even.

By the definition of a complete n-ary tree, we obtain

$$(1 + n + n^2 + \cdots + n^t) + (1 + n + n^2 + \cdots + n^t - 1) = 2^m - 1 \quad \text{or}$$
$$(2n^{(t+1)} - n - 1)/(n-1) = 2^m - 1 \quad \text{or} \quad (n^{(t+1)} - 1)/(n-1) = 2^{(m-1)},$$

which implies that n is odd and hence t is odd. Thus, from $(n^{(t+1)} - 1)/(n-1) = 2^{(m-1)}$, we obtain $(1 + n + n^2 + \cdots + n^t) = 2^{(m-1)}$ or

$$(1+n)(1 + n^2 + n^4 + \cdots + n^{t-1}) = 2^{(m-1)} \tag{10}$$

which implies that $(1+n) = 2^\alpha$, α is a positive integer. Thus, from (10) we obtain

$$(1 + n^2 + n^4 + \cdots + n^{t-1}) = 2^{(m-\alpha-1)}. \tag{11}$$

One can write (11) as

$$(1+n^2)(1 + n^4 + n^8 + \cdots + n^{t-3}) = 2^{(m-\alpha-1)},$$

which implies that $1 + n^2 = 2^\beta$. Substituting the value of n from $(1+n) = 2^\alpha$, we obtain

$$1 + (2^\alpha - 1)^2 = 2^\beta, \quad \text{or}$$
$$2^{2\alpha} - 2^{\alpha+1} + 2 = 2^\beta, \quad \text{or}$$
$$2^{2\alpha-1} - 2^\alpha + 1 = 2^{\beta-1}, \quad \text{or}$$
$$2^{2\alpha-1} - 2^{\beta-1} = 2^\alpha - 1$$

which implies that $2^\alpha - 1$ is even, or $\alpha = 0$, or $n = 0$, a contradiction. Thus, $1 + n = 2^\alpha$ or $n = 2^\alpha - 1$. Also, from (11) we obtain $1 = 2^{(m-\alpha-1)}$ or $m = \alpha + 1$ and also $t = 1$.

Conversely, suppose that $n = 2^\alpha - 1$ and $t = 1$. Then G reduces to the star $K_{1,2^\alpha-1}$. Let $X = \{1, 2, \ldots, m\}$, $X_1 = \{1\}$ and $X_2 = \{2, 3, \ldots, m\}$. Assign the set X_1 to the central vertex and all the nonempty subsets of X_2 to the remaining vertices of the star in a one-to-one manner. Then it is not hard to verify that the assignment is a *strong set coloring* of $K_{1,2^\alpha-1}$. □

A similar proof proves the following theorem.

Theorem 10. *The nontrivial complete n-ary tree G is properly set colorable if and only if $n = 2^\alpha - 1$ and $t = 1$.*

Theorem 11. *The complete bipartite graph $K_{a,b}$ is strongly set colorable if and only if $(a+1)(b+1) = 2^m$, where m is a positive integer.*

Proof. Let $K_{a,b}$ be strongly set colorable with respect to a set X of cardinality m. Then it follows that $|V(K_{a,b})| + |E(K_{a,b})| = 2^m - 1$, i.e., $a + b + ab = 2^m - 1$, which yields

$$(a + 1)(b + 1) = 2^m.$$

Conversely, assume that

$$(a + 1)(b + 1) = 2^m, \tag{12}$$

for some positive integers a, b and m, where m is the cardinality of the set X.

Taking the logarithm to base 2 on both sides of (12), we obtain

$$m = \log_2(a + 1) + \log_2(b + 1).$$

Hence, there exists a partition $\{X_1, X_2\}$ of X such that $|X_1| = \log_2(a + 1)$ and $|X_2| = \log_2(b + 1)$. Let A_1 and A_2 constitute the bipartition of the vertex set of $K_{a,b}$. Assign the nonempty subsets of X_i to the vertices in A_i, $i = 1, 2$, in a one-to-one manner. Then one can verify that the resulting assignment is indeed a strong set coloring of $K_{a,b}$. □

Conjecture 2. *The complete bipartite graph $K_{a,b}$ is properly set colorable if and only if it is a star with $a = 1$ and $b = 2^{n-1}$.*

Next, we give some results on the construction of strongly (properly) set colored graphs and show their embeddings.

Let G be the given planar graph with n vertices. Let T be a spanning tree of G. Introduce a new vertex v and join it to a vertex of G which is in the exterior face. As T is a spanning tree, let the new tree with v as the additional vertex be the tree T_l. Draw the tree T_l as a rooted tree with root v. Let $X = \{1, 2, \ldots, n\}$ be a set of cardinality n. Let the vertices of T be v_i, which are in ascending order in T_l. Assign the set X to v and the single-element subsets $\{i\}$ of X to the remaining vertices v_i such that $f(v_i) = \{i\}$. Let $\{A_{i,j} : v_iv_j \; \varepsilon \; E, i < j\}$ be the t two-element subsets of X which are already obtained on the edges of G and $B_l, B_2, \ldots, B_k$ be the remaining two-element subsets of X such that $t + k = (n - 1)(n)/2$. We know that $(n - 2)$-element subsets are the complements of 2-elements, $(n - 3)$-element subsets are the complements of 3-elements, ..., $(n - 2)/2$-element subsets are the complements of $(n + 2)/2$-elements if n is even and $(n - 1)/2$-element subsets are the complements of $(n + 1)/2$-elements if n is odd. Hence introducing the required number of new vertices, joining them to the vertex v and assigning the subsets of cardinality $(n - 3)$, $(n - 4)$, $(n - 5)$, etc., up to $(n - 1)/2$-element subsets, if n is odd (up to half of the $(n/2)$-element subsets, if n is even), we can obtain all the subsets of cardinality $3, 4, 5, \ldots, (n - 1)/2$ if n is odd (and $(n/2)$ if n is even). Thus, we have exhausted all the subsets except $(n(n-1)/2-t)$, two-element subsets, $(n-2)$-element subsets and $(n - 1)$, $(n - 1)$-element subsets.

Introduce k new vertices and join them to the vertex v. Then assign the sets $B_l, B_2, \ldots, B_k$ to these newly introduced vertices in a one-to-one manner. This assignment will generate the remaining k two-element subsets on these new edges. Thus, we have covered X, all the single-element subsets, one $(n - 1)$-element subset and all the two-element subsets of X. We have to obtain the remaining $(n-1)(n-1)$-element subsets and the remaining $(n-2)$-element subsets, either on vertices or edges.

To generate $(n - 1)$-element subsets, consider the internal vertices of T_1 (internal vertices are the vertices of degree at least 2). Starting from the first layer of T, whenever v_iv_j is an edge, introduce a new vertex and assign the complement of the set $A_{i,j}$, and join the new vertex to the internal vertex of T_l which was assigned the single-element subset containing the element i, which will yield a $(n - 1)$-element subset on the new edge. Continue the procedure until the last but one layer and until all the internal vertices of T_1 are exhausted. Thus, we have generated the $(n - 1)$-, $(n - 2)$-element subsets and $(n - 1)$-element subsets of X. Still we have to cover $(t - (n - 1))$, $(n - 2)$-element subsets. Introduce $t + 1 - n$ new vertices and assign the remaining $(t - (n - 1))$ $(n - 2)$-element subsets to these isolated vertices. Thus the resulting graph G_l is strongly set colorable and planar. □

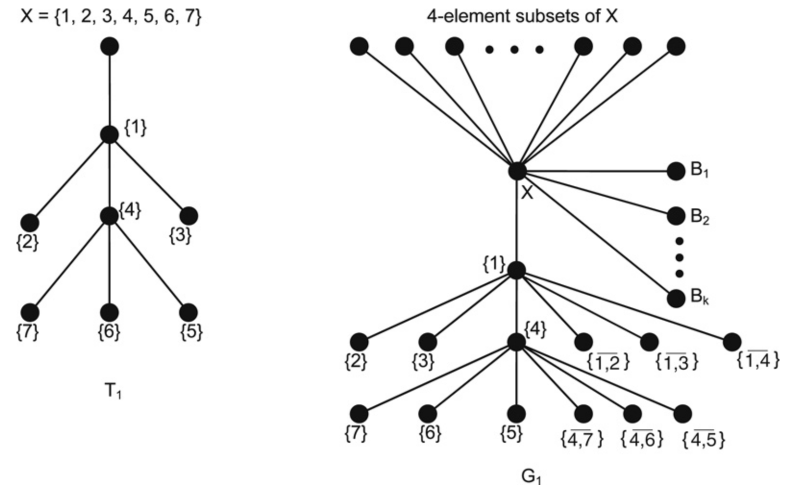

Fig. 3. Illustration of the procedure in the construction.

Remark 2. From the construction in Fig. 3 it follows that every planar graph can be embedded as an induced subgraph of a strongly set colored planar one. Also, it follows that any tree can be embedded as an induced subgraph of a strongly set colorable tree, as there will not be any isolated vertices if G itself is a tree.

Given below is a construction of a bigger properly set colored tree from a properly set colored tree.

Suppose that a tree T is properly set colored with respect to a set X of cardinality m. Then all the 2^m subsets of X appear on the vertices of T. Introduce $2^n - 1$ isolated vertices and join them to the vertices of T. Assign the nonempty subsets of a set $X'(X \cap X' = \emptyset)$ of cardinality n to the newly introduced vertices in a one-to-one manner. Then it is not hard to verify that all the $2^{m+n} - 1$ nonempty subsets of the set $Y = X \cup X'$ of cardinality $m + n$ will appear on the edges of the resulting graph. Hence the resulting graph is properly set colored.

The construction Fig. 4 proves that every tree can be embedded as an induced subgraph of a properly set colored tree.

Let T be the tree with n vertices. We prove the result by induction on the number of edges of T. One can easily see that trees with one or two edges can be embedded as an induced subgraph of a properly set colorable tree. Suppose that the result is true for a tree T_1 with $n - 2$ edges, where T_1 is obtained from T by removing a pendant edge uv such that v is in T_1. This means a tree with $n - l$ vertices and $n - 2$ edges can be embedded as an induced subgraph of a properly set colorable tree, say, T_2.

Let X be the set of cardinality m with respect to which T_2 has a proper set coloring f. Since T_2 is a tree, $f(T_2) = 2^X$. Then join the edge uv. Add an element y to all the 2^X sets which are assigned to the vertices of T_2 and assign the set $\emptyset$ to the vertex u of T. Let $f(v) = S \subset X$ (where S is a subset of X). Note the set $S \cup \{y\}$ has been assigned to v. The set $S \cup \{y\}$ is obtained on the edge uv. Then introduce a new vertex and join it to v. Assign the set S to the newly introduced vertex and then the set $\{y\}$ is generated on the new edge. Let $X_I = X \cup \{y\}$. Introduce $2^m - 2$ new vertices and join them to the vertex w where $f(w) = \emptyset$ and assign all the elements of $Y(X) - S$ to these newly introduced vertices in a one-to-one manner.

Thus, we have obtained all the nonempty subsets of X_l on the edges of the resulting graph, say, T_3. Hence, T_3 is properly set colorable.

We conclude the paper with a generalization of the notion of strong (proper) set colorings in the special case when the defining set X is taken to be a subset of the set N of nonnegative integers (or, for that matter, any linearly ordered set in place of N): An injective set assignment $f : V(G) \cup E(G) \rightarrow 2^X, X \subseteq N$ (or $f : V(G) \rightarrow 2^X$) of a (p, q)-graph G is called a k-semi-strong (or k-semi-proper) set coloring of G if it satisfies the following conditions:

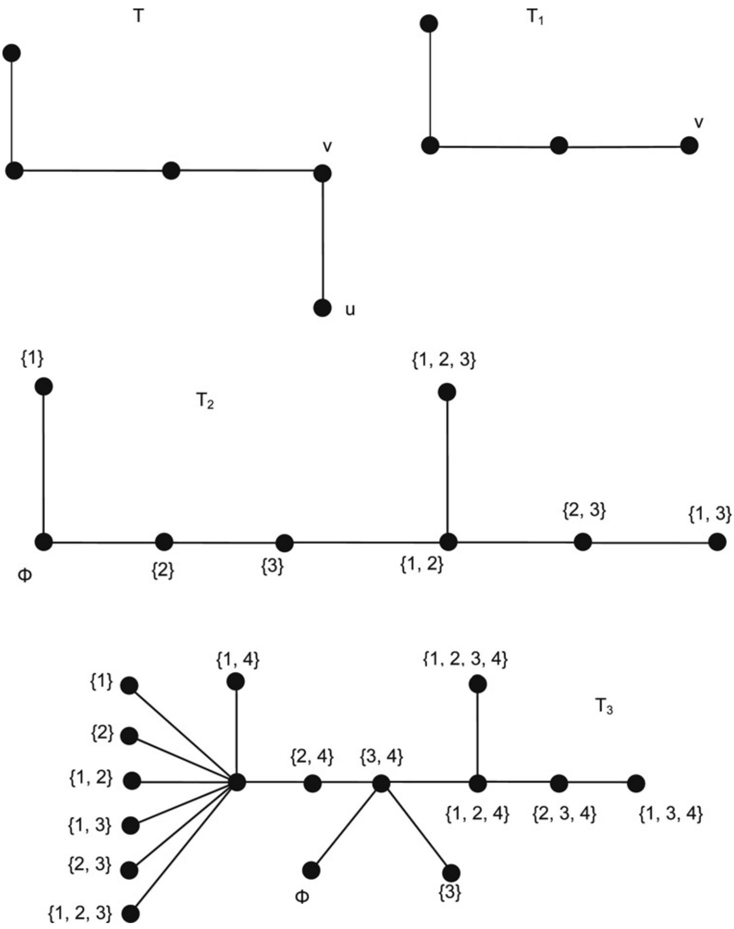

Fig. 4. Construction of a properly set colorable tree.

(i) $f^{\oplus}(uv) = f(u) \oplus f(v)$, $\forall uv \in E(G)$,

(ii) $f(G) \cup f^{\oplus}(G) = \{A_1, A_2, \ldots, A_{p+q}\}$ (or $f^{\oplus}(G) = \{A_1, A_2, \ldots, A_q\}$) where:

(a) $A_1 < A_2 < \cdots < A_{p+q}$ with "$<$" defined on 2^X by setting
$A < B \Leftrightarrow A, B \in 2^X$, either $|A| < |B|$ or
$|A| = |B|$ and $\min(A - B) < \min(B - A)$,

(b) for any $A \in 2^X$, if $A_i < A$ and $A < A_j$ for $i < j$, $i, j \in \{1, 2, 3, \ldots, p+q\}$ then $A = A_m$ for some $m \in \{1, 2, 3, \ldots, p+q\}$,

(iii) $|A_1| = k$.

The graph G is called a *k-SSS graph* if it admits a k-semi-strong set coloring and is called a *k-SPS graph* if it admits a k-semi-proper set coloring. In particular, a 1-SSS (1-SPS) coloring of G is simply called an SSS (SPS) coloring of G. Obviously then, an SSS coloring f of G is a strong (proper) set coloring of G if and only if $X \subseteq N$ and $f(G) \cup f^{\oplus}(G) = Y(X)$ ($f^{\oplus}(G) = Y(X)$). Fig. 5 displays *k-SSS* (*k-SPS*) graphs for some values of k.

For any graph G, $\beta_k(G)$ will denote the least cardinality of a set $X \subseteq N$ with respect to which G has a k-SSS coloring. From the very definition, it follows that for any k-SSS coloring f with respect to a set

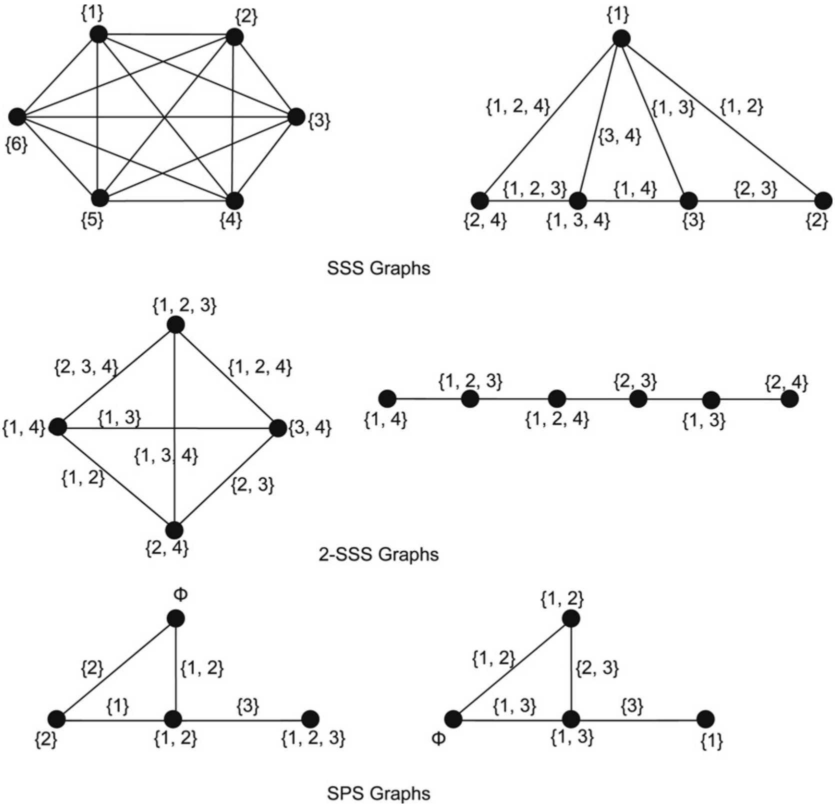

Fig. 5.

X of a graph G one must have

$$p + q \leq 2\beta_k - \sum_{j=0}^{k-1} \binom{\beta_k}{j} - L_k,$$

where L_k is the number of k-subsets of X which do not belong to $f(G) \cup f^{\oplus}(G)$. Furthermore, the bound is best possible.

Acknowledgements

I would like to thank the Director, Institute of Mathematical Sciences, Chennai, India, for giving me a Visiting Fellowship during my visit. I would like to thank Prof. V. Arvind, Dept. of Theoretical Computer Science, for valuable suggestions/discussions during the visit. Also, I would like to thank the referee for valuable suggestions/comments leading to improvement of the paper.

This work was initiated when the author was visiting the Institute of Mathematical Sciences (MATSCIENCE), Chennai, India.

References

[1] B.D. Acharya, MRI Lecture Notes in Applied Mathematics, vol. 2, Mehta Research Institute, Allahabad, India, 1993.
[2] P.N. Balister, O.M. Riordan, R.S. Schelp, Vertex distinguishing edge colorings of graphs, J. Graph Theory 42 (2003) 95–109.
[3] F Harary, Graph Theory, Addison–Wesley, Academic press, 1972.
[4] J.E. Hopcroft, M.S. Krishnamurthy, On the harmonious colorings of graphs, SIAM J. Alg. Discrete Math. 4 (3) (1983) 306–311.
[5] D.B. West, Introduction to Graph Theory, Prentice Hall of India, New Delhi, 2003.