Indian J. Pure Appl. Math., **50**(1): 1-13, March 2019

DOI: 10.1007/s13226-019-0302-2

UNIFIED CONVERGENCE FOR MULTI-POINT SUPER HALLEY-TYPE METHODS WITH PARAMETERS IN BANACH SPACE

Ioannis K. Argyros* and Santhosh George**

**Department of Mathematical Sciences, Cameron University, Lawton, OK* 73505, *USA*

***Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka*, *Mangaluru* 575 025, *India*

e-mails: iargyros@cameron.edu; sgeorge@nitk.ac.in

(*Received* 24 *March* 2017; *after final revision* 23 *January* 2018; *accepted* 24 *January* 2018)

We present a local convergence analysis of a multi-point super-Halley-like method in order to approximate a locally unique solution of an equation in a Banach space setting. The convergence analysis in earlier works was based on hypotheses reaching up to the third derivative of the operator. In the present study we expand the applicability of the Super-Halley-like method by using hypotheses only on the first derivative. We also provide: A computable error on the distances involved and a uniqueness result based on Lipschitz constants. The convergence order is also provided for these methods. Numerical examples are also presented in this study.

Key words : Halley-type method; Newton's methods; Banach space; local convergence.

1. Introduction

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces and also let Ω be a convex subset of $\mathcal{B}_1$. Numerous problems in computational disciplines can be written as an equation of the form

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \longrightarrow \mathcal{B}_2$ is a Fréchet-differentiable operator using mathematical modeling, [1-3, 5, 10-18, 23, 25-28]. Then, a locally unique solution p is sought in closed form. However, this is attainable only in special cases. That explains why most solution methods for these equations are usually iterative. Most of the iterative methods are essentially connected to Newton-like methods [1-28]. There exist many studies dealing with the local and semi-local convergence analysis of Newton-like methods such as [1-28]. In order to obtain a higher order of convergence Newton-like methods have been studied such as Potra-Ptǎk [23], Chebyshev, Cauchy, Halley [27] and Ostrowski method [28].

We present the local convergence analysis of a multi-point super Halley-type method (MSHTM) defined by:

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - \frac{\delta}{2\beta}Q_n(I+\gamma Q_n)^{-1}(y_n - x_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point, $\alpha, \delta, \gamma \in \mathbb{R}$, $\beta \in (0,1]$,

$$Q_n = F'(u_n)^{-1}[F'(x_n + \beta(y_n - x_n)) - F'(x_n)]$$

and $u_n = x_n$ or $u_n = y_n$. MSHTM is new and reduces to other popular methods studied in the literature under various assumptions. Our convergence analysis uses generalized Lipschitz conditions allowing all these methods and also new methods to be studied in a uniform way.

Newton's method (Take $\alpha = 1, \delta = 0$):

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n). \tag{1.3}$$

Super-Halley -type [13, 15] method (Take $\alpha = \delta = 1, \gamma = 0$ and $u_n = y_n$):

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - \frac{1}{2\beta}Q_n(y_n - x_n). \end{aligned} \tag{1.4}$$

Other choice of $\alpha, \beta, \gamma, \delta$, are possible [2, 5, 6, 22, 24, 26-28]. Method (1.4) was studied in [13, 15] under Lipschitz or Hölder continuity conditions. MSHT avoids the computation of the expensive in general $F''(x_n)$ required in the Super-Halley method defined for each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - \frac{1}{2}(I + H_n)^{-1}H_n(y_n - x_n), \end{aligned} \tag{1.5}$$

where $H_n = F'(x_n)^{-1}F''(x_n)(y_n - x_n)$. The semi-local convergence of these methods are shown using hypotheses given in non-affine invariant form by [7, 9, 11, 13-15, 24]

$(\mathcal{C}_1)$ $F : \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ is a thrice continuously differentiable operator;

($\mathcal{C}_2$) There exists $x_0 \in \Omega$ such that $F'(x_0)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and $\|F^{-1}(x_0)\| \leq \beta$; there exist $\eta \geq 0,\ \beta_1 \geq 0,\ \beta_2$ and $\beta_3 \geq 0$ such that

($\mathcal{C}_3$) $\|F'(x_0)^{-1}F'(x_0)\| \leq \eta;$

($\mathcal{C}_4$) $\|F''(x)\| \leq \beta_1$ for each $x \in \Omega;$

($\mathcal{C}_5$) $\|F'''(x)\| \leq \beta_2$ for each $x \in \Omega$;

($\mathcal{C}_6$) $\|F'''(x) - F'''(y)\| \leq \beta_3\|x - y\|$ for each $x, y \in \Omega$.

The hypotheses for the local convergence analysis of these methods are the same but x_0 is replaced by p. Notice however that hypotheses ($\mathcal{C}_5$) and ($\mathcal{C}_6$) limit the applicability of these methods. As a motivational example, Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0, 1]$. Let

$$x(s) = \int_0^1 K(s,t)\left(\frac{1}{2}x(t)^{\frac{3}{2}} + \frac{x(t)^2}{8}\right)dt, \tag{1.6}$$

where the kernel K is the Green's function defined on the interval $[0, 1] \times [0, 1]$ by

$$K(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases} \tag{1.7}$$

Define $F : C[0, 1] \longrightarrow C[0, 1]$ by

$$F(x)(s) := x(s) - \int_0^1 K(s,t)\left(\frac{1}{2}x(t)^{\frac{3}{2}} + \frac{x(t)^2}{8}\right)dt \tag{1.8}$$

and consider

$$F(x)(s) = 0. \tag{1.9}$$

Notice that F'' is not Lipschitz. Hence, the results in [8-17, 22, 24, 26, 28] cannot be used to solve equation (1.9). But our result can solve equation (1.9) (see Example 3.3).

Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations defined on $\mathcal{B}_1$ [1-28]. These results show that if the initial point x_0 is sufficiently close to the solution p, then the sequence $\{x_n\}$ converges to p. But how close to the solution p the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.2) in Section 2. The same technique can be used to study other methods. In the present study we extend the applicability of methods (1.2) by using hypotheses up to the first derivative of function F. The results obtained here are the same, if $Q_n, (I + \gamma Q_n)^{-1}$ are switched in (1.2). Moreover we avoid Taylor expansions and hypotheses on the second or higher Fréchet-derivatives (see ($\mathcal{C}_5$), ($\mathcal{C}_6$)). This way we do not have to use higher order derivatives to show the convergence of these methods.

Furthermore, we do compute the order of convergence but without using more smoothness on operator F as it is traditionally done in these type of studies [1-28]. Indeed, to achieve this and use only the first Fréchet-derivative we compute the computational order of convergence and the approximate computational order of convergence (see Remark 2.2 (5) that follows).

The paper is structured as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and a uniqueness result. Special cases and numerical examples are presented in the concluding Section 3.

2. Local Convergence Analysis

Let $\alpha,\gamma,\delta\in\mathbb{R}$ and $\beta\in(0,1]$ be given parameters. It is convenient for us to introduce some scalar functions needed in the local convergence analysis that follows. Let also $w_0:\mathbb{R}_+\cup\{0\}\longrightarrow\mathbb{R}$ be a continuous nondecreasing function satisfying $w_0(0)=0$. Let ρ_0 be defined by

$$\rho_0=\sup\{t\in[0,+\infty):w_0(t)<1\}.$$

Let $w,v:[0,\rho_0)\longrightarrow\mathbb{R}$ be continuous and nondecreasing functions with $w(0)=0$. Suppose that

$$|1-\alpha|v(0)<1. \tag{2.1}$$

Define functions φ_1 and ψ_1 on $[0,\rho_0)$ by

$$\varphi_1(t)=\frac{\int_0^1[w((1-\theta)t)+|1-\alpha|v(\theta t)]d\theta}{1-w_0(t)}$$

and

$$\psi_1(t)=\varphi_1(t)-1.$$

We have by (2.1) that $\psi_1(0)=|1-\alpha|v(0)-1<0$ and $\psi_1(t)\longrightarrow+\infty$ as $t\longrightarrow\rho_0^-$. Then, the intermediate value theorem guarantees the existence of at least one solution for equation $\psi_1(t)=0$. Denote by ρ_1 the smallest solution of equation $\psi_1(t)=0$ in $(0,\rho_0)$. Define functions $\bar{w}_0,h$ and h_1 on the interval $[0,\rho_0)$ by

$$\bar{w}_0(t)=\begin{cases}w_0(t), & u_n=x_n\\ w_0(\varphi_1(t)t), & u_n=y_n,\end{cases}$$

$$\frac{h(t)}{|\gamma|}=\begin{cases}\frac{w(|\beta|(1+\varphi_1(t))t)}{1-\bar{w}_0(t)}, & \gamma\neq0,u_n=x_n\\ \text{or} & \\ \frac{[w_0(((1-\beta)+\beta\varphi_1(t))t)+w_0(t)]}{1-\bar{w}_0(t)} & \gamma\neq0,u_n=y_n,\end{cases}$$

and $h_1(t) = h(t) - 1$. We have $h_1(0) = -1 < 0$ and $h_1(t) \longrightarrow +\infty$ as $t \longrightarrow \rho_0^-$. Denote by ρ_{h_1} the smallest solution of equation $h(t) = 0$. Define functions φ_2 and ψ_2 on the interval $[0, \rho_{h_1})$ by

$$\varphi_2(t) = \begin{cases} \varphi_1(t) + \frac{|\alpha\delta| h(t) \int_0^1 v(\theta t) d\theta}{2\beta|\gamma|(1-h(t))(1-w_0(t))}, & \gamma \neq 0 \\ \varphi_1(t) + \frac{|\alpha\delta| h(t) \int_0^1 v(\theta t) d\theta}{2\beta|\gamma|(1-w_0(t))}, & \gamma = 0 \end{cases}$$

and

$$\psi_2(t) = \varphi_2(t) - 1.$$

We get that $\psi_2(t) = |1-\alpha|v(0) - 1 < 0$ and $\psi_2(t) \longrightarrow +\infty$ as $t \longrightarrow \rho_{h_1}^-$. Denote by ρ_2 the smallest solution of equation $\psi_2(t) = 0$. Define the radius of convergence ρ by

$$\rho = \min\{\rho_1, \rho_2\} \tag{2.2}$$

Then, we have that for each $t \in [0, \rho)$

$$0 \leq \varphi_1(t) < 1 \tag{2.3}$$

$$0 \leq \varphi_2(t) < 1 \tag{2.4}$$

and

$$0 \leq h(t) < 1. \tag{2.5}$$

Set $B(q, \lambda) = \{x \in \mathcal{B}_1 : \|x - q\| < \lambda\}$. Denote by $\bar{B}$ the closure of B. Next, the local convergence of MSHT is presented where the preceding terminology is used and conditions ($\mathcal{A}$):

($\mathcal{A}_1$) $F : \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ is a continuously Fréchet differentiable operator.

($\mathcal{A}_2$) There exists $p \in \Omega$ such that $F(p) = 0$ and $F'(p)^{-1} \in L(\mathcal{B}_2, \mathcal{B}1)$;

($\mathcal{A}_3$) There exists function $w_0 : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}$ continuous and nondecreasing with $w_0(0) = 0$ such that for all $x \in \Omega$

$$\|F'(p)^{-1}(F'(x) - F'(p))\| \leq w_0(\|x - p\|).$$

Set $\Omega_0 = \Omega \cap B(p, \rho_0)$.

($\mathcal{A}_4$) There exists functions $w : [0, \rho_0) \longrightarrow \mathbb{R}$, $v : [0, \rho_0) \longrightarrow \mathbb{R}$ continuous and nondecreasing with $w(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|F'(p)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|)$$

and

$$\|F'(p)^{-1}F'(x)\| \leq v(\|x - p\|).$$

($\mathcal{A}_5$) Let $\alpha \in \mathbb{R}$. Then, the following holds

$$|1-\alpha|v(0) < 1.$$

($\mathcal{A}_6$) $\bar{B}(p,\rho) \subseteq \Omega$, where ρ is given in (2.2).

***Theorem* 2.1** — *Suppose that the conditions ($\mathcal{A}$) hold. Then, the sequence $\{x_n\}$ starting from $x_0 \in B(p,\rho) - \{p\}$ and generated by MSHT exists, stays in $B(p,\rho)$ for all $n = 0, 1, 2, ...$ and converges to p such that*

$$\|y_n - p\| \leq \varphi_1(\|x_n - p\|)\|x_n - p\| < \|x_n - p\| \leq \rho \tag{2.6}$$

and

$$\|x_{n+1} - p\| \leq \varphi_2(\|x_n - p\|)\|x_n - p\| < \|x_n - p\|, \tag{2.7}$$

where the functions φ_1 and φ_2 are defined previously. Moreover, if there exists $\rho^* \geq \rho$ such that

$$\int_0^1 w_0(\theta\rho^*)d\theta < 1, \tag{2.8}$$

then, p is the only solution of equation $F(x) = 0$ in $\Omega_1 = \Omega \cap \bar{B}(p,\rho^*)$.

PROOF : We shall first show estimates (2.6) and (2.7) hold for $n = 0$. By hypothesis $x_0 \in B(p,\rho) - \{p\}$, ($\mathcal{A}_3$) and $x \in B(p,\rho)$ we have that

$$\|F'(p)^{-1}(F'(x) - F'(p))\| \leq w_0\|x - p\| < w_0(\rho) < w_0(\rho_0) = 1. \tag{2.9}$$

Estimate (2.9) and the Banach perturbation lemma [2, 18, 23] guarantee that $F'(x)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(x)^{-1}F'(p)\| \leq \frac{1}{1 - w_0(\|x - p\|)}. \tag{2.10}$$

In particular for $x = x_0, y_0$ exists by the first sub-step of MSHT for $n = 0$ and also (2.10) holds (for $x = x_0$). We can write by ($\mathcal{A}_2$) that

$$F(x) = F(x) - F(p) = \int_0^1 F'(p + \theta(x - p))(x - p)d\theta. \tag{2.11}$$

Note that $\|p + \theta(x - p) - p\| = \theta\|x - p\| < \rho$ for all $\theta \in [0, 1]$. Then, using ($\mathcal{A}_4$) and (2.11), we get that

$$\|F'(p)^{-1}F(x)\| = \|\int_0^1 F'(p)^{-1}F'(p+\theta(x-p))(x-p)d\theta\| \leq \int_0^1 v(\theta\|x-p\|)d\theta\|x-p\|. \tag{2.12}$$

Using the first substep of method MSHT for $n = 0$, (2.3), ($\mathcal{A}_4$), (2.10) and (2.12), we obtain in turn that

$$\begin{aligned}
\|y_0 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1}F(x_0) + (1-\alpha)F'(x_0)^{-1}F(x_0)\| \\
&\le \|F'(x_0)^{-1}F'(p)\| \| \int_0^1 F'(p)^{-1}(F'(p+\theta(x_0-p)) - F'(x_0))(x_0-p)d\theta\| \\
&+ |1-\alpha| \|F'(x_0)^{-1}F'(p)\| \|F'(p)^{-1}F(x_0)\| \qquad (2.13)\\
&\le \frac{\int_0^1 w((1-\theta)\|x_0-p\|)d\theta \|x_0-p\|}{1-w_0(\|x_0-p\|)} + \frac{|1-\alpha| \int_0^1 v(\theta\|x_0-p\|)\|x_0-p\|}{1-w_0(\|x_0-p\|)} \\
&= \varphi_1(\|x_0-p\|)\|x_0-p\| < \|x_0-p\| < \rho, \qquad (2.14)
\end{aligned}$$

so (2.6) holds for $n = 0$ and $y_0 \in B(p, \rho)$. Next, we shall show that $(I + \gamma Q_0)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Let $u_0 = x_0$. Then, using ($\mathcal{A}_4$), (2.10) and (2.14), we get in turn that

$$\begin{aligned}
\|\gamma Q_0\| &\le |\gamma| \|F'(x_0)^{-1}F'(p)\| \\
&\quad \|F'(p)^{-1}(F'(x_0+\theta(y_0-x_0)) - F'(x_0))\| \\
&\le |\gamma| \frac{w(\beta\|(x_0-p)+(p-y_0)\|)}{1-w_0(\|x_0-p\|)} \\
&\le |\gamma| \frac{w(\beta(1+\varphi_1(\|x_0-p\|)\|p-y_0\|))}{1-w_0(\|x_0-p\|)} \qquad (2.15)
\end{aligned}$$

and similarly for $u_0 = y_0$

$$\begin{aligned}
\|\gamma Q_0\| &\le |\gamma| \|F'(y_0)^{-1}F'(p)\| \\
&\quad [\|F'(p)^{-1}(F'(x_0+\theta(y_0-x_0)) - F'(x_0))\| \\
&\quad + \|F'(p)^{-1}(F'(x_0) - F'(p))\|] \\
&\le |\gamma| \frac{[w_0(((1-\beta)+\beta\|x_0-p\|)\|x_0-p\|) + w_0(\|x_0-p\|)]}{1-w_0(\varphi_1(\|x_0-p\|)\|x_0-p\|)} \\
&\le |\gamma| \frac{[w_0((1-\beta)+\beta\varphi_1(\|x_0-p\|)\|x_0-p\|) + w_0(\|x_0-p\|)]}{1-w_0(\varphi_1(\|x_0-p\|)\|x_0-p\|)} \qquad (2.16)
\end{aligned}$$

In either case (2.15) or (2.16), we have that

$$\|\gamma Q_0\| \le h(\|x_0-p\|) \le h(\rho) < 1, \qquad (2.17)$$

so $(I + \gamma Q_0)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|(I+\gamma Q_0)^{-1}\| \le \frac{1}{1-h(\|x_0-p\|)}. \qquad (2.18)$$

Moreover, x_1 is well defined by the second sub-step of MSHT for $n = 0$. Furthermore, using second substep of MSHT for $n = 0$, (2.4), (2.10), (2.12), (2.14) and (2.18), we have in turn that

$$\begin{aligned}
\|x_1 - p\| &\leq \|y_0 - p\| + \frac{|\delta|}{2\beta}\|Q_0\| \\
&\quad \|(I+\gamma Q_0)^{-1}\||\alpha|\|F'(x_0)^{-1}F(p)\|\|F'(p)^{-1}F(x_0)\| \\
&\leq \left[\varphi_1(\|x_0 - p\|) + \frac{|\alpha\delta|}{2\beta|\gamma|}\frac{h(\|x_0 - p\|)\int_0^1 v(\theta\|x_0 - p\|d\theta)}{(1-h(\|x_0 - p\|))(1-w_0(\|x_0 - p\|))}\right]\|x_0 - p\| \\
&= \varphi_2(\|x_0 - p\|)\|x_0 - p\| \leq \|x_0 - p\| < \rho, \qquad (2.19)
\end{aligned}$$

so (2.7) holds for $n = 0$ and $x_1 \in B(p, \rho)$. The induction for (2.6) and (2.7) is completed by using x_k, y_k, u_k, x_{k+1} for x_0, y_0, u_0, x_1 in the preceding estimates. It then, follows from the estimate

$$\|x_{k+1} - p\| \leq c\|x_k - p\| < \rho, \quad c = \varphi_2(\|x_0 - p\|) \in [0, 1)$$

that $\lim_{k\to\infty} x_k = p$ and $x_{k+1} \in B(p, \rho)$. The uniqueness part is shown by using $T = \int_0^1 F'(p + \theta(p^* - p))d\theta$ for some $p^* \in \Omega_1$ with $F(p^*) = 0$. Using $(\mathcal{A}_3)$ and (2.8) we get that

$$\begin{aligned}
\|F'(p)^{-1}(T - F'(p))\| &\leq \int_0^1 w_0(\theta\|p - p^*\|d\theta) \\
&\leq \int_0^1 w_0(\theta p^*)d\theta < 1,
\end{aligned}$$

so $T^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Then, from the identity $0 = F(p^*) - F(p) = T(p^* - p)$, we deduce that $p^* = p$. □

Remark 2.2 : 1. In view of the estimate

$$\begin{aligned}
\|F'(p)^{-1}F'(x)\| &= \|F'(p)^{-1}(F'(x) - F'(p)) + I\| \\
&\leq 1 + \|F'(p)^{-1}(F'(x) - F'(p))\| \leq 1 + w_0(\|x - p\|),
\end{aligned}$$

we can set

$$v(t) = 1 + w_0(t)$$

or $v(t) = 2$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [2, 5, 18] of the form

$$F'(x) = G(F(x))$$

where $G : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous operator. Then, since $F'(x^*) = G(F(p)) = G(0)$, we can apply the results without actually knowing p. For example, let $F(x) = e^x - 1$. Then, we can choose: $G(x) = x + 1$.

3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies in discretization studies [2, 5].

4. If $w_0(t) = L_0 t$ and $w(t) = Lt$, then, the parameter $r_A = \frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton's method [3, 6]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \ \text{ for each } \ n = 0, 1, 2, \cdots \tag{2.20}$$

under the conditions $(\mathcal{A}_1)$–$(\mathcal{A}_4)$. It follows from the definitions of radii r that the convergence radius r of these preceding methods cannot be larger than the convergence radius r_A of the second order Newton's method (2.20). As already noted in [2, 5] r_A is at least as large as the convergence ball given by Rheinboldt [25]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \ \ as \ \ \frac{L_0}{L} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [27].

5. It is worth noticing that the studied methods are not changing when we use the conditions of the preceding Theorems instead of the stronger conditions used in [8-19, 22, 24, 26-28]. Moreover, the preceding Theorems we can compute the computational order of convergence (COC) [28] defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence without resorting to the computation of higher order derivatives appearing in the method or in the sufficient convergence criteria usually appearing in the Taylor expansions for the proofs of those results.

3. Numerical Examples

We present numerical examples in this section. In the first two examples, we show that the radii are larger than the ones in old approaches, whereas in the last example results from old approaches cannot be used.

Example 3.1 : Let us consider a system of differential equations governing the motion of an object and given by

$$F_1'(x) = e^x,\ F_2'(y) = (e-1)y + 1,\ F_3(z) = 1$$

with initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. Let $F = (F_1, F_2, F_3)$. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, D = \bar{U}(0,1), p = (0,0,0)^T$. Define function F on ω for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

The Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the ($\mathcal{A}$) conditions, we get $w_0(t) = (e-1)t, w(t) = e^{\frac{1}{e-1}}t, v(t) = e^{\frac{1}{e-1}}$. The radii for $\alpha = 1 - \frac{1}{2L}, \beta = \gamma = \delta = 0.5$ are

$$\rho_1 = 0.1913 = \rho,\ \rho_2 = 0.2375.$$

Using old approaches we must set $w_0(t) = w(t) = et$ and $v(t) = e$. Then, the radii are

$$\tilde{\rho}_1 = 0.1226 = \tilde{\rho}, \tilde{\rho}_2 = 0.1493.$$

Example 3.2 : Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ be equipped with the max norm. Let $\omega = \overline{U}(0,1)$. Define function F on ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \omega.$$

Then, we get that $p = 0$, so $w_0(t) = 7.5t, w(t) = 15t$ and $v(t) = 2$. Then the radii for $\alpha = 1 - \frac{1}{2L}, \beta = \gamma = \delta = 0.5$ are

$$\rho_1 = 0.0333 = \rho,\ \rho_2 = 0.0374.$$

Using old approaches we must set $w_0(t) = w(t) = 15t$ and $v(t) = 2$. Then, the radii are

$$\tilde{\rho}_1 = 0.0222 = \tilde{\rho}, \tilde{\rho}_2 = 0.0262.$$

Example 3.3 : Returning back to the motivational example at the introduction of this study, we have that

$$F'(x)\mu(s) = \mu(s) - \int_0^1 K(s,t)(\frac{3}{4}x(t)^{\frac{1}{2}} + \frac{x(t)}{4})\mu(t)dt.$$

Notice that $p(s) = 0$ is a solution of (1.9). Using (1.7), we obtain

$$\| \int_0^1 K(s,t)dt \| \leq \frac{1}{8}. \tag{3.2}$$

Then, by (1.7) and (3.2), we have that

$$\|F'(x) - F'(y)\| \leq \frac{1}{32}(3\|x-y\|^{\frac{1}{2}} + \|x-y\|). \tag{3.3}$$

We have $w_0(t) = w(t) = \frac{1}{32}(3t^{1/2} + t)$ and $v(t) = 1 + w_0(t)$. Then the radii for $\alpha = \beta = \gamma = \delta = 0.5$ are

$$\rho_1 = 3.1973, \rho_2 = 0.0190 = \rho.$$

In view of (3.3) earlier results requiring hypotheses on the second Fréchet derivative or higher (see the $(\mathcal{C})$ conditions) cannot be used to solve equation (1.9).

References

1. S. Amat, M. A. Hernández, and N. Romero, Semilocal convergence of a sixth order iterative method for quadratic equations, *Applied Numerical Mathematics*, **62** (2012), 833-841.
2. I. K. Argyros, *Computational theory of iterative methods. Series: Studies in computational mathematics*, **15**, Editors: C. K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.
3. I. K. Argyros, A semilocal convergence analysis for directional Newton methods, *Math. Comput.*, **80** (2011), 327-343.
4. I. K. Argyros and S. Hilout, Weaker conditions for the convergence of Newton's method, *J. Complexity*, **28** (2012), 364-387.
5. I. K. Argyros and Said Hilout, *Computational methods in nonlinear analysis. Efficient algorithms, fixed point theory and applications*, World Scientific, 2013.
6. I. K. Argyros and H. Ren, Improved local analysis for certain class of iterative methods with cubic convergence, *Numerical Algorithms*, **59** (2012), 505-521.

7. I. K. Argyros, D. Chen, and Q. Quian, A local convergence theorem for the Halley method in a Banach space, *Appl. Math. Letters*, **7**(5) (1994), 49-52.

8. V. Candela and A. Marquina, Recurrence relations for rational cubic methods I: The Halley method, *Computing*, **44** (1990), 169-184.

9. V. Candela and A. Marquina, Recurrence relations for rational cubic methods II: The Chebyshev method, *Computing*, **45**(4) (1990), 355-367.

10. J. A. Ezquerro and M. A. Hernández, Recurrence relations for Chebyshev-type methods, *Appl. Math. Optim.*, **41**(2) (2000), 227-236 .

11. R. Ezzati and E. Azandegan, A simple iterative method with fifth order convergence by using Potra and Ptak's method, *Mathematical Sciences*, **3**(2) (2009), 191-200.

12. J. A. Ezquerro, J. M. Gutiérrez, and M. A. Hernández, Avoiding the computation of the second-Fréchet derivative in the convex acceleration of Newton's method, *J. Comput. Appl. Math.*, **96** (1998), 1-12.

13. J. A. Ezquerro and M. A. Hernández, Multipoint super-Halley type approximation algorithms in Banach spaces, *Numer. Funct. Anal. Optimiz.*, **21**(7&8) (2000), 845-858.

14. J. A. Ezquerro and M. A. Hernández, A modification of the super-Halley method under mild differentiability condition, *J. Comput. Appl. Math.*, **114** (2000), 405-409.

15. J. M. Gutiérrez and M. A. Hernández, A family of Chebyshev-Halley-type methods in Banach space, *Bull. Austral. Math. Soc.*, **55** (1997), 113-130.

16. M. A. Hernández and M. A. Salanova, Modification of the Kantorovich assumptions for semi-local convergence of the Chebyshev method, *Journal of Computational and Applied Mathematics*, **126** (2000), 131-143.

17. J. M. Gutiérrez, A. A. Magreñán, and N. Romero, On the semi-local convergence of Newton-Kantorovich method under center-Lipschitz conditions, *Appl. Math. Comput.*, **221** (2013), 79-88.

18. L. V. Kantorovich and G. P. Akilov, *Functional analysis*, Pergamon Press, Oxford, 1982.

19. J. S. Kou, Y. T. Li, and X. H. Wang, A modification of Newton method with third-order convergence, *Appl. Math. Comput.*, **181** (2006), 1106-1111.

20. A. A. Magreńãn, Different anomalies in a Jarratt family of iterative root finding methods, *Appl. Math. Comput.*, **233** (2014), 29-38.

21. A. A. Magreńãn, A new tool to study real dynamics: The convergence plane, *Appl. Math. Comput.*, **248** (2014), 29-38.

22. M. S. Petkovic, B. Neta, L. Petkovic, and J. Džunič, *Multipoint methods for solving nonlinear equations*, Elsevier, 2013.

23. F. A. Potra and V. Pták, Nondiscrete induction and iterative processes, In: *Research Notes in Mathematics*, **103**, Pitman, Boston, 1984.

24. H. Ren and Q. Wu, Convergence ball and error analysis of a family of iterative methods with cubic convergence, *Appl. Math. Comput.*, **209** (2009), 369-378.

25. W. C. Rheinboldt, An adaptive continuation process for solving systems of nonlinear equations, In: *Mathematical models and numerical methods*, (A. N. Tikhonov *et al.* eds.) pub. 3, (1977), 129-142 Banach Center, Warsaw Poland.

26. J. R. Sharma, P. K. Guha, and R. Sharma, An efficient fourth order weighted-Newton method for systems of nonlinear equations, *Numerical Algorithms*, **62**(2) (2013), 307-323.

27. J. F. Traub, *Iterative methods for the solution of equations*, AMS Chelsea Publishing, 1982.

28. S. Weerakoon and T. G. I. Fernando, A variant of Newton's method with accelerated third-order convergence, *Appl. Math. Lett.*, **13** (2000), 87-93.