A STUDY ON LABELINGS OF DIRECTED GRAPHS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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July, 2016

Dedicated to my family and friends

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **A STUDY ON LABELINGS OF DIRECTED GRAPHS** which is being submitted to the National Institute of Technology Karnataka, Surathkal in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me.* The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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Date: 15.07.2016

CERTIFICATE

This is to *certify* that the Research Thesis entitled **A STUDY ON LABELINGS OF DIRECTED GRAPHS** submitted by **Mrs. Kumudakshi**, (Register Number: 135028MA13F01) as the record of the research work carried out by her, is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Prof. S. M. Hegde
Research Guide

Chairman - DRPC
(Signature with Date and Seal)

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ABSTRACT OF THE THESIS

In this thesis, two types of graph labeling problems has been studied namely, graceful and sequential labeling problems of digraphs.

The use of modular arithmetic in these labeling ties them to a variety of algebraic problems. Using some of the algebraic structures such as (v, k, λ) difference set, complete mapping and partition theory the gracefulness of some known class of digraphs has been proved.

A construction of bigraphs from digraphs (vice-versa) are given using an adjacency matrix. Using this constructive method it is proved that, the graceful labelings of some class of bigraphs gives rise to graceful digraphs and vice-versa.

Further, a relationship between the sequential labelings and graceful labelings of digraphs has been given. With this relation it is proved that, the sequential digraphs are related to near complete mappings and cyclic multiplicative groups.

Also, for some more class of digraphs its gracefulness and sequentialness has been proved.

Keywords: ***Algebraic structures, Zero-sequencing, Tournament, Partitions, Subset sum problems,***

Contents

List of Tables

List of Figures

Chapter 1

INTRODUCTION

In this chapter, we present most of the terminology and definition used in the thesis as well several basic results. These are not only used frequently in other chapters, but also serves application of labeled directed graphs.

Graph theory is a very popular area of discrete mathematics with numerous applications in the areas of mathematics, computer science, physics, chemistry, operation research and social networks. It is a research area, which is still relatively young, but is growing rapidly with many highly desirable results over the last couple of decades.

The areas of undirected graphs and directed graphs are considered to be the two branches of theory of graphs. Both areas have immense potential for applications for various reasons. Although, both areas have numerous important applications, undirected graphs have been studied much more extensively than directed graphs. Despite of this, the theory of directed graphs has developed enormously within the last three decade. More than 3000 papers can be found extensively in literature on digraphs. Most of these papers contain, theoretical aspects and also important algorithms as well as applications.

1.1 Some basic definitions and terminologies

For standard notations and terminologies in graph theory we follow (Chartrand and Lesniak, 2010), (Harary, 1969) and (West, 2001).

A *graph* G is a finite non empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex set of G is denoted by $V(G)$, while the edge set is denoted by $E(G)$.

A *directed graph* or *digraph* D is a finite non empty set of objects called vertices together with a set of ordered pairs of distinct vertices of D called arcs. As with graphs, the vertex set of D is denoted by $V(D)$ and the arc set is denoted by $E(D)$.

In a directed graph D an arc (x, y) is called a *symmetric arc* if both (x, y) and (y, x) are arcs in D.

Underlying graph G of a directed graph D is the graph in which two vertices x and y are adjacent if and only if either (x, y) or (y, x) (or both) is an arc in D.

A *symmetric digraph* G based on (underlying) graph G has the same vertex set as G, but has arcs (x, y) and (y, x) replacing each edge xy of G.

Each edge of G can be oriented in one or both of two directions in D. Hence there are at most 3^e digraphs associated with each underlying graph G on e edges.

A digraph D_1 is *isomorphic* to a digraph D_2 if there exists a one-to-one mapping ϕ, called an isomorphism, from $V(D_1)$ onto $V(D_2)$ such that $(x, y)E(D_1)$ if and only if $(\phi x, \phi y)E(D_2)$.

Tournament is a directed graph in which every pair of vertices is connected by a single directed edge.

Ditree is a digraphs whose underlying graph is a tree.

A digraph D is said to be *degree balanced*, if the in-degree and out-degree of each vertex is same.

Unidirectional cycles (or unicycles) are connected digraphs in which every vertex has indegree = outdegree =1.

A directed path is *unidirectional* if all internal vertices have indegree =outdegree = 1.

1.2 Graph labelings

A graph labeling is an assignment of real values or subsets of a set to the vertices or edges or both subject to certain conditions. A graph can be labeled in infinitely many ways. But when conditions are posed on certain parameters like vertices, edges or both then such problems are of some interest. With regard to this graph labeling problems have become a tool to solve many real life problems. A large variety of these labeling techniques have been studied such as graceful labeling, sequential labeling, cordial labeling, harmonious labeling and set labeling etc. An extensive literature on labeling problems is given by J. A Gallian (Gallian, 2014).
Labeled graphs are becoming an increasingly useful family of mathematical models for a broad range of applications. The usage of these labeled graphs are found in various coding theory problems, including the design of good radar-type codes. They have also been applied to determine ambiguities in X-ray crystallographic analysis, to design a communication network addressing system, to determine optimal circuit layouts and to problems in analytic number theory. Bloom and Golomb (Bloom and Golomb, 1977) have described the applications of labeled graphs.

1.2.1 Graceful labeling of graphs

Rosa (Rosa, 1966) introduced in 1966 the very first types of graph labelings or valuations (α, β, γ, and ρ valuations) as tools to solve one of the famous conjectures on graph theory due to Ringel that K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges. The β- valuation was renamed 'graceful' by Golomb (Golomb, 1972) and this name has become more popular than the original one.

DEFINITION 1.2.1. *(Rosa, 1966) An undirected graph with e edges is gracefully labeled if each vertex v is assigned a distinct value $f(v)$ from $\{0, 1, ..., e\}$ in such a way that the set of edge labels equals $\{1, 2, ..., e\}$ when edge uv is labeled by $f(u, v) = |f(u) - f(v)|$. A graph is said to be graceful (undirected) graph if it can be gracefully labeled.*

EXAMPLE 1.2.2. *The following is an example of a graceful and a non-graceful graph.*

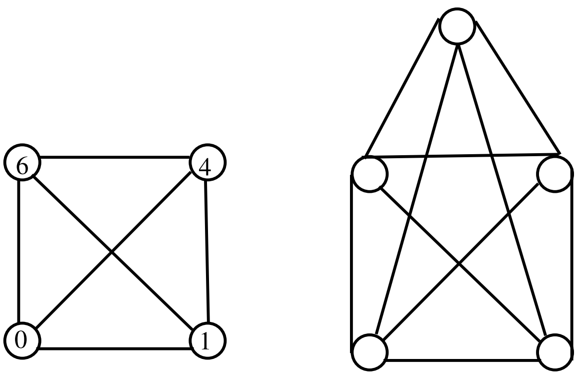

Figure 1.1: A graceful K_4 and a non-graceful K_5

Although numerous families of graceful graphs are known, the problem of characterizing graceful graphs introduced by Rosa (Rosa, 1966) still remains unsolved. Also, it is not known whether "All trees are graceful".
The following are some known results on the graceful graphs.

THEOREM 1.2.3. *(Rosa, 1966) The cycle C_n is graceful if and only if $n \equiv 0, 3(mod 4)$.*

THEOREM 1.2.4. *(Abrham and Kotzig, 1996) The disjoint cycles $C_p \cup C_q$ is graceful iff $p + q \equiv 0, 3(mod 4)$.*

THEOREM 1.2.5. *(Rosa, 1966) The path P_n on n vertices is graceful.*

THEOREM 1.2.6. *(Traetta, 2013) The graph $C_s \cup P_n$ is graceful iff $s + n \geq 6$.*

THEOREM 1.2.7. *(Rosa, 1966) The complete bipartite graph $K_{m,n}$ is graceful.*

1.2.2 k- Sequential labeling of graphs

The concept of k-sequential labeling was introduced in 1981 by Peter J. Slater (Slater, 1981) as follows.

DEFINITION 1.2.8. *A graph G with $\mid V(G) \cup E(G) \mid = t$ is called k-sequential if there is a bijection $f : V(G) \cup E(G) \longrightarrow \{k, k+1, .., t+k-1\}$ such that for each edge $e = xy$ in $E(G)$ one has $f(e) = \mid f(x) - f(y) \mid$. A graph that is 1-sequential is called simply sequential.*

EXAMPLE 1.2.9. *The following figure is an example of a 1-sequential graph.*

The following are some known results on 1-sequential graphs (Bange et al., 1979).

THEOREM 1.2.10. *Graph G is 1-sequential if and only if $G + v$ is graceful by a total numbering f with $f(v) = 0$.*

THEOREM 1.2.11. *If every vertex of G has odd degree, and if $\mid E(G) \mid + \mid V(G) \mid \equiv 1$ or $2(mod\ 4)$, then G is not 1-sequential.*

THEOREM 1.2.12. *A tree T is graceful if and only if T is 1-sequential via a function f' such that $f'(v)$ is odd for each $v \in V(T)$.*

THEOREM 1.2.13. *K_n is 1-sequential if and only if $n \leq 3$.*

THEOREM 1.2.14. *Cycle C_n is 1-sequential.*

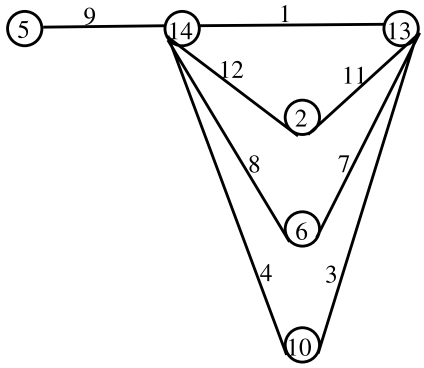

Figure 1.2: A 1-sequential graph

1.3 Graceful labeling of digraphs

The extension of graceful labelings to directed graphs arose in the characterization of some algebraic structures (Hsu and Keedwell, 1984). The relation between graceful digraphs and variety of algebraic structures including cyclic difference sets, sequential groups, generalized complete mappings, near complete mappings finite neofields etc., are discussed in (Bloom and Hsu, 1985).

DEFINITION 1.3.1. *A digraph D with p vertices and q arcs is labeled by assigning a distinct integer value $g(v)$ from $\{0, 1, 2, ..., q\}$ to each vertex v. The vertex values, in turn, induce a value $g(u, v)$ on each arc (u, v) where $g(u, v) = (g(v) - g(u)) \ (mod\ q + 1)$. If the arc values are all distinct, then the labeling is called a graceful labeling of a digraph.*

EXAMPLE 1.3.2. *Figure 1.3 is an example of a graceful and a non-graceful digraph.*

Bloom and Hsu in their introductory paper (Bloom and Hsu, 1985) obtained the following results.

- Start with any gracefully labeled undirected graph G with vertex labeling $f(u)$ for vertex u. Simply orienting the edges of G to point towards the larger vertex value produces a graceful digraph D with G as its underlying graph. Thus, if $f(u) > f(v)$, then the edge uv is labeled as

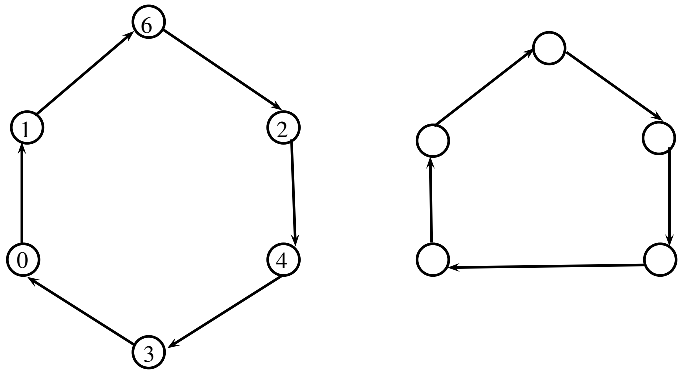

Figure 1.3: A graceful $\vec{C_6}$ and a non-graceful $\vec{C_5}$

$f(u,v) = \mid f(u) - f(v) \mid$ which results in the same value being assigned to the corresponding edges in G and D.

- Although a graceful graph always gives rise to a graceful digraph, an ungraceful graph may underlie a graceful directed one; moreover, not all orientations of an undirected graph are graceful, regardless of whether the underlying graph is graceful or not.

- An undirected graph is termed *digraceful* if some orientation of its edges produces a graceful directed graph. Every graceful graph and some non-graceful ones are digraceful by the "trivial orientation". Nevertheless, not every graph is digraceful, i.e., has an orientation of its edges that yields a graceful digraph.

A class of non-digraceful graphs are specified in Proposition 1.3.3.

PROPOSITION 1.3.3. *A graph with e arcs having even degrees at each vertex and $e \equiv 1(mod\ 4)$ is not digraceful.*

- In addition to the gracefully labeled simple orientations of graphs, there is another set of digraphs that are immediately gracefully labeled from a graceful labeling of their underlying graphs i.e., a symmetric digraph $\overleftrightarrow{G}$.

PROPOSITION 1.3.4. *If gracefully labeled graph G has e edges, then $\overleftrightarrow{G}$ is graceful with the same vertex labels.*

- A graceful digraph D does not have a unique graceful labeling, since adding a constant modulo $(e+1)$ to all of the vertex labels of a digraph preserves the arc labels and therefore generates a new graceful labeling of D.

- Graceful labeling of the vertices of D, $\theta_1(V(D))$ and $\theta_2(V(D))$ are termed equivalent if $\theta_1(V(D)) \equiv \ \theta_2(V(D)) + k(mod(e+1))$.

- A set of $(e+1)$ equivalent graceful labeling of D is called *complete*. It is easily seen that a complete set of equivalent graceful labeling of $\vec{C_4}$ results from adding constants to the sequence of labels $\{0, 4, 1, 2\}$ or from $\{0, 4, 2, 3\}$. It is useful to be able to choose a representative labeling from a complete set of equivalent graceful labeling. The canonical representative graceful labeling of a complete set of equivalent labeling will be chosen so that the edge labeled e is directed from the vertex label 0 to the vertex label e, i.e., $\theta(0, e) = e$.

- A further obvious but useful implication of equivalent graceful labelings is the *rotatability* of vertex labels. Any desired vertex label may be assigned to any desired vertex of a graceful digraph by adding an appropriate constant. It is also important to realize that not all graceful labelings of a digraph are equivalent.

- If D is a digraph, its corresponding reversed digraph $(-D)$ can be obtained from D by replacing each edge (u, v) by its reversed edge (v, u). Clearly, if $\theta(V)$ is a graceful vertex labeling for D, then it is also a graceful vertex labeling for $(-D)$.

These observations are immediate corollaries to proposition 1.3.6, for which the following definition is needed.

DEFINITION 1.3.5. *Digraphs D_1 and D_2 with common underlying graph G are said to be similar (or edge-pair similar) if there is an identical vertex labeling which is graceful for both.*

PROPOSITION 1.3.6. *Each digraph with e edges having graceful vertex labeling $\theta(V)$ is a representative of a family of edge-pair similar digraphs containing no more than $2^{\frac{e}{2}}$ distinct graceful digraphs, all having identically labeled vertices in common underlying graph.*

1.3.1 Complete symmetric digraphs

The only graceful complete graphs are those with no more than four vertices. Nevertheless, many complete symmetric digraphs are graceful.

PROPOSITION 1.3.7. *$\vec{K_n}$ has a graceful labeling if and only if there exists a cyclic (v, k, λ)-difference set with $v = n^2 - n + 1$, $k = n$, and $\lambda = 1$.*

Giving a complete list of the values of n for which $\vec{K_n}$ is graceful is not yet possible. The well known Singer theorem asserts that there exists a cyclic $(v, k, 1)$-difference set when $k - 1$ is any prime power, but the conjecture remains unsolved that no $(\lambda = 1)$-cyclic difference set exists for other values of k. Consequently, despite the conjecture for the necessity of the condition the following is the strongest statement that can be made.

PROPOSITION 1.3.8. *If $n - 1$ is a prime power, then $\vec{K_n}$ is graceful.*

Examples of graceful labeling of complete digraphs $\vec{K_4}$, $\vec{K_5}$ and $\vec{K_9}$ can respectively be gracefully labeled with $\{1, 2, 4, 10\}$, $\{0, 3, 4, 9, 11\}$, and $\{0, 1, 3, 7, 15, 31, 36, 54, 63\}$.

1.3.2 Trees

EXAMPLE 1.3.9. *The following figure gives a graceful and a non-graceful oriented P_3.*

From Figure 1.4 one can see that one orientation of the tree with three vertices is graceful and the other is not. The most studied problem of graceful labeling of undirected graphs is to determine, if all trees are graceful.

Beyond the facts that the graceful trees trivially give graceful directed trees and that all trees similar to these are graceful, little is known about

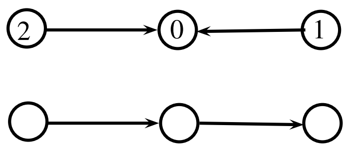

Figure 1.4: A graceful orientation of P_3 and a non-graceful orientation of P_3

general, arbitrarily oriented trees. Only one infinite class of graceful directed trees has been characterized.

PROPOSITION 1.3.10. *The unidirectional path $\vec{P_n}$ on n vertices is graceful if and only if n is even.*

A non equivalent graceful labeling of a unidirectional path can also be generated by the process of sequencing the elements of a sequenceable cyclic group.

The procedure for using sequenceable cyclic groups to generate graceful labeling for the unidirectional path can be viewed as a special class of "ruler models" using the additive group of integers *modulo n*.

First, segments of the intended ruler are created of lengths $1, 2, ..., n-1$, i.e., their lengths are equal to the nonzero elements of Z_n. These segments are then put into a linear sequence to form a ruler of length $\sum_{i=0}^{n-1} i = n(n-1)/2$, such that the set of $n-1$ measurements made between one designated end vertex of the ruler and each of the other $n-1$ ruler marks are all distinct when calculated *modulo n*. Thus, if the sequence of segments is $s_0 = 0, s_1, s_2, ..., s_{(n-1)}$ and measurements $d_0, d_1, ..., d_{(n-1)}$ are made from end vertex d_0, then Z_n is termed as sequenceable if $s_i = \{d_i = \sum_{k=0}^{i} s_k (mod\ n)\} = Z_n$. This is equivalent to saying that for a sequenceable group, assigning d_i as the vertex label of the i^{th} vertex, gives s_i as the distinct edge labels and automatically yields a graceful labeling.

The following is an alternative way of stating the above theorem.

THEOREM 1.3.11. *The unidirectional path $\vec{P_n}$ is graceful if and only if Z_n is sequenceable.*

EXAMPLE 1.3.12. *The set $s_i = \{0, 1, 6, 3, 4, 5, 2, 7\}$ is a sequencing of the cyclic group Z_8.*
Consequently, $\{d_0, d_1, d_2, ..., d_7\} = \{0, 1, 7, 2, 6, 3, 5, 4\}$ is used to label the vertices of $\vec{P_8}$.

1.3.3 Union of unicycles

Some unicycles are graceful and some are not. Moreover, some collections of disjoint unicyclic components are graceful and some are not.

The results reported below gives a relation between graceful unicycles and complete mappings by establishing the relation of each to a particular class of permutations.

EXAMPLE 1.3.13. *If the arc labels are ignored, Figure 1.5 can be regarded as the permutation* $(1\ 8\ 4)(2\ 3\ 6\ 5\ 7)$ *of* $Z_9 \backslash \{0\}$.

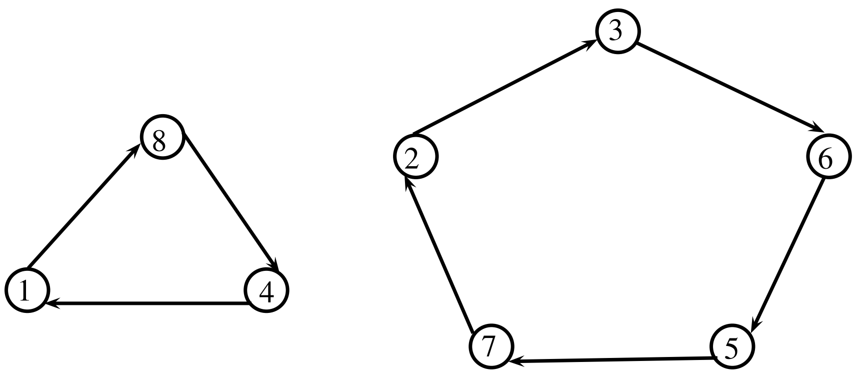

Figure 1.5: A graceful $\vec{C_3} \cup \vec{C_5}$ using Z_9

The idea of a complete mapping was introduced by H.B.Mann (Mann, 1942) and studied later by L.J.Paige (Paige et al., 1951).

DEFINITION 1.3.14. *(Mann, 1942) A* ***complete mapping*** *of a group G is a permutation $g \to \theta(g)$ of the elements of G such that $\phi : g \to g\theta(g)$ is again a permutation of the elements of G. In this case, the mapping ϕ is called an* ***orthomorphism*** *of G.*

Further Hsu and Keedwell in (Hsu and Keedwell, 1985) generalized the concept of complete mapping and called it as a (k, λ) complete mapping.

DEFINITION 1.3.15. *For a specified integer λ and sequence $K = \{k_1, k_2, ..., k_t\}$ in which the k_i are integers such that $\Sigma_{i=1}^{t} k_i = \lambda(n-1)$, a (K, λ) complete mapping is an arrangement of λ copies of the nonzero elements of Z_n into t cyclic sequences of lengths $k_1, k_2, .., k_t$ say $(g_{11}, g_{12}, ..., g_{1k_1})(g_{21}, g_{22}, ..., g_{2k_2})........(g_{t1}, g_{t2}, ..., g_{tk_t})$, such that the following distinct difference property holds. For $i = 1, 2, ..., t$ and $g_{i(k_i+1)} = g_{i1}$, the set of differences $\{g_{i(j+1)} - g_{ij}\}$ comprises λ copies of the nonzero elements of Z_n.*

In other words, as a special case for $\lambda = 1$, a $(K, 1)$ complete mapping is a permutation of $Z_n \backslash \{0\}$ with t cycles, in which the set of modular differences between successive elements in the cycles equals $Z_n \backslash \{0\}$. (In the above figure it is shown that the example is a permutation which satisfies the distinct difference property). In fact, when $\lambda = 1$, the distinct difference property is equivalent to requiring that all arc labels be distinct in the graphical representation of the permutation cycles. Consequently, as a direct result of the definition, the following characterization holds.

THEOREM 1.3.16. *A graceful labeling for $\bigcup_{i=1}^{t} C_{k_i}$, where $\Sigma_{i=1}^{t} k_i = e$ exists if and only if there exists a $(K, 1)$ complete mapping of Z_{e+1} where $K = \{k_1, k_2, ...k_t\}$.*

The study of complete mappings also gave the following results:

THEOREM 1.3.17. *Let $\vec{G} = \bigcup_{i=1}^{t} \vec{C_i}$, the union of t disjoint identical unicycles on n vertices. $\vec{G}$ is graceful if*

1. $t = 1$ *and* n *is even*

2. $t = 2$ *or*

3. *if* $n = 2$ *or* $n = 6$, *moreover* $\vec{G}$ *is not graceful if* tn *is odd.*

EXAMPLE 1.3.18. $(1,6,5,7),(2,8,3,4)$ *is a* $(k,1)$ *complete mapping of* Z_9 *where* $K = \{4,4\}$ *which gives rise to the graceful labeling of the unidirectional* $\vec{C_4} \cup \vec{C_4}$.

1.3.4 Collections of unicycles and paths

One can see that the assignment (1 6 7)(2 5 3)(0 4) shows a gracefully labeled three component digraph of two unidirectional 3-cycles and a single arc path. In a manner similar to that of above section, graceful unions of unicycles and unidirectional paths can be characterized.

Since the components considered are not all cycles, it cannot be viewed as representing a permutation in the way it was done for union of unicycles before; however, it is almost a permutation. That is, the mapping is almost bijective, going from $Z_8\backslash\{4\}$ to $Z_8\backslash\{4\}$. The "almost permutation" character of the above example corresponds to the following algebraic structure for the case $\lambda = 1$.

DEFINITION 1.3.19. *For a given integer* λ *and sequence* $K = \{k_1, k_2, ...k_r; h_1, h_2, ...h_s\}$ *such that the* k_i *and* h_j *are integers satisfying* $\Sigma_{i=1}^{r} k_i + \Sigma_{j=1}^{s} h_j = \lambda n$, *a* (K,λ) *near-complete mapping is an arrangement of* λ *copies of elements of* Z_n *into* r *cyclic sequences with lengths* $k_1, k_2, ...k_r$ *and* s *sequences of lengths* $h_1, h_2, ...h_s$ *say* $(g_{11}, g_{12}, ...g_{1k_1})$, $(g_{21}, g_{22}, ...g_{2k_2}), ..., (g_{r1}, g_{r2}, ...g_{rk_r})$, $[g^1_{11}, g^1_{12}, ...g^1_{1h_1}], ...\ ,[g^1_{s1}, g^1_{s2}, ...g^1_{sh_s}]$ *such that the following distinct difference property holds for* $i = 1,2,...,r; j = 1,2,...,s$ *and* $g_{i(k_i+1)} = g_{i1}$ *the sets of differences* $\{g_{i,j+1} - g_{i,j}\}$ *and* $\{g^1_{i,j+1} - g^1_{i,j}\}$ *together comprise* λ *copies of* Z_n.

The correspondence between near-complete mappings and graceful digraphs is given by the following result.

Table 1.1: A permutation π_1 on N_8

x	0	1	a	a^2	a^3	a^4	a^5	a^6
$\pi_1(x)$	1	0	a^2	a^4	a^6	a	a^3	a^5

THEOREM 1.3.20. *A graceful digraph D comprising a collection of both unicycles and unidirectional paths must contain exactly one path and contains an odd total number of arcs.*

THEOREM 1.3.21. *Let n be the total number of vertices and e be the total number of arcs in a digraph. A graceful labeling of* $(\bigcup_{i=1}^{r}\vec{C}_{k_i}) \cup (\bigcup_{j=1}^{s}\vec{P}_{h_j})$, *where* $\Sigma_{i=1}^{r}k_i + \Sigma_{j=1}^{s}h_j = n = e + s$, *occurs iff there exists a* $(K, 1)$ *near complete mapping of* $Z_n = Z_{e+s}$, *where* $K = \{k_1, k_2, ...k_r; h_1, h_2, ...h_s\}$.

1.3.5 Digraph models for cyclic groups and other structures

In this section we see a relation between graceful digraphs and Latin squares, Abelian groups, Galois fields and neofields.

Let H_n be the cyclic multiplicative group of order n, $H_n = \{1, a, a^2, ...a^{n-1}\}$ with generator a.

Augment this group with a zero element 0 $(x * 0 = 0 * x = 0)$ to form $N_{n+1} = H_n \cup \{0\}$.

A permutation π_1 is defined on N_{n+1} in which for reasons of convenience $\pi_1(0) = 1$.
The permutation can be represented as a digraph which is termed the *unit addition digraph* A_1 for (N_{n+1}, π_1).
A second operation, addition is defined upon N_{n+1} in terms of the permutation for $x \in N_{n+1}$ $1 + x = \pi_1(x)$
It is required that $x(y+z) = xy+xz$ and $(y+z)x = yx+zx$ for $x, y, z \in N_{n+1}$; but no requirements of commutative or associativity are made.

Table 1.2: A permutation π_1 whose addition table is not a normalized Latin square on N_8

x	0	1	a	a^2	a^3	a^4	a^5	a^6
$\pi_1(x)$	1	0	a^2	a^3	a	a^5	a^6	a^4

Table 1.3: An addition table for N_8

$+$	0	1	a	a^2	a^3	a^4	a^5	a^6
0	0	1	a	a^2	a^3	a^4	a^5	a^6
1	1	0	a^2	a^4	a^6	a	a^3	a^5
a	a	a^6	0	a^3	a^5	1	a^2	a^4
a^2	a^2	a^5	1	0	a^4	a^6	a	a^3
a^3	a^3	a^4	a^6	a	0	a^5	1	a^2
a^4	a^4	a^3	a^5	1	a^2	0	a^6	a
a^5	a^5	a^2	a^4	a^6	a	a^3	0	1
a^6	a^6	a	a^3	a^5	1	a^2	a^4	0

These relations allow the calculation of addition table for $(N_{n+1}, +)$ as shown in Table1.3 on N_8.

The next theorem highlights a graph theoretical techniques for calculating and representing the rows of the addition table for any $(N_{n+1}, +)$.

THEOREM 1.3.22. *Given a cyclic group $(H_n, *)$ of order n with generator a, the k th row of the addition table for $(N_{n+1}, +)$ corresponds to the labelled permutation digraph A_{k+1} generated by multiplying the node values of A_1 by a^k.*

Table 1.3 is of some interest, since no elements of N_8 is repeated in any row or column of the addition table, this table is a *Latin square.*

The structure $(N_{n+1}, *, +)$, whose addition table is normalized Latin square is called a *Cyclic neofield.*

DEFINITION 1.3.23. *A neofield $(S, *, +)$ consists of a set S upon which two binary operations $+$ and $*$ are defined provided that*

1. *the table for $(S,+)$ can be written as a normalized Latin square.*

2. *$(S \setminus \{0\}, *)$ is a group.*

3. *multiplication distributes over addition on both left and right.*

A neofield is cyclic if $(S \setminus \{0\}, *)$ is a cyclic group.

A neofield is a finite field in which the associativity and commutativity of addition are not required.

The next theorem implies the interrelationships between graceful digraphs and cyclic neofields.

THEOREM 1.3.24. *Let H_n be a cyclic group of order n. Let $N_{n+1} = H_n \cup \{0\}$, $\pi_1(x) = 1 + x$, π_k, A_1, A_{k+1} be defined as before. For any fixed k, let v_0 be the node with label 0 in A_{k+1} and let A be the digraph A_{k+1} with its node labels removed. Let $D = A - v_0$. Then π_1 defines a cyclic neofield $(N_{n+1}, *, +)$ with $N_{n+1} = H_n \cup \{0\}$, if and only if the digraph D is graceful.*

THEOREM 1.3.25. *The n labelings of the labeled digraph A, generated from the n rows of a cyclic neofield addition table based on H_n correspond to the set of n equivalent graceful numberings of $A - v_0$.*

1.3.6 Conjectures on graceful digraphs

Hegde and Shivarajkumar (Hegde et al., 2013) proved the two conjectures that "All unicyclic wheels are graceful" and "For any positive even n and any integer $m \geq 14$, the digraph $n - \vec{C_m}$ is graceful".
The following results were used in proving the above Conjectures.

THEOREM 1.3.26. *For any even m and even n, the elements of the set $S = \{1, 2, ...r-1, r+1, ..(m-1)n+1\}$, where $r = \dfrac{(m-1)n+2}{2}$ can be partitioned into n disjoint subsets of cardinality $m-1$, so that the sum of the elements of each subset is equal to $\dfrac{2mn(\frac{m}{2}-1)+2m+n-2}{2}$.*

THEOREM 1.3.27. *Let $S = \{1, 2, ...r-1, r+1, ..(m-1)n+1\}$ where m is odd, n is even and $r = \dfrac{(m-1)n+2}{2}$. Then S can be partitioned into n disjoint subsets each of cardinality, $m-1$ in such a way that*

1. *the sum of the elements of each of $\dfrac{n}{2}$ subsets is equal to $\dfrac{m^2n - 3mn + 2m + 2n - 4}{2}$,*
2. *the sum of the elements in each of the remaining $\dfrac{n}{2}$ subsets is equal to $\dfrac{m^2n - mn + 2m + 2m}{2}$.*

Using the system of linear congruences they also obtained the following results.

LEMMA 1.3.28. *Let D be a graceful digraph with p vertices and q arcs. Suppose that $\vec{C_n}$ is contained in the digraph D. Then the sum of the labels on the arcs of $\vec{C_n}$ is congruent to zero (mod (q+1)).*

LEMMA 1.3.29. *$\vec{C_n}$ is graceful iff the sum of the elements $1, 2, ..., n$ is congruent to zero(mod $(n+1)$) and there exists an arrangement of these elements in a circular way, with the sum of $m(m < n)$ consecutive elements not congruent to zero(mod $(n+1)$).*

1.4 k-Sequential labeling of digraphs

The idea of k-sequential labeling of undirected graphs introduced by P.J. (Slater, 1981) was extended to directed graphs by Shivarajkumar (Shivarajkumar, 2013) as follows.

DEFINITION 1.4.1. *A digraph D with $|V(D) \cup E(D)| = t$ is called k-sequential if there exists a bijection $g : V(D) \cup E(D) \longrightarrow \{k, k+1, ..., t+k-1\}$ such that each arc (x, y) is labeled with $g(x, y) = (g(y) - g(x))(mod(t+k))$. If digraph D admits a k-sequential labeling then D is a k-sequential digraph. A digraph that is 1-sequential is called sequential.*

EXAMPLE 1.4.2. *Figure 1.6 represents a $4-$sequential labeling of an unicycle $\vec{C_3}$.*

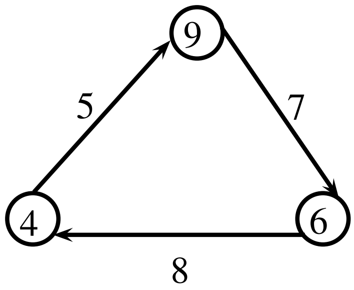

Figure 1.6: A 4-sequential labeling of unicycle $\vec{C_3}$

The following are some known results on sequential digraphs.

THEOREM 1.4.3. *If n is even, then the unidirectional path $\vec{P_n}$ on n vertices is sequential.*

THEOREM 1.4.4. *All unicycles $\vec{C_n}$ are sequential.*

THEOREM 1.4.5. *A complete symmetric digraph $\vec{K_n}$ is not k-sequential for $k = 1$ and for any $k \geq n + 2$.*

THEOREM 1.4.6. *A digraph D is sequential if and only if $D+v$ is graceful.*

1.5 Outline of the thesis

In chapter 2, we prove the gracefulness of some class of digraphs using (v, k, λ)-difference set, complete mappings and partition theory.

In chapter 3, we give further results on graceful digraphs. Infact, we consider the special cases of subset sum problems to find the arc labels of some class of digraphs and hence prove their gracefulness.

In chapter 4, we construct bigraphs from digraphs and vice-versa using the concept of adjacency matrices and establish a relation between them in terms of graceful labeling.

In chapter 5, we prove the existence of strong links between graceful and sequential digraphs. Using these links we give the sequential labeling of many more classes of digraphs.

In chapter 6, we conclude the thesis by giving some scopes for future work and research.

Chapter 2

GRACEFUL DIGRAPHS AND ALGEBRAIC STRUCTURES

In this chapter, we use some known algebraic structures such as cyclic (v, k, λ) difference set, complete mappings and set partitioning of subset sum problems to get the arc labels of some class of digraphs and hence prove that they are graceful.

2.1 Introduction

Labelings are related in a variety of ways to another area of mathematics such as algebra, combinatorics and number theory. Use of modular arithmetic ties them to a variety of algebraic problems. According to Beineke and S. M. Hegde (Beineke and Hegde, 2001) graph labeling serves as a frontier between number theory and the structure of graphs. Normally in graph labeling problems, we label the vertices and then accordingly get the labels on the edges. It is known that if gracefully labelled graph has e edges, then its symmetric digraph is graceful with the same vertex labels. Although, the cycle C_m is not graceful for $m \equiv 1, 2(mod\ 4)$ we show that the symmetric digraph based on cycle C_m i.e., the *double cycle*, DC_m which is constructed from a m-cycle by replacing each edge by a pair of arcs, edge xy gives rise to arcs (x, y) and (y, x), is graceful for any m vertices specifically for $m \equiv 1, 2(mod\ 4)$. Also, using partition theory we obtain the graceful labeling of some more class of digraphs.

2.2 Graceful digraphs and difference sets

In this section we construct graceful digraphs using cyclic difference sets.

From Proposition 1.3.7 it is understood that for every cyclic difference set with $\lambda = 1$ there exists a graceful complete symmetric digraph. The question is, "Is it possible to have a graceful digraph for $\lambda > 1$?", if so how this can be established from λ copies of elements of $Z_v \setminus \{0\}$?

An important topic while discussing collection of subsets of size greater than 2 is the existence of *System of distinct representatives* which was discussed in (Wallis and George, 2011).

DEFINITION 2.2.1. *If $D = \{B_1, B_2, ..., B_k\}$ are any k set, then a system of distinct representatives or SDR for D is defined to be a way of selecting a member x_i from each set B_i such that $x_1, x_2, ..., x_k$ are all different.*

For a (v, k, λ) difference set $D = \{d_1, d_2, d_3, ..., d_k\}$,
let $\mathcal{S} = \{x_{(d_i,d_j)} | d_j - d_i \equiv x(mod\ v) : \forall\ i, j\ 1 \leq i, j \leq k, i \neq j\}$ be the collection of $\lambda(v-1)$ differences of the elements of the difference set D.

Let $\mathcal{A} = \{B_1, B_2, ..., B_{v-1}\}$ represent the collection of $v-1$ sub-multisets of $\mathcal{S}$, with $B_x = \{x_{(d_i,d_j)} | d_j - d_i \equiv x(mod\ v), for\ some\ i, j,\ 1 \leq i, j \leq k, i \neq j\}$. Then each B_x, $1 \leq x \leq v-1$ is a collection of one of the elements of $Z_v \setminus \{0\}$ appearing exactly λ times for some $d_i, d_j \in D$, $i \neq j$.
One can observe that,

1. $B_i \cap B_j = \emptyset\ \forall$ i,j , $1 \leq i, j \leq v-1,\ \ i \neq j$,
2. $\bigcup_{x=1}^{v-1} B_x = \mathcal{S}$, which is the same λ copies of elements of $Z_v \setminus \{0\}$.

DEFINITION 2.2.2. *Let $\mathcal{A} = \{B_1, B_2, ..., B_{v-1}\}$ be the collection of $v-1$ sub-multisets of $\mathcal{S}$, where $\mathcal{S}$ is the collection of $\lambda(v-1)$ differences of the* *elements of the difference set* $D = \{d_1, d_2, ..., d_k\}$ *with* $B_x = \{x_{(d_i,d_j)} | d_j - d_i \equiv x(mod v)$*, for some* i, j, $1 \leq i, j \leq k, i \neq j\}$*. Then we define a* ***collection of different representative****(CDR) for* $\mathcal{A}$ *to be a way of selecting a member* $x_{(d_i,d_j)}$ *for some* $d_i, d_j \in D$*, from each* $B_x, 1 \leq x \leq v-1$ *such that they are all different.*

From the above construction, one can follow that,

1. since $B_i \cap B_j = \emptyset \; \forall \; i, j, \; 1 \leq i, j \leq v-1, \; \; i \neq j$, there always exists a CDR for $\mathcal{A}$.

2. Also, since each B_x has one of the elements of $Z_v \setminus \{0\}$, λ times, each CDR for $\mathcal{A}$ represents exactly one copy of the elements of $Z_v \setminus \{0\}$.

Using the above construction we prove the following.

THEOREM 2.2.3. *For a* (v, k, λ) *difference set* $D = \{d_1, d_2, d_3, ..., d_k\}$ *there exists a graceful digraph.*

Proof. For a (v, k, λ) difference set $D = \{d_1, d_2, d_3, ..., d_k\}$
let $\mathcal{A} = \{B_1, B_2, ..., B_{v-1}\}$, where each $B_x = \{x_{(d_i,d_j)} | d_j - d_i \equiv x(mod \; v)$, $forsomei, j, \; 1 \leq i, j \leq k, \; i \neq j\}$, is a collection of one of the elements of $Z_v \setminus \{0\}$ appearing exactly λ times for some $d_i, d_j \in D$, $i \neq j$.
If $\lambda = 1$, then each B_x, $1 \leq x \leq v-1$ consists exactly one element of $Z_v \setminus \{0\}$. Hence there exists only one CDR for $\mathcal{A}$ with one copy of the elements of $Z_v \setminus \{0\}$. Thus by Propositio 1.3.7 it is clear that there exists a graceful digraph.
If $\lambda > 1$ and a CDR for $\mathcal{A}$ is defined on the difference set D, then by taking the k distinct elements of the difference set $D = \{d_1, d_2, d_3, ..., d_k\}$ as the vertex labels, the elements of a CDR for $\mathcal{A}$ acts as the arc labels.
That is, orienting an arc from d_i to d_j, we getthe set $\{x : 1 \leq x \leq v-1\}$, as the collection of the arc labels for the constructed digraph.
Thus, this constructed digraph represents a graceful digraph with k vertices and $v-1$ arcs. Since each CDR for $\mathcal{A}$, gives the elements of $Z_v \setminus \{0\}$ exactly

once, from every CDR for $\mathcal{A}$ there exists a graceful digraph with k vertices and $v-1$ arcs .

□

EXAMPLE 2.2.4. *For a* $(11,5,2)$ *difference set* $D = \{1,3,4,5,9\}$*, let* $\mathcal{A} = \{B_1, B_2, ..., B_{10}\}$*, where* $B_1 = \{1_{(3,4)}, 1_{(4,5)}\}$*,* $B_2 = \{2_{(1,3)}, 2_{(3,5)}\}$*,...,*$B_{10} = \{10_{(4,3)}, 10_{(5,4)}\}$.
Consider a CDR for $\mathcal{A}$ *as* $\{1_{(3,4)}, 2_{(3,5)}, 3_{(9,1)}, 4_{(5,9)}, 5_{(9,3)}, 6_{(3,9)}, 7_{(5,1)}, 8_{(4,1)}, 9_{(3,1)}, 10_{(5,4)}\}$.

Then a graceful digraph for Example 2.2.4 is shown in Figure 2.1.

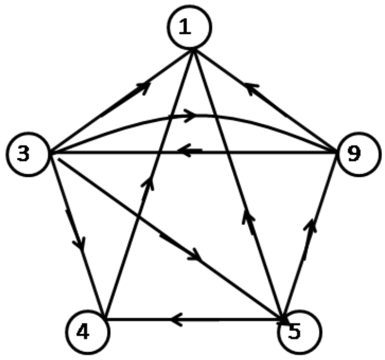

Figure 2.1: A graceful digraph with 5 vetrices and 10 arcs

In Example 2.2.4 instead of selecting $6_{(3,9)}$ as one of the element of B_6, if we select $6_{(9,4)}$ then, the digraph constructed from this possibility represents a graceful tournament.

Since complete graphs are graceful with no more than four vertices, tournament with 3 and 4 vertices can be gracefully labeled.
As a special case for $\lambda = 2$ we observe that, although there exists limited graceful complete graphs, there are some orientations of complete graphs (tournament) with $n > 4$ vertices which turns out to be graceful from the above construction.

PROPOSITION 2.2.5. *For a* (v, k, λ) *difference set* $D = \{d_1, d_2, d_3, ..., d_k\}$ *with* $v = (\frac{n^2-n}{2} + 1)$*,* v *odd,* $k = n$*,* $\lambda = 2$*, there exists a graceful tournament.*

Proof. A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges, hence there are $\frac{n(n-1)}{2}$ directed edges in a tournament. Suppose that there exists a (v, k, λ) difference set $D = \{d_1, d_2, ..., d_k\}$ with $v = (\frac{n^2-n}{2} + 1)$ v odd, $k = n$, $\lambda = 2$. Let $\mathcal{A} = \{B_1, B_2, ..., B_{v-1}\}$ be the collection of $v-1$ sub-multisets of $\mathcal{S}$, where $\mathcal{S}$ is the collection of $\lambda(v-1)$ differences of the elements of the difference set D, $B_x = \{x_{(d_i,d_j)} | d_j - d_i \equiv x(mod\ v),\ for some\ i, j,\ \ 1 \leq i, j \leq k, i \neq j\}$.
Suppose a CDR for $\mathcal{A}$ is chosen in such a way that, whenever $x_{(d_i,d_j)}$ is an element selected from B_x and if $d_i = d_s$, $d_j = d_r$ for some $1 \leq i, j, r, s \leq k$, $i \neq j$ then $y_{(d_r,d_s)}$ is not an element selected from B_y, $1 \leq x, y \leq v-1$, $x \neq y$. Since v is odd, one can see that the values x and y represented above acts as the inverse elements of each other w.r.t mod(v) and hence selecting a CDR for $\mathcal{A}$ in the above manner is always possible. Using this CDR for $\mathcal{A}$, one can construct a tournament which has a graceful labeling. This is achieved by assigning the elements of the difference set D as the vertex labels which in-turn gives the elements of a CDR for $\mathcal{A}$ (as constructed above) as the arc labels.
That is, orienting an arc from d_i to d_j, the set $\{x :\ 1 \leq x \leq v-1\}$ represents the arc labels for the constructed digraph.
These induced arc labels are all distinct and makes exactly one copy of the elements of $Z_v \setminus \{0\}$. Thus the constructed digraph represents a graceful tournament. □

EXAMPLE 2.2.6. *For a* $(11, 5, 2)$ *difference set* $D = \{0, 2, 5, 9, 10\}$*, let* $\mathcal{A} = \{B_1, B_2, ...B_{10}\}$*, where* $B_1 = \{1_{(9,10)}, 1_{(10,0)}\}$*,* $B_2 = \{2_{(0,2)}, 2_{(9,0)}\}$*,...,* $B_9 = \{9_{(0,9)}, 9_{(2,0)}\}$*,* $B_{10} = \{10_{(0,10)}, 10_{(10,9)}\}$*.*
Consider a CDR for $\mathcal{A}$ *given by* $\{1_{(9,10)}, 2_{(0,2)}, 3_{(2,5)}, 4_{(5,9)}, 5_{(0,5)}, 6_{(10,5)}, 7_{(2,9)}, 8_{(2,10)}, 9_{(0,9)}, 10_{(0,10)}\}$*, using which one can see that a tournament with 5 vertices is graceful.*

For the above example a graceful tournament with 5 vertices is given in Figure 2.2.

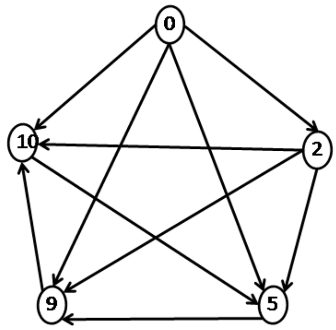

Figure 2.2: A graceful tournament with 5 vertices

NOTE 2.2.7. *A cyclic* $(37, 9, 2)$ *difference set with* $D = \{1, 7, 9, 10, 12, 16, 26, 33, 34\}$ *gives a graceful labeling of a tournament with 9 vertices.*

For a (v, k, λ) difference set with v odd, one can construct a graceful symmetric digraph. This is obtained by a CDR for $\mathcal{A}$ as given below.
For v odd, the elements x and y mentioned in the proof of Proposition 2.2.5 represents the inverse elements of each other. Hence it is always possible to choose a CDR for $\mathcal{A}$ in such a way that, $x_{(d_i,d_j)}$ is an element selected from B_x and if $d_i = d_s$, $d_j = d_r$ for some $1 \leq i, j, r, s \leq k$, $i \neq j$, then $y_{(d_r,d_s)}$ is selected as an element of B_y, $1 \leq x, y \leq v-1$, $x \neq y$.

Hence we have the following result.

PROPOSITION 2.2.8. *For a* (v, k, λ) *difference set with* v *odd and* λ *copies of elements of* $Z_v \setminus \{0\}$ *there exists a graceful symmetric digraph.*

For example, consider a $(11, 5, 2)$ difference set $D = \{0, 2, 5, 9, 10\}$ with $\mathcal{A} = \{B_1, B_2, ...B_{10}\}$, where $B_1 = \{1_{(9,10)}, 1_{(10,0)}\}$, $B_2 = \{2_{(0,2)}, 2_{(9,0)}\}$,..., $B_9 = \{9_{(0,9)}, 9_{(2,0)}\}$, $B_{10} = \{10_{(0,10)}, 10_{(10,9)}\}$.
Let $\{1_{(9,10)}, 2_{(0,2)}, 3_{(2,5)}, 4_{(5,9)}, 5_{(0,5)}, 6_{(5,0)}, 7_{(9,5)}, 8_{(5,2)}, 9_{(2,0)}, 10_{(10,9)}\}$ be a CDR for $\mathcal{A}$. Assigning the elements of the difference set D as the vertex labels and using the above CDR for $\mathcal{A}$ to orient an arc from d_i to d_j, $d_i, d_j \in D$, the elements of the CDR for $\mathcal{A}$ acts as the arc labels. Thus the digraph as-

sociated with this CDR for $\mathcal{A}$ represents a graceful symmetric digraph with 5 vertices and 10 arcs.

NOTE 2.2.9. *From Theorem 2.2.3, Proposition 2.2.5 and 2.2.8, it is clear that whenever there is a cyclic (v,k,λ) difference set with λ copies of elements of $Z_v \setminus \{0\}$, it is always possible to construct a graceful digraph.*

It is also noticed that, from the trivial difference set i.e., when $v=k=\lambda$ one can construct a graceful ditree.
For example, when $v=k=\lambda=7$ the trivial difference set $\{0,1,2,3,4,5,6\}$ with $\mathcal{A}=\{B_1,B_2,...B_6\}$, where $B_1=\{1_{(0,1)},1_{(1,2)},1_{(2,3)},1_{(3,4)},1_{(4,5)},1_{(5,6)},1_{(6,0)}\}$, $B_2=\{2_{(0,2)},2_{(1,3)},2_{(2,4)},2_{(3,5)},2_{(4,6)},2_{(6,1)},2_{(5,0)}\}$,...,$B_6=\{6_{(0,6)},6_{(1,0)},6_{(2,1)},6_{(3,2)},$ $6_{(4,3)},6_{(5,4)},6_{(6,5)}\}$ gives distinct graceful ditrees for each CDR for $\mathcal{A}$.
Suppose $\{1_{(0,1)},2_{(0,2)},3_{(1,4)},4_{(2,6)},5_{(0,5)},6_{(4,3)}\}$ is a CDR for $\mathcal{A}$, then the following figure represents a graceful ditree.

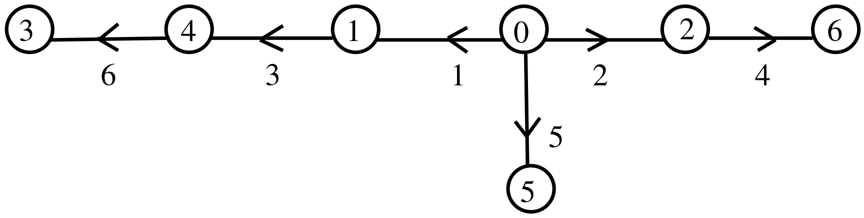

Figure 2.3: A graceful ditree with 7 vertices

2.3 Graceful digraphs and complete mappings

From Theorem 1.3.16, it is clear that the disjoint union of directed cycles are graceful iff there exists a complete mapping. Using a $(k,1)$ complete mapping of the cyclic group Z_n, where $n>1$ is odd we show that, not only the disjoint unoin of directed cycles but, it is also possible to have a graceful directed non-disjoint cycles.

NOTE 2.3.1. *(Friedlander et al., 1981) In the definition of complete mappings, when $\lambda = 1$ and k_i $(1 \le i \le t)$ are all equal, then such a mapping is called as k-regular complete mapping of the cyclic group Z_n.*

THEOREM 2.3.2. *(Friedlander et al., 1981) If $k = 2$ or $\frac{n-1}{2}$, then there exists a* ***k-regular complete mapping*** *of the cyclic group Z_n, where $n > 1$ is odd.*

From Theorem 2.3.2, it is clear that there always exists a $(k, 1)$ complete mapping of non-zero elements of Z_n (where $n > 1$ is odd) into 2 cyclic sequences of equal length.

A $(k, 1)$ complete mapping of $Z_n \setminus \{0\}$ (where $n > 1$ is odd) given in the form $(g_{11}, g_{12}, ..., g_{1k_1})$ $\quad ((g_{11})^{-1}, (g_{12})^{-1}, ..., (g_{1k_1})^{-1})$, is an arrangement of one copy of $Z_n \setminus \{0\}$ into 2 cyclic sequences of equal length $\{k_1 = k_2\}$, such that they follow the distinct difference property of the Definition 1.3.15. Then one can also observe that:

- The differences in each of the cyclic sequences represent a partition of non-zero elements of Z_n into 2 sets of equal cardinality $(k_1 = k_2)$,
- such that the sum of the elements in each set is $\equiv\ 0(mod\ n)$.

We show that the above partition of non zero elements of Z_n (where $n = 2m + 1$) gives rise to a graceful digraph which contains exactly two non-disjoint unidirectional cycles .

We denote $2 : \overrightarrow{C_m}$ as a digraph consisting of 2 directed cycles each of length m, with vertices $v_1^i, v_2^i, ..., v_{r-1}^i, v_r^i, v_{r+1}^i, ..., v_m^i$ for all i $(1 \le i \le 2)$, such that the directed cycles have v_1^i and v_r^i for all i $(1 \le i \le 2)$ as the two common vertices where the directed path length from v_1^i to v_r^i is same. $(m \ge 4,\ 3 \le r \le m - 1)$.

LEMMA 2.3.3. *If the digraph $2 : \overrightarrow{C_m}$ is graceful, then the sum of the labels on the arcs of $2 : \overrightarrow{C_m}$ is congruent to zero(mod $(2m + 1)$).*

Proof. By the definition of digraph $2:\overrightarrow{C_m}$ it is clear that, it consists of 2 directed cycles each of length m. Also from Lemma 1.3.28 one can see that the sum of the arc labels on each of the directed cycle of $2:\vec{C_m}$ is congruent to zero(mod $(2m+1)$). Hence the the sum of the labels on the arcs of $2:\overrightarrow{C_m}$ is congruent to zero(mod $(2m+1)$). □

THEOREM 2.3.4. *From every $(k,1)$ complete mapping of $Z_{2m+1}\setminus\{0\}$ into 2 cyclic sequences of equal length, there exists a graceful labeling of $2:\overrightarrow{C_m}$.*

Proof. suppose $(s_1, s_2, ..., s_m)$ and $(l_1, l_2, ..., l_m)$ are the differences of a $(k,1)$ complete mapping $(a_1, a_2, ..., a_m)(-a_1, -a_2, ..., -a_m)$ of $Z_{2m+1}\setminus\{0\}$.
`Step 1`: Rearrange the elements of $(s_1, s_2, ..., s_m)$ and $(l_1, l_2, ..., l_m)$ such that they satisfy the following conditions.
(i)$\sum_{k=1}^{i} s_k$ and $\sum_{k=1}^{i} l_k$ are not congruent to zero $(mod(2m+1))$ for $1 \le i \le m-1$. (ii) Also, for some r, $(2 \le r \le m-1)$ $\sum_{i=1}^{r} s_i = \sum_{i=1}^{r} l_i$ $(mod(2m+1))$ but no $(r^{'}) > r$ is considered such that $\sum_{i=1}^{r^{'}} s_i = \sum_{i=1}^{r^{'}} l_i$ $(mod(2m+1))$. One can arrange the elements in this way as there is no additive inverse of any of its elements.
`Step 2`: Using these rearranged elements of the differences as in step 1, find
(i) $d_i = \sum_{k=0}^{i} s_k$ $(mod(2m+1))$, $s_0 = 0$, $0 \leqslant i \leqslant m-1$, $d_m = d_0$ and
(ii) $d_i^{'} = \sum_{k=0}^{i} l_k$ $(mod(2m+1))$, $l_0 = 0$, $0 \leqslant i \leqslant m-1$, $d_m^{'} = l_0$
where, for $i = r-1$, $d_{r-1} = d_{r-1}^{'}(mod(2m+1))$, then each d_i and $d_i^{'}$, $(1 \leqslant i \leqslant m-1)$ is a distinct element of Z_{2m+1}, with $d_0 = d_0^{'} = 0$, $d_m = d_m^{'} = 0$ and $d_{r-1} = d_{r-1}^{'} \in Z_{2m+1}$.
`Step 3`: Use
(i) $d_0 = 0, d_1, ..., d_{r-1}, d_r, ..., d_{m-1}$ as the vertex labels of the directed cycle with vertices $v_1^1, v_2^1, ..., v_r^1, v_{r+1}^1, ..., v_m^1$ of $2:\vec{C_m}$ respectively and
(ii) $d_0^{'} = 0, d_1^{'}, ...d_{r-1}^{'}, d_r^{'}, ..., d_{m-1}^{'}$ as the vertex labels of the directed cycle with vertices $v_1^2, v_2^2, ..., v_r^2, v_{r+1}^2, ..., v_m^2$ of $2:\vec{C_m}$ respectively, with two common vertex labels $v_1^1 = 0$, $v_r^1 = d_{r-1}$ where $d_{r-1} = d_{r-1}^{'}$ $(mod(2m+1))$.
One can see that the sets $\{s_i\}$ and $\{l_i\}$ for all i, $(0 \leqslant i \leqslant m)$ represents the distinct arc labels and hence yields a graceful labeling of $2:\vec{C_m}$.

□

EXAMPLE 2.3.5. $(1,8,5,7,2)(10,3,6,4,9)$ *is a* $(k,1)$ *complete mapping* *of non-zero elements of* Z_{11} *into 2 cyclic sequences of equal length* $\{k_1 = k_2 = 5\}$*. Then an arrangement* $(1, 9, 3, 4, 5)$ *and* $(8, 2, 10, 7, 6)$ *of the distinct differences* $(7, 8, 2, 6, 10)$ *and* $(4, 3, 9, 5, 1)$ *obtained from the given* $(k, 1)$ *complete mapping of non-zero elements of* Z_{11} *gives a graceful digraph as given in Figure 2.4.*

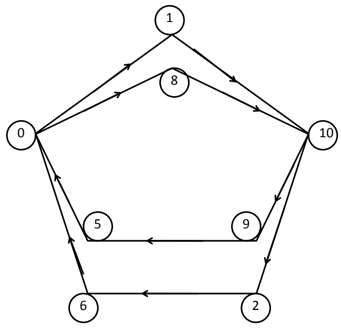

Figure 2.4: A graceful digraph using a $(k, 1)$ complete mapping of Z_{11}

Hence one can observe, how a partition of $Z_{2m+1} \setminus \{0\}$ deduced from a complete mapping gives a graceful digraph other than the one discussed in Theorem 1.3.16 for cycles of length 2.

OBSERVATION 2.3.6. *One can observe that, the differences in each of the cyclic sequences represent a partition of non-zero elements of* Z_n *into* t *sets of cardinalities* k_i $(1 \leq i \leq t)$ *with sum of the elements in each set* $\equiv 0(mod\ n)$*. Using these differences one can construct a graceful digraph.*

For example, consider the permutation (1 8 4)(2 3 6 5 7) of $Z_9 \setminus \{0\}$ given in Example 1.3.13 . An arrangement (7 5 6)(1 3 2 8 4) of the differences in each of the cyclic sequences of the above permutation gives a graceful digraph as in Figure 2.5.

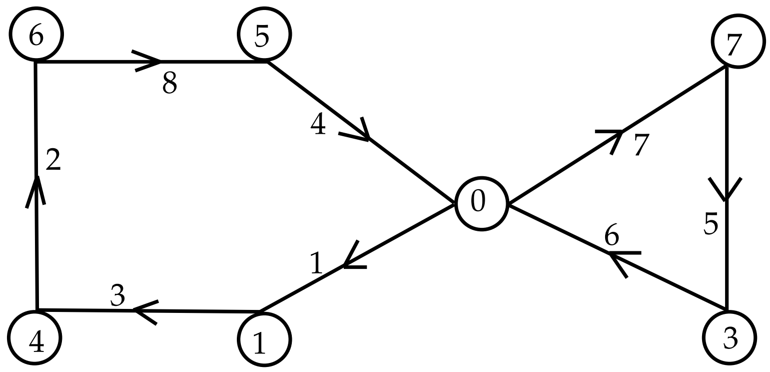

Figure 2.5: A graceful digraph obtained from a complete mapping

2.4 Graceful digraphs and subset sum problems

In this section we discuss the gracefulness of *double cycles* and some more class of directed cycles.

Let C_m and $\vec{C_m}$ denote the cycle and directed cycle on m vertices respectively.

DEFINITION 2.4.1. *(Marr and Wallis, 2012) Double cycle DC_m is constructed from a m-cycle by replacing each edge by a pair of arcs: edge xy gives rise to arcs (x, y) and (y, x).*

OBSERVATION 2.4.2. *From Theorem 2.2.3 and Proposition 1.3.4 it follows that DC_m is graceful for $m \equiv 0, 3(mod\ 4)$.*

NOTE 2.4.3. *We show that there exists a graceful labeling of DC_m for all m vertices in particular for $m \equiv 1, 2(mod\ 4)$, whereas C_m is not graceful for $m \equiv 1, 2(mod\ 4)$.*

Using Lemma 1.3.28 we give the following result for DC_m.

LEMMA 2.4.4. *If DC_m is graceful, then the sum of the labels on the arcs of DC_m is congruent to $0(mod\ 2m + 1)$.*

Proof. If DC_m is gracefully labeled then, the sum of the labels on the arcs of each of the directed cycle (one in clockwise direction another in anticlockwise direction) of length m, must add up to $0(mod\ 2m+1)$. Hence the sum of the arc labels on DC_m is congruent to $0(mod\ 2m+1)$. □

Using the definition of DC_m we have:

LEMMA 2.4.5. *If DC_m is graceful, then the arc labels on each of the directed cycles of length m are inverse elements of each other (in the same order) under $(mod\ 2m+1)$.*

To get these arc labels as a collection of two subsets of $Z_{2m+1} \setminus \{0\}$, we use the result given between partitions of groups and complete mappings in (Friedlander et al., 1981).

These partitions are considered to be the special cases of subset sum problems. Subset sum problem is to find subset of elements that are selected from a given set whose sum adds up to a given number k (Hwang et al., 1985). The subset sum problem is a member of the NP-complete class of computational problems.

THEOREM 2.4.6. *Friedlander et al. (1981) Suppose n is odd and $k \mid n-1$, where $k > 1$. Then the nonzero residues (mod n) can be partitioned into $\frac{n-1}{k}$ sets of cardinality k, so that the sum of the elements of each set is $\equiv 0(mod\ n)$.*

From the above theorem, one can see that, for a group of residue classes under addition modulo $2m+1$, with $k=2$ the required sets are just $\{1, 2m\}$, $\{2, 2m-1\}, \ldots.., \{m, m+1\}$.

Also, the set $\{1, 3, 5, ..., 2m-1\}$ is the absolute differences, between the elements in each of the above sets.

DEFINITION 2.4.7. *A set $D = \{s_1, s_2, ..., s_m\}$ which is a collection of m residues modulo $2m+1$ is termed as a Zero-sequencing of Z_{2m+1} if it has the following properties;*

1. *each s_i, $1 \leqslant i \leqslant m$ is one of the elements taken from the set $\{i, i^{-1}\}$ where $i \in Z_{2m+1} \setminus \{0\}$, $1 \leqslant i \leqslant m$.*

2. *$\sum_{i=1}^{m} s_i \equiv 0 \ (mod \ 2m+1)$.*

NOTE 2.4.8. *If $D = \{s_1, s_2, ..., s_m\}$ is a Zero-sequencing of Z_{2m+1}, then the complement of this set $\bar{D}$ is again a Zero- sequencing of Z_{2m+1} and $D \cup \bar{D} = Z_{2m+1} \setminus \{0\}$.*

The following result proves the existence of such a Zero-sequencing.

THEOREM 2.4.9. *There always exists a Zero-sequencing of Z_{2m+1} for $m > 2$.*

Proof. Let $D = \{s_1, s_2, ..., s_m\}$ be a collection of $m(> 2)$ residues modulo $2m+1$.
Suppose $s_i = i, \ \ 1 \leqslant i \leqslant m$.
Then $\sum_{i=1}^{m} s_i = \sum_{i=1}^{m} i = \dfrac{m(m+1)}{2} \not\equiv 0(mod \ 2m+1)$.

One can see that, if for some $j \geq 1$ there exists a d_j, $0 < d_j < m^2$, such that $\sum_{i=1}^{m} i \ + d_j \equiv 0 \ (mod \ 2m+1)$ then

$$d_j = (2m+1)\{\left\lfloor \frac{m(m+1)}{2(2m+1)} \right\rfloor + j\} - \frac{m(m+1)}{2}.$$

We consider the following three possible cases, for $j = 1, \ 0 < d_1 \leq 2m-1$.

Case 1: Suppose d_1 is even and $d_1 \neq 2$.
Express d_1 as the sum of any two or more distinct elements of the set $\{1, 3, 5, ..., 2m-1\}$. Then corresponding to each of these elements,(which represents the absolute differences between the elements in each of the set $\{i, i^{-1}\}$, $i \in Z_{2m+1} \setminus \{0\}$, $1 \leqslant i \leqslant m$) replace s_i by s_i^{-1} in D, $1 \leqslant i \leqslant m$.

Then the resultant set D becomes a Zero-sequencing of Z_{2m+1}.

Case 2: Suppose d_1 is odd.
Then $d_1 \in \{1, 3, 5, ..., 2m-1\}$ and for some $i, 1 \leqslant i \leqslant m$, $d_1 = \mid s_i - s_i^{-1} \mid$. Replacing this s_i by s_i^{-1} in D the set $\{1, 2, ..., s_{i-1}, s_i^{-1}, s_{i+1}, ..., m\}$ forms a Zero-sequencing of Z_{2m+1} satisfying both the properties of the Definition 2.4.7.
Also, for $d_1 > 7$ the method given in case 1 can be used to get a Zero-sequencing of Z_{2m+1}.

Case 3: Suppose $d_1 = 2$ (or for $j > 1$).
Find some $j > 1$, for which $2m - 1 < d_j < m^2$. Express this d_j as the sum of any two or more distinct elements of the set $\{1, 3, 5, ..., 2m-1\}$. Using the method given in case 1 we get a Zero-sequencing of Z_{2m+1}.
Hence the proof. □

From the above result we observe that a zero-sequencing of Z_{2m+1} is not unique.

NOTE 2.4.10. *If $D = \{s_1, s_2, ..., s_m\}$ is a Zero-sequencing of Z_{2m+1}, then for any $m > 2$, one can see that, D does not contain the additive inverse of any of its elements. Hence, there exists an arrangement of the elements of D such that $\sum_{k=0}^{i} s_k$ (mod $2m+1$), with $s_0 = 0$, $0 \leqslant i \leqslant m-1$, are all distinct.*

Using the Zero-sequencing of Z_{2m+1}, we prove the following theorem for double cycles.

THEOREM 2.4.11. *The double cycle DC_m is graceful if and only if $\{s_1, s_2, ..$
$., s_m\}$ is a Zero-sequencing of Z_{2m+1}, $m > 2$.

Proof. Suppose the double cycle DC_m is graceful, then from Lemma 2.4.4 and 2.4.5 it follows that the arc labels on one of the directed cycles of DC_m

is a collection say, $\{s_1, s_2, ..., s_m\}$ satisfying both the properties of Definition 2.4.7 hence forms a Zero-sequencing of Z_{2m+1}, $m > 2$.
Conversely, let $D = \{s_1, s_2, ..., s_m\}$ be a Zero-sequencing of Z_{2m+1}. The elements of D are arranged in such a way that $d_i = \sum_{k=0}^{i} s_k \ (mod\ 2m + 1)$, with $s_0 = 0$, where each d_i, $0 \leqslant i \leqslant m - 1$, is a distinct element of Z_{2m+1} and $d_m = d_0$. Then assigning these d_i's to the vertex labels of DC_m, we get $\{s_1, s_2, ..., s_m\}$ as the arc labels on one of the directed cycles of DC_m and $\{s_1^{-1}, s_2^{-1}, ..., s_m^{-1}\}$ as the arc labels on another directed cycle of DC_m, where $s_i^{-1} \in Z_{2m+1} \setminus \{0\}$, $1 \leqslant i \leqslant m$. Hence DC_m is graceful. □

EXAMPLE 2.4.12. *A graceful double cycle with 5 vertices is shown in Figure 2.6.*

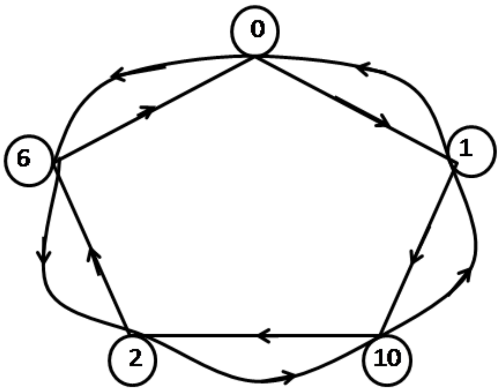

Figure 2.6: A graceful double cycle with 5 vertices

From Theorem 2.4.11 it follows that DC_m is gracefully labeled even though the cycle C_m is not graceful for $m \equiv 1, 2 (mod\ 4)$.

There is another class of directed cycle $\overrightarrow{C_m}(r)$ which can be gracefully labelled for even m and $r = \frac{m}{2} + 1$ by using the concept of Zero-sequencing.

DEFINITION 2.4.13. $\overrightarrow{C_m}(r)$ *is a directed cycle having vertices* $\{v_1, v_2, ..., v_{r-1}, v_r, v_{r+1}, ..., v_m\}$ *and a pair of arcs* (x, y) *and* (y, x) *which connects vertex* v_1 *with* v_r.

$\overrightarrow{C_m}(r)$ can also be viewed as having two directed cycles, one of the form $v_1, v_r, v_{r+1}, ..., v_m, v_1$ and the other is a directed cycle of the form $v_1, v_2, ...,$ v_{r-1}, v_r, v_1.

Suppose that m is even and $r = \frac{m}{2} + 1$ then $\overrightarrow{C_m}(r)$ has two directed cycles each of length r, with two common vertices and $2r$ arcs.

PROPOSITION 2.4.14. *If $\{s_1, s_2, ..., s_r\}$ is a Zero-sequencing of Z_{2r+1} for $r > 2$, then the directed cycle $\overrightarrow{C_m}(r)$ with even m and $r = \frac{m}{2} + 1$ is graceful.*

Proof. Let $D = \{s_1, s_2, ..., s_r\}$ be a Zero-sequencing of Z_{2r+1}, $r > 2$.

Step 1:

Arrange the elements of D and $\bar{D}$ such that $d_i = \sum_{k=0}^{i} s_k (mod\, 2r+1)$, $s_0 = 0$, $s_1 = 1$, $0 \leqslant i \leqslant r$, where each d_i, $0 \leqslant i \leqslant r-1$ is a distinct element of Z_{2r+1}, $d_r = d_0$ and $f_i = \sum_{k=0}^{i} s_k^{-1} (mod\ 2r+1)$, $s_0^{-1} = 0$, $0 \leqslant i \leqslant r$, where each f_i, $0 \leqslant i \leqslant r-1$ is a distinct element of Z_{2r+1}, $f_{r-1} = d_1$, $f_r = f_0$, with $\{d_i\} \cap \{f_j\} = \{0, 1\}$, $0 \leqslant i, j \leqslant r$.

Step 2:

Assigning $d_0, d_1, ..., d_{r-1}, d_r$ to the vertices $v_1, v_r, v_{r+1}, ..., v_m, v_1$ and $f_0, f_1, ...,$ f_{r-1}, f_r to the vertices $v_1, v_2, v_3, ..., v_r, v_1$ respectively, with common vertex labels $v_1 = 0$ and $v_r = 1$ (i.e., $f_0 = d_0$ and $d_1 = f_{r-1}$), we get $\{s_i\}$ and $\{s_i^{-1}\}$ $1 \leqslant i \leqslant r$, as the arc labels of $\overrightarrow{C_m}(r)$. Hence $\overrightarrow{C_m}(r)$ is graceful. $\square$

EXAMPLE 2.4.15. *Figure 2.7 is an example of a graceful $\overrightarrow{C_6}(4)$.*

NOTE 2.4.16. *For example, assigning $d_0 = 0, d_1 = 4, d_2 = 2, d_3 = 10, d_4 = 1$ and $f_0 = 0, f_1 = 1, f_2 = 7, f_3 = 3,$ $f_4 = 6$ as the vertex labels of $\overrightarrow{C_8}(5)$, we get a graceful labeling of $\overrightarrow{C_8}(5)$. Then it is noted that the arc labels $(4, 9, 8, 2, 10)$ and $(1, 6, 7, 3, 5)$ of the graceful $\overrightarrow{C_8}(5)$ do not satisfy one of the properties of Definition 2.4.7. Hence they do not form a Zero-sequencing of Z_{2r+1}. Therefore, the converse of Proposition 2.4.14 is not always true.*

Using Theorem 2.4.6 we give the following subset partition, which helps us to find the gracefulness of another class of digraph.

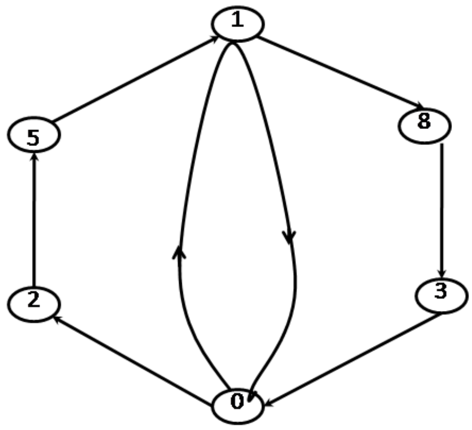

Figure 2.7: A graceful directed cycle

Suppose for example let $S_k = \{k, 3n-k+1, 4n-k+1, n+k\}$, $(1 \le k \le n)$ be a partition of Z_{4n+1}, into n subsets each of cardinality 4 such that sum of the elements of each set is $\equiv 0(mod(4n+1))$.

Then one can use the above partition of Z_{4n+1} to find the gracefulness of the digraph $n : \overrightarrow{C_4}$, where, $n : \vec{C_4}$ is a digraph having n $(n > 1)$ copies of the directed cycles each of length 4 with 2 non-adjacent vertices as common vertices.

The digraph $n : \overrightarrow{C_4}$ has $4n$ edges and for graceful labeling edge labeling are computed *modulo* $4n+1$. Hence the above subsets gives a graceful $n : \vec{C_4}$.

THEOREM 2.4.17. *The digraph $n : \overrightarrow{C_4}$ is graceful for any $n > 1$.*

Proof. Let v_j^i be the jth vertex of the ith directed cycle in $n : \vec{C_4}$, where $1 \le i \le n$, $0 \le j \le 3$. Let $v_0^i = v_0^{i+1}$ and $v_2^i = v_2^{i+1}$ for all i, $1 \le i \le n-1$.
Let e_{ks} be the sth arc of the kth directed cycle in $n : \overrightarrow{C_4}$ $(1 \le k \le n$, $1 \le s \le 4)$.
Label the $4n$ arcs of $n : \overrightarrow{C_4}$ as $e_{k1} = k,\ e_{k2} = 3n-k+1,\ e_{k3} = 4n-k+1$ *and* $e_{k4} = n+k$ $(1 \le k \le n)$.
To generate the labels on the vertices, let $v_0^i = 0\ \forall\ i$, $1 \le i \le n$, then for

each i, $1 \leq i \leq n$ we get $v_j^i = v_{j-1}^i + e_{ij}$, where j varies from 1 to 3.
To prove that the above vertex and arc labels form a graceful labeling, it is necessary only to show that the $2n+2$ vertex values are distinct and that the $4n$ arc values comprise $Z_{4n+1} \backslash \{0\}$. One can see that the vertex labels as given above gives $\{v_0^i\} = \{0\}, \{v_1^i\} = \{1, 2, 3, ..., n-1, n\}, \{v_2^i\} = \{3n+1\}, \{v_3^i\} = \{3n, 3n-1, ..., 2n+2, 2n+1\}$ $\forall i$, $1 \leq i \leq n$.
Hence $v_j^i \cap v_p^i = \phi$ for any $0 \leq j, p \leq 3$ and $1 \leq i \leq n$.
Also we have $\cup e_{ks} = \{1, 2, 3, ..., 4n-1, 4n\}$ $(1 \leq k \leq n)$, $(1 \leq s \leq 4)$.
Hence $n : \overrightarrow{C_4}$ is graceful for any $n > 1$. □

EXAMPLE 2.4.18. *For* $n = 3$, $Z_{4n+1} \setminus \{0\}$ *is partitioned into 3 subsets of cardinality 4 as* $(1, 9, 12, 4)(2, 8, 11, 5)(3, 7, 10, 6)$ *. Using the elements of these subsets as the arc labels of* $3 : \overrightarrow{C_4}$*, we get a graceful labeling of* $3 : \overrightarrow{C_4}$*, as shown in Figure 2.8 .*

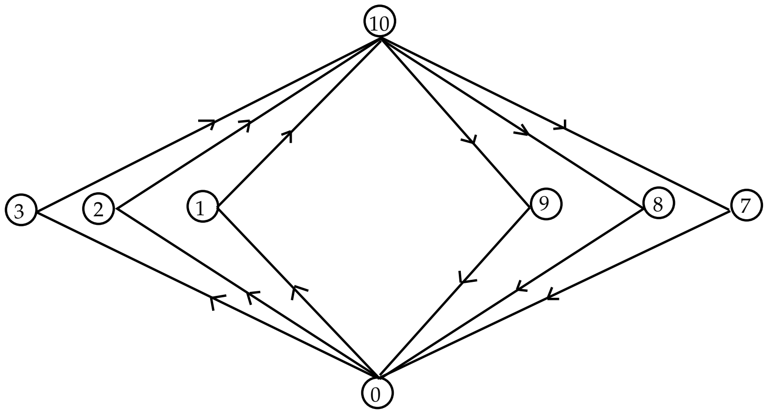

Figure 2.8: A graceful directed graph

Chapter 3

CLASSES OF GRACEFUL DIGRAPHS

In this chapter, we prove the gracefulness of some class of digraphs using the special cases of subset sum problems.

3.1 Introduction

We find a sequence of subsets of Z_n for any $n > 2$ such that there exists at least one common element between the consecutive subsets, with same loads, i.e., sum of the elements in each subsets are same. These cases are considered to be the special cases of subset sum problems (Wu and Yao, 2007). Using the above sequence of subsets, we prove the gracefulness of the digraph $\overrightarrow{P_m \Box P_n}$ which is a particular type of an orientation of a planar grid graph $P_m \Box P_n$ and some more class of digraphs.

3.2 Gracefulness of an oriented grid graph $P_m \Box P_n$

Grid graph is a graph obtained by the cartesian product of two paths P_m and P_n.
Acharya and Gill (Acharya and Gill, 1981) have investigated the graceful labeling for the grid graph $P_m \Box P_n$.

NOTE 3.2.1. *Since the graph $P_m \Box P_n$ is graceful one can see that there* *exists at least one orientation of $P_m \Box P_n$ which is graceful.*

DEFINITION 3.2.2. *We define $\overrightarrow{P_m \Box P_n}$ as an orientation of a planar grid $P_m \Box P_n$ in which each cell is a unicycle of length four (as given in Figure 3.1).*

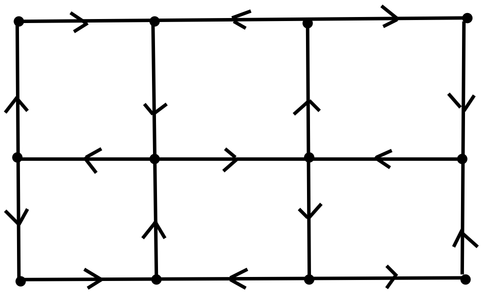

Figure 3.1: An orientation of the planar grid graph $P_4 \Box P_3$

Using Lemma 1.3.28 and the structure of the digraph $\overrightarrow{P_m \Box P_n}$ we prove the following Lemma 3.2.3, which is used in proving the gracefulness of $\overrightarrow{P_m \Box P_n}$.

LEMMA 3.2.3. *Let $S = \{1, 2, 3, ..., 2mn - m - n\}$ where $m, n > 1$. Then there exists $(m-1)(n-1)$ subset S_{ij} $(1 \leq i \leq n-1,\ 1 \leq j \leq m-1)$ of S, each of cardinality 4 such that for $n > 1$ and $m > 1$*

1. $| S_{ij} \cap S_{i(j+1)} |= 1$, $(1 \leq i \leq n-1,\ 1 \leq j \leq m-2)$,
2. $| S_{ij} \cap S_{(i+1)j} |= 1$, $(\ 1 \leq i \leq n-2,\ 1 \leq j \leq m-1)$

where sum of the elements in each subset S_{ij} $(1 \leq i \leq n-1,\ 1 \leq j \leq m-1)$, $\equiv 0\ mod(2mn - m - n + 1)$.

Proof. Let $S = \{1, 2, 3, ..., 2mn - m - n\}$ where $m, n > 1$.
Suppose $S_{ij} = \{h_{(i-1)j}, h_{ij}, v_{i(j-1)}, v_{ij}\}$ $(1 \leq i \leq n-1,\ 1 \leq j \leq m-1)$ are the $(m-1)(n-1)$ subsets of S each of cardinality 4, where $h_{rs} = (2m-1)r + s$, $(0 \leq r \leq n-1,\ 1 \leq s \leq m-1)$ and $v_{kl} = (2m-1)(n-k) - l$, $(1 \leq k \leq n-1,\ 0 \leq l \leq m-1)$, then we show that the subset S_{ij} $(1 \leq i \leq n-1,$

$1 \leq j \leq m-1$) satisfies the conditions of the given lemma.
If $n > 1$ and $m > 1$, then

1. for each i, $(1 \leq i \leq n-1)$ we have $S_{ij} \cap S_{i(j+1)} = \{h_{(i-1)j}, h_{ij}, v_{i(j-1)}, v_{ij}\} \cap \{h_{(i-1)(j+1)}, h_{i(j+1)}, v_{ij}, v_{i(j+1)}\} = \{v_{ij}\} \in S$, where j varies from 1 to $m-2$.

2. for each j, $(1 \leq j \leq m-1)$ we have $S_{ij} \cap S_{(i+1)j} = \{h_{(i-1)j}, h_{ij}, v_{i(j-1)}, v_{ij}\} \cap \{h_{ij}, h_{(i+1)j}, v_{(i+1)(j-1)}, v_{(i+1)j}\} = \{h_{ij}\} \in S$, where i varies from 1 to $n-2$.

Hence the subsets S_{ij} $(1 \leq i \leq n-1,\ 1 \leq j \leq m-1)$ satisfies the two conditions of the given lemma.
Next we show that the sum of the elements in each of the subset $S_{ij} \equiv 0\ mod(2mn - m - n + 1)$.
For any i, j $(1 \leq i \leq n-1,\ 1 \leq j \leq m-1)$ sum of the elements in S_{ij}
$=(h_{(i-1)j} + h_{ij} + v_{i(j-1)} + v_{ij})(1 \leq i \leq n-1,\ 1 \leq j \leq m-1)$
$=(2mi - 2m - i + 1 + j) + (2mi - i + j) + (2mn - 2mi - n + i - j + 1) + (2mn - 2mi - n + i - j)$
$=4mn - 2m - 2n + 2$
$=2(2mn - m - n + 1)$
$\equiv 0\ mod(2mn - m - n + 1)$.
Hence there exists $(m-1)(n-1)$ subsets of S which satisfies the conditions of the lemma. $\square$

THEOREM 3.2.4. *For any $m > 1$ and $n > 1$, the digraph $\overrightarrow{P_m \square P_n}$ is graceful.*

Proof. Let a_{ij} denote the mn vertices of the digraph $\overrightarrow{P_m \square P_n}$ $(0 \leq i \leq n-1,\ 0 \leq j \leq m-1)$.
Let h_{rs} and v_{kl} be the horizontal and vertical arcs of $\overrightarrow{P_m \square P_n}$ respectively $(0 \leq r \leq n-1,\ 1 \leq s \leq m-1)$ $(1 \leq k \leq n-1,\ 0 \leq l \leq m-1)$ where $q = 2mn - m - n$.
Using Lemma 3.2.3, we get the required labels on the arcs of the digraph $\overrightarrow{P_m \square P_n}$.

To generate the labels on the vertices, let $a_{00} = 2mn - m - n$, then for each i, $(i = 0, 1, 2, 3, ..., n-1)$, we get a recurrence relation $a_{ij} = a_{i(j-1)} \pm h_{ij}$ where j varies from $1, 2, 3, 4, ..., m-1$. (for all even(odd) i, the above relation starts with plus(minus) sign and alternatively varies with j). Also $a_{i0} = a_{(i-1)0} \pm v_{i0}$ where i varies from $1, 2, 3, 4, ..., n-1$ i.e., same as $a_{00} + h_{01} = a_{01}$, $a_{01} - h_{02} = a_{02}$, ...etc. (for all odd(even) i, the above relation starts with minus(plus) sign).

One can see that, vertex labels as given above gives rise to a graceful labeling of the digraph $\overrightarrow{P_m \square P_n}$. i.e., the mn vertex values are all distinct, this is because a mapping from $V(\overrightarrow{P_m \square P_n})$ into $[0, 2mn - m - n]$ is an injective mapping where $a_{ij} \neq a_{ks}$ for any (i, j, k, s) such that $i \neq k$ and $j \neq s$, $0 \leq i, k \leq n-1$, $0 \leq j, s \leq m-1$. Otherwise, the above recurrence relation indicates that the h_{ij} and v_{i0} values are all same which is not so from Lemma 3.2.3. Also, from Lemma 3.2.3 we have $h_{rs} \cup v_{kl} = \{1, 2, ..., 2mn - m - n\}$ which comprise $Z_{2mn-m-n+1} \setminus \{0\}$. Hence the digraph $\overrightarrow{P_m \square P_n}$ is graceful for any $m > 1$ and $n > 1$. $\square$

3.2.1 Illustration with an example

Consider the digraph $\overrightarrow{P_6 \square P_4}$. Here $m = 6$, $n = 4$ and $q = 38$.

- From Lemma 3.2.3, we get $S = \{1, 2, ..., 38\}$, $(m-1)(n-1) = 15$, $S_{i,j} = \{h_{(i-1)j}, h_{ij}, v_{i(j-1)}, v_{ij}\}$ $(1 \leq i \leq 3,\ 1 \leq j \leq 5)$,
- where $h_{01} = 1,\ h_{02} = 2,\ h_{03} = 3,\ h_{04} = 4,\ h_{05} = 5$
  $h_{11} = 12,\ h_{12} = 13,\ h_{13} = 14,\ h_{14} = 15,\ h_{15} = 16$
  $h_{21} = 23,\ h_{22} = 24,\ h_{23} = 25,\ h_{24} = 26,\ h_{25} = 27$
  $h_{31} = 34,\ h_{32} = 35,\ h_{33} = 36,\ h_{34} = 37,\ h_{35} = 38$,
- $v_{10} = 33,\ v_{11} = 32,\ v_{12} = 31,\ v_{13} = 30,\ v_{14} = 29,\ v_{15} = 28$
  $v_{20} = 22,\ v_{21} = 21,\ v_{22} = 20,\ v_{23} = 19,\ v_{24} = 18,\ v_{25} = 17$
  $v_{30} = 11,\ v_{31} = 10,\ v_{32} = 9,\ v_{33} = 8,\ v_{34} = 7,\ v_{35} = 6$.

Using these h_{ij} and v_{kl} values as the arc labels of the digraph $\overrightarrow{P_6 \square P_4}$, one can see that, the arc labels in each cell represents the elements of the subsets

S_{ij}'s of S. The vertex labels a_{ij}'s of $\overrightarrow{P_6 \Box P_4}$ are obtained using these arc labels, which gives a graceful labeling of $\overrightarrow{P_6 \Box P_4}$ as shown in Figure 3.2.

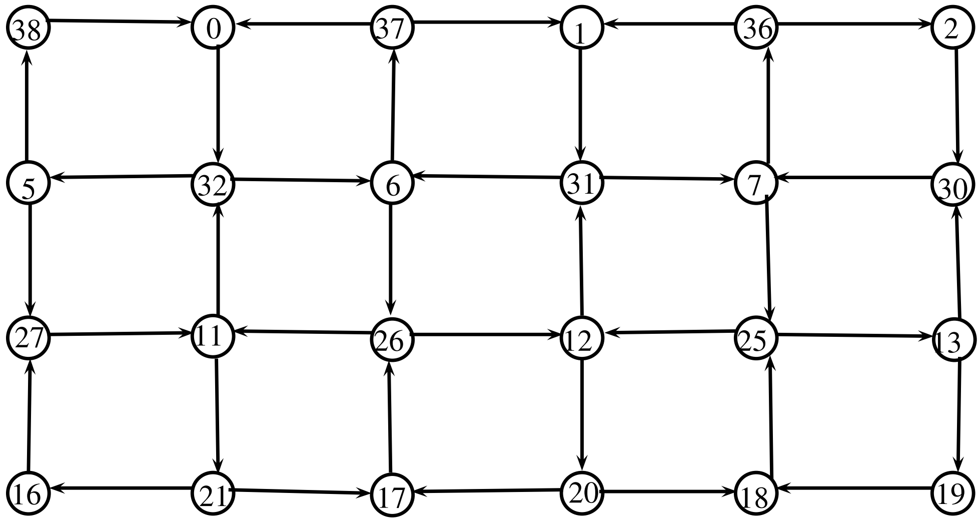

Figure 3.2: A graceful labeling of an oriented $P_6 \Box P_4$

3.3 Gracefulness of an oriented shell graph $C(n, n-3)$

Deb and Limaye (Deb and Limaye, 2002) have defined a shell graph as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex. Shell graphs are denoted as $C(n, n-3)$.
In (Sethuraman and Dhavamani, 2000) it is proved that, for $k \geq 1$ and $n \geq 4$ the generalized shell graph $G(n, n-3, k)$ is graceful. Hence the shell graph $G(n, n-3, 1)$ is graceful.

We define $\vec{C}(n, n-3)$ as an orientation of the shell graph $C(n, n-3)$, such that each shell is a unicycle of length 3, as given in Figure 3.3.

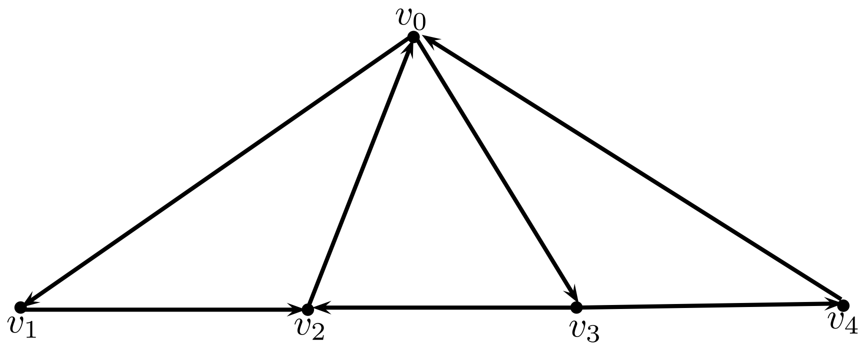

Figure 3.3: A digraph $\vec{C}(5,2)$.

First we prove the Lemma 3.3.1 which helps us to prove the gracefulness of $\vec{C}(n, n-3)$ for any even $n \geq 4$.

LEMMA 3.3.1. *For any even n, the set $S = \{1, 2, 3, ..., 2n-3\}$, can be arranged as a sequence $s_1, s_2, .., s_r$, $(r = n-2)$, of 3 element subsets such that $\mid s_i \cap s_{i+1} \mid = 1$ and sum of the elements in each such subset is $\equiv 0(mod\ 2(n-1))$.*

Proof. Let $s_i = \{2i-1, 2i+1, 2n-2-4i\}$ $(1 \leq i \leq r; r = n-1)$.
Then $s_i \cap s_{i+1} = \{2i+1\}$, also $\sum_{i=k} s_i = 2(n-1) \equiv 0(mod\ 2(n-1)$ $(1 \leq k \leq r)$.
Hence the sequence of subsets $s_1, s_2, ..., s_r$ of S given by $(1, 3, 2n-6)$, $(3, 5, 2n-10)$, ..., $(2r-1, 2r+1, 2n-2-4r)$ with $-x \equiv 2(n-1) - x$ $(mod\ 2(n-1))$ satisfies the condition of the given lemma. Hence the proof. □

THEOREM 3.3.2. *For any even n, the digraph $\vec{C}(n, n-3)$ is graceful.*

Proof. Let D be a digraph $\vec{C}(n, n-3)$ with n vertices and $2n-3$ arcs. We describe the digraph D as follows: In D, start denoting the vertices as $v_1, v_2...., v_n$ from the left of the apex, and apex as v_0. Start denoting the n arcs from the left of the apex as $e_1, e_2, ..., e_n$; e_i $(1 \leq i \leq n)$ and the $n-3$ arcs which are the oriented chords as $e_{n+1}, e_{n+2}, e_{n+3}, ..., e_{2n-3}$.

Using Lemma 3.3.1 the labels on the arcs of $\vec{C}(n, n-3)$ can be obtained as follows: Label the arcs $(e_1, e_{n+1}, e_{n+2}, ...e_{2n-3}, e_n)$ as $(1, 3, 5, ..., 2n-5, 2n-3)$ and $(e_1, e_{n+1}, e_{n+2}, ...e_{2n-3}, e_n)$ with $2n - 2 - 4i$ $\forall i, 1 \leq i \leq r$ respectively. Then one can observe that the arc labels on each of the unicycle of length 3 as a subset obtained in the previous lemma. To generate the labels on the vertices let $v_0 = 0$, then $v_1 = v_0 + e_1$, $v_2 = v_0 - e_{n+1}$, $v_3 = v_0 + e_{n+2}$ etc. w.r.t $(mod(2(n-1))$ are obtained. Hence $v_0 = 0, v_1 = 1, v_2 = 2n-5, ..., v_{n-1} = 2n - 3$ which are all distinct elements of $Z_{2(n-1)} \setminus 0$. Hence the digraph $\vec{C}(n, n-3)$ is graceful for even n. $\square$

EXAMPLE 3.3.3. *Consider* $\vec{C}(8, 5)$*, then from Lemma 3.3.1 the set* $S = \{1, 2, 3, ..., 13\}$ *has 6 subsets* $(1, 3, 10), (3, 5, 6), (5, 7, 2), (7, 9, 12), (9, 11, 8),$ $(11, 13, 4)$ *using which a graceful digraph* $\vec{C}(8, 5)$ *is constructed as given in Figure 3.4.*

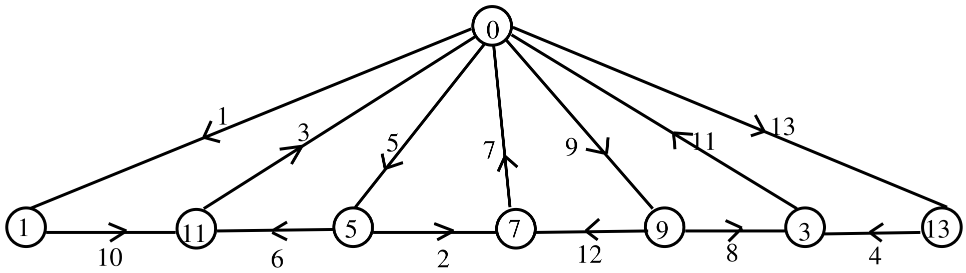

Figure 3.4: A graceful $\vec{C}(8, 5)$

NOTE 3.3.4. *The following example gives the gracefulness of* $\vec{C}(n, n-3)$ *for odd n, with this we strongly believe that:*

Conjecture 3.3.5. *For any odd n, the digraph* $\vec{C}(n, n-3)$ *is graceful.*

EXAMPLE 3.3.6. *The following Figure 3.5 represents the graceful digraphs* $\vec{C}(5, 2)$ *and* $\vec{C}(7, 4)$*.*

OBSERVATION 3.3.7. *The element distribution given in Lemma 3.3.1 can also be used to find the gracefulness of another class of digraph* $\vec{P}_r^2$*.*

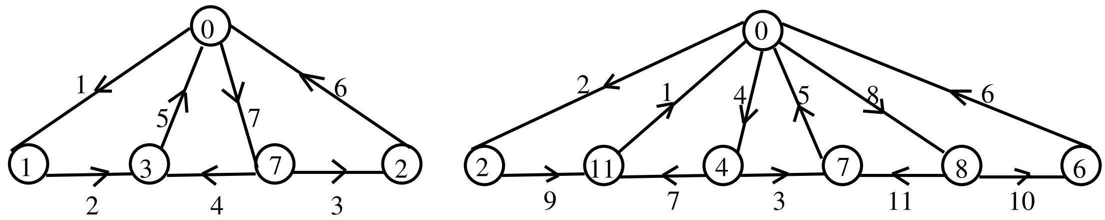

Figure 3.5: A graceful $\vec{C}(5,2)$ and a graceful $\vec{C}(7,4)$

3.4 Gracefulness of an oriented P_r^2

The graph P_r^2 (Wang et al., 2008) is obtained from the path P_r, where every pair of vertices of distance two or less in P_r is connected by an edge.

We define $\vec{P}_r^2$ as an orientation of P_r^2 where each cycle of length 3 is a unicycle.
Let the n vertices of $\vec{P}_r^2$ be denoted as $v_0, v_1, ..., v_{n-2}, v_{n-1}$. Assigning the values $i^2(mod\ (2(n-1))$ for all i, $(0 \leq i \leq n-1)$ to the vertices v_i, one can see the arc labels as the elements of the subsets of S as given above. This yields a graceful $\vec{P}_r^2$ when r is even.

EXAMPLE 3.4.1. *Figure 3.6 is an illustrative example of a graceful $\vec{P}_6^2$.*

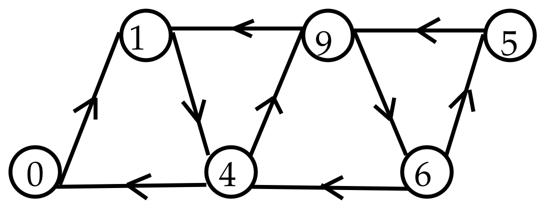

Figure 3.6: A graceful $\vec{P}_6^2$

REMARK 3.4.2. *If for some $i \neq j$ $(0 \leq i, j \leq n-1)$, $i^2 = j^2$ w.r.t $mod(2(n-1))$, then the above element distribution does not give a graceful* *labeling of $\vec{P}_r^2$, for even r.*

For example, if we consider $\vec{P}_{10}^2$, then $i^2 = j^2$ w.r.t. $(mod\ (18))$ for $i = 0$ and for $j = 6$. Hence the digraph $\vec{P}_{10}^2$ is not graceful with the above method.

The above three sections only deals with some particular cases of element distribution given for a set. But, it is also possible to generalize this concept of element distribution for any n.
We strongly believe that, the following result is true. If such partition exists, then it is possible to construct a graceful directed graphs from it.

Conjecture 3.4.3. *Suppose that, $(k-1)|(n-2)$ for any n and $k > 2$. Then the nonzero residues $(mod\ n)$ can be partitioned into i subset S_i $(1 \leq i \leq \frac{n-2}{k-1})$ each of cardinality k, such that $\mid s_i \cap s_{i+1} \mid = 1$ $(1 \leq i \leq \frac{n-2}{k-1} - 1)$ and sum of the elements in each such subset is $\equiv 0(mod\ n)$.*

EXAMPLE 3.4.4. *Suppose $n = 18$ and $k = 5$, then $4 \mid 16$. Then the following sequence of subsets of Z_{18} satisfy the conditions of Conjecture 3.4.3,* (5 4 6 13 8) (5 2 7 3 1) (7 12 11 15 9) (15 14 10 16 17).

NOTE 3.4.5. *The above conjecture follows in the below cases:*
(i) $k = \frac{n}{2}$ or $k = \frac{n-1}{2}$ (Hegde et al., 2013).
(ii)$k = 4$ (Lemma 3.2.3).

We define $\vec{C}_k^r$ as the directed graph obtained from r copies of $\vec{C}_k$, in which any two consecutive directed cycles have a common arc.

The following illustration gives the relation between Conjecture 3.4.3 and the gracefulness of digraph $\vec{C}_k^r$.
Using Example 3.4.4 we construct a graceful directed graph $\vec{C}_5^4$ which consists 4 copies of $\vec{C}_5$ and have a common arc between any two consecutive directed cycles, as shown in Figure 3.7

EXAMPLE 3.4.6. *Suppose $n = 17$, $k = 6$, then a sequence of subsets* (1 3 4 6 8 12) (2 5 8 13 14 9) (7 10 11 15 16 9) *of $Z_{17}/\{0\}$ gives a graceful directed graph $\vec{C}_6^3$ which consists 3 copies of $\vec{C}_6$ and have a common arc between any two consecutive directed cycles.*

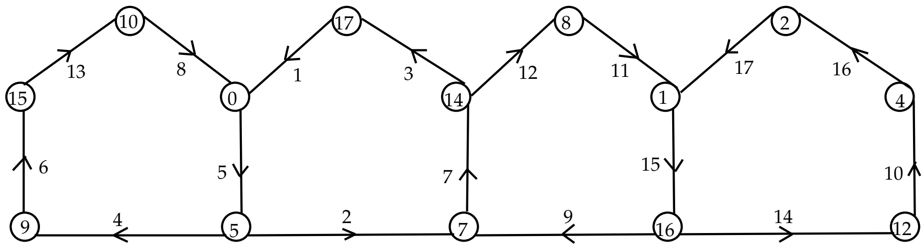

Figure 3.7: A graceful directed graph $\vec{C}_5^4$

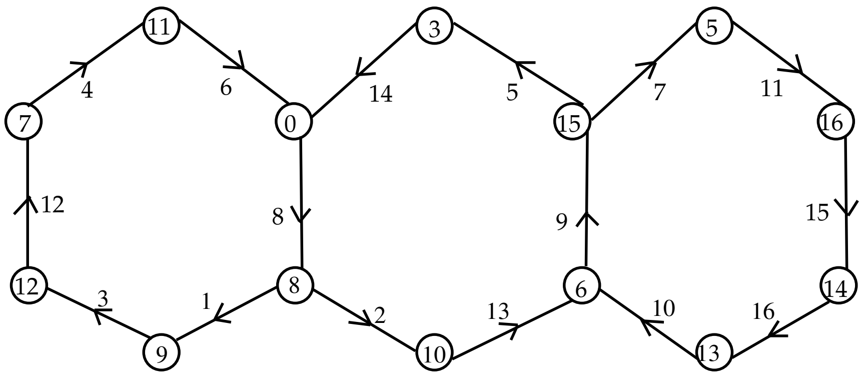

Figure 3.8: A graceful digraph $\vec{C}_6^3$

Also, from the same sequence of subsets but, with different arrangement of elements (within the subsets) i.e., (1 3 4 6 8 12) (8 2 13 9 14 5) (9 7 11 15 16 10) one can get another graceful labeling for the same directed graph $\vec{C}_6^3$.

NOTE 3.4.7. *It is noticed that from each such sequence of subsets of Z_n one can construct graceful directed graph $\vec{C}_k^r$. What matters is the arrangement of these elements which gives the arc labels for the constructed graceful directed graph $\vec{C}_k^r$.*

Hence we strongly believe that:

Conjecture 3.4.8. *For any $k \geq 4$ and $r \geq 2$, the digraph $\vec{C}_k^r$ is graceful.*

Chapter 4

RELATION BETWEEN BIGRAPHS AND DIGRAPHS IN TERMS OF LABELINGS

In this chapter we give a construction of digraph and bigraph, which is the modified form of one-to-one correspondence given between bigraphs and digraphs by Dulmage and Mendelsohn in (Brualdi et al., 1980). Using this construction we show that some class of bigraphs and digraphs are graceful.

4.1 Introduction

Dulmage and Mendelsohn (Brualdi et al., 1980) while working on matrix reducibility observed that, there is a one-to-one correspondence between bigraphs and digraphs using the adjacency matrices. In this chapter we obtain a relation between bigraphs and digraphs in terms of labelings. It is known that a graceful graph always gives rise to a graceful digraph. Using the construction of bigraphs and digraphs, we show that the gracefulness of some classes of digraphs also gives rise to the gracefulness of its associated bigraph and vice-versa. i.e., the directed graph $D(A)$ associated with A (where A is a $n \times n$ binary matrix) is graceful iff the bigraph $G(A)$ associated with A is graceful.

4.2 Construction of digraphs and bigraphs

Let D be a digraph with n vertices $(v_1, v_2, ..., v_n)$ and e arcs, no self loops and no more than one arc directed from any one vertex x to any other vertex y. Using D we construct a bigraph G having $2n$ vertices say, $(t_1, t_2, ..., t_n; s_1, s_2, ..., s_n)$. The edge set of G is obtained by joining t_i with s_i and t_i is joined with s_j iff $v_i v_j$ is an arc in D, for i, j $(1 \leq i, j \leq n)$.

The above relation given between bigraphs and digraphs can also be determined by the utilization of the adjacency matrix of digraphs(or bigraphs).

DEFINITION 4.2.1. *(Harary, 1969) Let D be a digraph with n vertices. The adjacency matrix of D, $A(D)$ is a $n \times n$ binary matrix (a_{ij}) with*

$$a_{ij} = \begin{cases} 1 & if \;\; v_i v_j \text{ is an arc of } D, \\ 0 & otherwise. \end{cases}$$

Construction of digraph $D(A)$ using A

Let $A = (a_{ij})$ be a $n \times n$ binary matrix with all diagonal entries zeros. Then the digraph $D(A)$ is constructed as follows:

Let $v_1, v_2, ..., v_n$ be the n vertices of $D(A)$ and any two vertices v_i is adjacent to v_j in $D(A)$ [there is an arc (v_i, v_j) from v_i to v_j] iff $a_{ij} = 1$ $(1 \leq i, j \leq n)$. Thus one can see that A represents the adjacency matrix of a loop less digraph $D(A)$.

Construction of Bigraph $B(A)$ using A and $D(A)$

As we give a relation between bigraphs and digraphs, we also give a relation between A and $G(A)$. Let $(t_1, t_2, ..., t_n; s_1, s_2, ..., s_n)$ be the vertices of $G(A)$. Then t_i and s_j are adjacent (there is an edge joining t_i with s_j) iff for $i \neq j$, $a_{ij} = 1$ $(1 \leq i, j \leq n)$; t_i and s_i are adjacent (there is an edge joining t_i with s_i) iff for $i = j$, $a_{ij} = 0$ $(1 \leq i, j \leq n)$.
Hence one can observe that, the adjacency matrix of $G(A)$ is the partitioned matrix $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$ where $B = A + I$, I is an identity matrix of order $n \times n$.

EXAMPLE 4.2.2. *Let* $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$.

Then the directed graph $D(A)$ and the bigraph $G(A)$ associated with this binary matrix A are given in Figure 4.1.

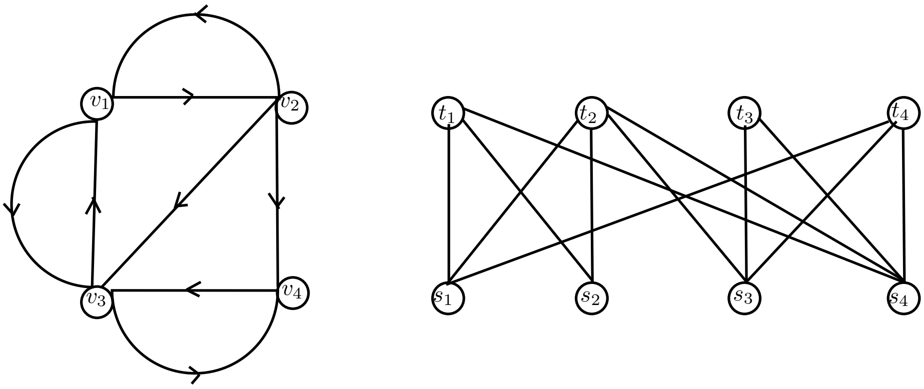

Figure 4.1: The directed graph $D(A)$ and the bigraph $G(A)$ associated with A.

OBSERVATION 4.2.3. *One can observe that, by permuting the rows or columns of A we get different digraphs $D(A)$. Hence the digraph $D(PAQ)$ and $D(A)$ are not isomorphic for some permutation matrices $P_{n\times n}$ and $Q_{n\times n}$.*

Using the above construction of digraphs and bigraphs, we show how the gracefulness of some classes of digraphs gives rise to graceful bigraphs and vice-versa.

4.3 Relation between bigraphs and digraphs in terms of graceful labeling

DEFINITION 4.3.1. *(Biggs, 1974) A $n\times n$ matrix A is said to be a circulant matrix if row i of A is obtained from the first row of A by a cyclic shift of $i-1$ steps, and so any circulant matrix is determined by its first row.*

The following theorem gives a relation between THEOREM 2.2.3 and THEOREM 1.3.17.

THEOREM 4.3.2. *Let A be a $n \times n$ (n even) circulant matrix with first row $\begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \end{pmatrix}$, then the directed graph $D(A)$ is graceful if and only if the bigraph $G(A)$ is graceful.*

Proof. Suppose that, A is a $n \times n$ circulant matrix with first row $\begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \end{pmatrix}$, then its digraph $D(A)$ represents a directed cycle $\vec{C_n}$ (n even) with vertices $v_1, v_2, ..., v_n$.
This digraph $D(A)$ can be gracefully labeled as follows:

Case 1: Let $n \equiv 0(mod\ 4)$.

Define a function $g : V(D(A)) \longrightarrow Z_{n+1}$ as follows:

$$g(v_i)=\begin{cases} \dfrac{i-1}{2} & if \quad i = 1, 3, 5, ..., n-1, \\ n+1-\dfrac{i}{2} & if \quad i = 2, 4, 6, ..., \dfrac{n}{2}, \\ n-\dfrac{i}{2} & if \quad \dfrac{n}{2}+2, \dfrac{n}{2}+4, ..., n. \end{cases}$$

Case 2: Let $n \equiv 2(mod\ 4)$.

Define a function $g : V(D(A)) \longrightarrow Z_{n+1}$ as follows:

$$g(v_i)=\begin{cases} \dfrac{i-1}{2} & if \quad i=1,3,5,...,\dfrac{n}{2}, \\ \dfrac{i-1}{2}+1 & if \quad i=\dfrac{n}{2}+2,\dfrac{n}{2}+4,...,n-1, \\ n+1-\dfrac{i}{2} & if \quad i=2,4,6,...,\dfrac{n}{2}-1,, \\ n-\dfrac{i}{2}+1 & if \quad i=\dfrac{n}{2}+1,\dfrac{n}{2}+3,...,n. \end{cases}$$

The vertex labels $g(v_i) = g(v_j) \implies i = j \ \ (1 \leq i, j \leq n)$.
Thus the vertex function g defined by the above two cases, induces a bijective edge function $g^* : E(D(A)) \longrightarrow \{1, 2, ..., n\}$. Hence g is graceful labeling of the directed graph $D(A)$ in both the cases.
Using the labeling of digraph $D(A)$ the bigraph $G(A)$ associated with A can be gracefully labeled as follows.
Let the $2n$ vertices of $G(A)$, $(t_1, t_2, ..., t_{n-1}, t_n; s_1, s_2, s_3, ..., s_{n-1}, s_n)$ are denoted as $(v_1, v_3, ..., v_{2n-3}, v_{2n-1}; v_{2n}, v_2, v_4, v_6, ..., v_{2n-4}, v_{2n-2})$ respectively.
Then for $n \equiv 0, 2(mod\, 4)$, define a function $g : V(G(A)) \longrightarrow Z_{2n+1}$ as follows:

$$g(v_i)=\begin{cases} \dfrac{i-1}{2} & if \quad i=1,3,5,...,2n-1, \\ 2n+1-\dfrac{i}{2} & if \ i=2,4,6,\ldots,n, \\ 2n-\dfrac{i}{2} & if \ i=n+2,n+4,...,2n. \end{cases} \quad . \qquad (4.3.1)$$

With the vertices labeled in this manner, the edges attain the values $(1, 2, 3, ...\ n, n+1, ..., 2n-1, 2n)$. Thus, gives a graceful labeling of the bigraph $G(A)$.

Conversely, for the given circulant matrix A, its associated bigraph $G(A)$ is a cycle C_{2n} which can be gracefully labeled for any even n using expression 4.3.1. Using this labeling function of C_{2n}, the digraph $D(A)$ obtained from A is also gracefully labeled. For $n \equiv 0(mod\, 4)$ replace $2n$ by n and n by $n/2$ in expression 4.3.1. Hence we get case 1 as the graceful labeling function for $D(A)$. Similarly for $n \equiv 2(mod\, 4)$ replace $2n$ by n and n by $n/2$ in expression 4.3.1 and split the interval for odd and even values as given in case 2, which gives a graceful labeling function for $D(A)$.

□

EXAMPLE 4.3.3. *Let* $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. *Then the graceful directed graph* $D(A)$ *and the graceful bigraph* $G(A)$ *associated with this binary matrix* A *are given in the following Figure 4.2.*

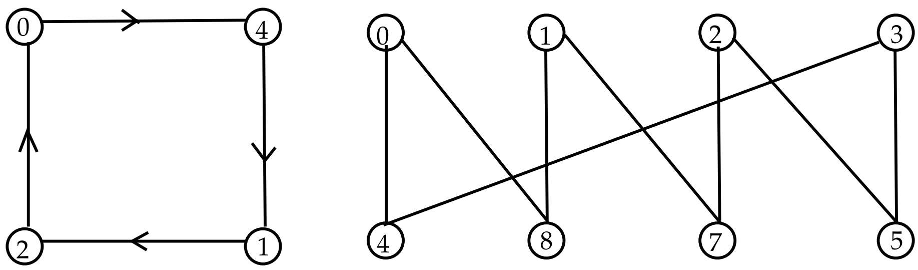

Figure 4.2: A graceful $\vec{C_4}$ and C_8 associated with given A

NOTE 4.3.4. *Suppose* A *is a* $n \times n$ *binary matrix as given in Theorem 4.3.2 but with* n *odd, then both the digraph* $D(A)$ *and its associated bigraph* $G(A)$ *are not graceful.*

As a consequences of Theorem 4.3.2 we have the following corollary.

COROLLARY 4.3.5. *Let* $A = \begin{pmatrix} A_{1(r\times r)} & 0_{r\times s} \\ 0_{s\times r} & A_{2(s\times s)} \end{pmatrix}$ *be a partitioned matrix with* $r+s$ *even, where* 0 *is the zero matrix,* A_1 *is a* $r \times r$ *and* A_2 *is a* $s \times s$ *circulant matrix each with first row* $\begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \end{pmatrix}$. *Then* $D(A)$ *is graceful iff* $B(A)$ *is graceful.*

NOTE 4.3.6. *Since the union of two (or more) unicycles to be graceful the total number of arcs in the digraph must be even. Also the disjoint cycles* $C_p \cup C_q$ *is graceful iff* $p + q \equiv 0, 3(mod4)$. *Thus the condition* $r + s$ *is even implies that* $2(r + s)$ *is always* $\equiv\ 0(mod\ 4)$ *hence the corollary is true.*

Theorem 4.3.7 gives a relation between the directed path $\vec{P_n}$ and the path P_n in terms of graceful labelings.

THEOREM 4.3.7. *If $A = (a_{ij})$ is a $n \times n$ (n even) binary matrix with*
$$a_{ij} = \begin{cases} 1 & for \quad j = i+1,\ 1 \leq i \leq n-1, \\ 0 & otherwise, \end{cases}$$

then the digraph $D(A)$ is graceful iff the bigraph $G(A)$ is graceful.

Proof. Let $A = (a_{ij})$ be a $n \times n$ (n even) binary matrix as given above. Then the digraph $D(A)$ associated with matrix A is a unidirectional path $\vec{P_n}$ on n vertices $v_1, v_2, ..., v_n$. This digraph $D(A)$ is gracefully labeled by defining a function $g : V(D(A)) \longrightarrow Z_{n+1}$ as follows:

$$g(v_i) = \begin{cases} i - j & for \quad i = (1, 3, 5, ..., n-1),\ j = (1, 2, 3, .., \frac{n}{2})\ resp., \\ n - (i - j) & for\ i = (2, 4, 6, ..., n),\ j = (1, 2, 3, .., \frac{n}{2})\ resp. \end{cases} \tag{4.3.2}$$

The vertex labels $g(v_i) = g(v_j) \Longrightarrow i = j \quad (1 \leq i, j \leq n)$.
Thus the vertex function g defined above induces a bijective edge function $g^* : E(D(A)) \longrightarrow \{1, 2, ..., n\}$. Hence g is graceful labeling of the directed graph $D(A)$.
Using the above graceful labeling of the digraph $D(A)$, the bigraph $G(A)$ with $2n$ vertices (n even) is gracefully labeled as given below.
Suppose the $2n$ vertices of $G(A)$, $(t_1, t_2, ..., t_{n-1}, t_n; s_1, s_2, s_3, ..., s_{n-1}, s_n)$ are denoted as $(v_2, v_4, ..., v_{2n-2}, v_{2n}; v_1, v_3, v_5, ..., v_{2n-3}, v_{2n-1})$ respectively. Then define a function $g : V(G(A)) \longrightarrow \{0, 1, ..., 2n\}$ as follows:

$$g(v_i) = \begin{cases} i - j & for \quad i = (1, 3, 5, ..., 2n-1),\ j = (1, 2, 3, .., n)\ resp., \\ 2n - (i - j) & for\ i = (2, 4, 6, ..., 2n),\ j = (1, 2, 3, .., n)\ resp. \end{cases} . \tag{4.3.3}$$

With the vertices labeled in this manner, the edges attain the values $\{1, 2, 3, ..., n, n+1, ..., 2n-2, 2n-1\}$. Thus, gives a graceful labeling of the bigraph $G(A)$.

Conversely, for the given binary matrix A its associated bigraph $G(A)$ represents the path P_{2n} on $2n$ vertices. This can be gracefully labeled by assigning the values for the vertices v_i as given in expression 4.3.3. Using this labeling function of $B(A)$, the digraph $D(A)$ associated with A is gracefully labeled. Replace $2n$ by n and n by $n/2$ in expression 4.3.3 which gives a graceful labeling for the digraph $D(A)$. Hence the proof. $\square$

EXAMPLE 4.3.8. *Let* $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Then the graceful directed graph $D(A)$ and the graceful bigraph $G(A)$ associated with this binary matrix A are given in the following Figure 4.3.

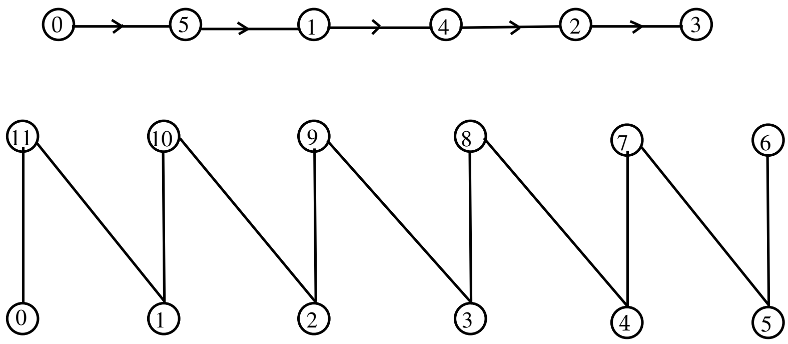

Figure 4.3: A graceful $\vec{P_6}$ and a graceful P_{12} associated with given A

The following corollary is a consequences of Theorem 4.3.7.

COROLLARY 4.3.9. *Let* $A = \begin{pmatrix} A_{1(s\times s)} & 0_{s\times n} \\ 0_{n\times s} & A_{2(n\times n)} \end{pmatrix}$ *be a partitioned matrix with $s+n = l$, l is even, $l \geq 6$, where 0 is the zero matrix, A_1 is a $s \times s$ circulant matrix with first row* $\begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \end{pmatrix}$ *and $A_2 = (a_{2(ij)})$ is a $n \times n$ binary matrix with*
$(a_{2(ij)}) = \begin{cases} 1 & for \;\; j = i+1, \; 1 \leq i \leq n-1, \\ 0 & otherwise \end{cases}$. *Then $D(A)$ is graceful iff $B(A)$ is graceful.*

NOTE 4.3.10. *Since the directed graph $D(A)$ associated with given A represents disjoint union of an unicycle and an unidirectional path i.e., $\vec{C}_s \cup \vec{P}_n$. Hence it has $s+n-1$ arcs and since $(s+n)$ is even(≥ 6) this directed graph is graceful. Similarly the bigraph $G(A)$ associated with given A is a graph $C_{2s} \cup P_{2n}$ with $2(s+n)$ vertices. Again the condition $(s+n)$ is even(≥ 6), implies that this graph is graceful.*

The following theorem gives a relation between the complete symmetric digraph and the complete bipartite graph in terms of labelings.

THEOREM 4.3.11. *If $A = (a_{ij})$ is a $n \times n$ ($n-1$ is a prime power) binary matrix with* $(a_{ij}) = \begin{cases} 1 & for \ \ i \neq j, \ 1 \leq i, j \leq n, \\ 0 & for \ \ i = j, \ 1 \leq i, j \leq n. \end{cases}$

Then the directed graph $D(A)$ is graceful if and only if the bigraph $G(A)$ is graceful.

Proof. If $A = (a_{ij})$ is a $n \times n$ $(n-1$ is a prime power) binary matrix as given above, then the directed graph $D(A)$ associated with A represents a complete symmetric digraph $\vec{K}_n$. Since $n-1$ is a prime power, from Proposition 1.3.8 the digraph $D(A)$ is graceful. The gracefulness of $\vec{K}_n$ also implies that there exists a cyclic (v, k, λ) difference set $D = \{d_1, d_2, ..., d_k\}$ of Z_v with $v = n^2 - n + 1, \ k = n, \ and \ \lambda = 1$.

One can see that the bigraph $G(A)$ constructed from A has n^2 edges. Consider the two subsets $\{0, 1, 2, ..., n-1\}$ and $\{n, 2n, 3n, ..., (n-1)n, n^2\}$ of Z_{n^2+1}. Then one can also see that, the set of all the absolute differences obtained from the elements of the first subset with all the elements of the second subset equals $Z_{n^2+1} \setminus \{0\}$. Using the elements of these two subsets of Z_{n^2+1} as the $2n$ vertex labels of $G(A)$ we obtain a graceful labeling of $G(A)$.

Conversely, for the given binary matrix $A = (a_{ij})$, its associated bigraph $G(A)$ is a complete bipartite graph $K_{n,n}$, which can be gracefully labeled by considering the two subsets of Z_{n^2+1} as given above.
Clearly it is observed that the digraph $D(A)$ constructed from A has n^2-n+1

arcs and n vertices. Since $n-1$ is a prime power, the well known Singer theorem asserts that, there exists a cyclic $(v, k = n, 1)$-difference set. Hence using the elements of this difference set as the n vertex labels of the digraph $D(A)$, gives a graceful digraph $D(A)$. Hence the proof. $\square$

EXAMPLE 4.3.12. *Let* $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

Then the graceful directed graph $D(A)$ and the graceful bigraph $G(A)$ associated with this binary matrix A are given in the following Figure 4.4.

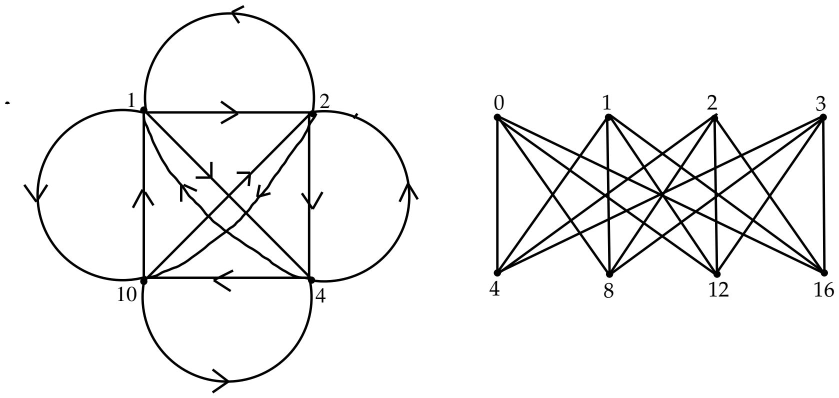

Figure 4.4: A graceful complete symmetric digraph $\vec{K_4}$ and graceful $K_{4,4}$ associated with A

The next theorem gives a relation between directed star $\vec{K_{1,n}}$ and a spider.

DEFINITION 4.3.13. *(Gallian, 2014) A spider is a tree that has at most one vertex (called the center) of degree greater than 2. A leg of a spider tree is any one of the paths from the central vertex to a leaf of the tree.*

THEOREM 4.3.14. *If $A = (a_{ij})$ is a $n \times n$ binary matrix with exactly one row as* $\begin{pmatrix} 0 & 1 & 1 & . & . & . & 1 \end{pmatrix}$ *and all the other $n-1$ rows with zero entries, then the directed graph $D(A)$ is graceful if and only if the bigraph $G(A)$ is graceful.*

Proof. Let $A = (a_{ij})$ be a $n \times n$ binary matrix with exactly one row, say first row as $\begin{pmatrix} 0 & 1 & 1 & . & . & . & 1 \end{pmatrix}$ and all the other $n-1$ rows with zero entries. Then the digraph $D(A)$ associated with A represents a directed star $\vec{K_{1,n}}$. This directed star can be gracefully labeled by assigning 0 to the central vertex v_0 and the other n vetrices $v_{11}, v_{21}, ..., v_{n1}$ which are 1 distance apart from central vertex v_0 as $1, 2, ..., n$ respectively.
In the same manner we try to label the bigraph $G(A)$.
It is clear from the construction that the bigraph $G(A)$ associated with A has $2(n+1)$ vertices.
Let us denote the $2(n+1)$ vertices $(t_1, t_2, ..., t_{n-1}, t_n; s_1, s_2, s_3, ..., s_{n-1}, s_n)$ of $G(A)$ as $(v_0, v_{12}, v_{22}, v_{32}..., v_{(n-1)2}, v_{n2}; v_{(n+1)1}, v_{11}, v_{21}, v_{31}, ..., v_{(n-1)1}, v_{n1})$ respectively, where the vertices v_{i1} and v_{i2} $(1 \leq i \leq n)$ are respectively 1 distance, 2 distance apart from v_0. To prove the gracefulness of the bigraph $G(A)$ we consider the following 2 possible cases.
Case 1: When $n+1$ is odd.
Label v_0 as 1 and the other $2n+1$ vertices respectively be labeled as

$$v_{i1} = \begin{cases} 2n+2,\ 2n+1,\ 2n,\ ...,(2n+2) - \left\lfloor \dfrac{n+1}{2} \right\rfloor \\ \qquad\qquad\qquad for \quad i = 1, 3, 5, ..., n-1, \\ 2,\ 3,\ 4,\ ...,(2n+1) - 3\left\lfloor \dfrac{n+1}{2} \right\rfloor \\ \qquad\qquad\qquad for \quad i = 2, 4, 6, ..., n \end{cases}$$

and

$$v_{i2} = \begin{cases} (2n+2) - 3\left\lfloor \dfrac{n+1}{2} \right\rfloor,\ (2n+3) - 3\left\lfloor \dfrac{n+1}{2} \right\rfloor,\ ....,n+1 \quad for \quad i = 1, 3, 5, ..., n-1, \\ (2n+1) - \left\lfloor \dfrac{n+1}{2} \right\rfloor,\ (2n) - \left\lfloor \dfrac{n+1}{2} \right\rfloor,\ ....,n+2 \qquad for \quad i = 2, 4, 6, ..., n \end{cases}$$

which gives a graceful labeling of $G(A)$.
Case 2: When $n+1$ is even.
Label v_0 as 1 and the other $2n+1$ vertices respectively be labeled as

$$v_{i1} = \begin{cases} 2n+1,\ 2n,\ 2n-1,\ ...,(2n+2) - \left\lfloor \dfrac{n+1}{2} \right\rfloor \quad for \quad i = 1, 3, 5, ...n-1, \\ 2,\ 3,\ 4,\ ...,(2n+2) - 3\left\lfloor \dfrac{n+1}{2} \right\rfloor,\ (2n+2) \quad for \quad i = 2, 4, 6, ..., n \end{cases}$$

and

$$v_{i2} = \begin{cases} (2n+3) - 3\left\lfloor \dfrac{n+1}{2} \right\rfloor,\ (2n+4) - 3\left\lfloor \dfrac{n+1}{2} \right\rfloor,\ .\ .\ .\ .\ , n,\ n+1 \quad for \quad i = 1,3,5,...,n \\ (2n+1) - \left\lfloor \dfrac{n+1}{2} \right\rfloor,\ (2n) - \left\lfloor \dfrac{n+1}{2} \right\rfloor,\ .\ .\ .\ .\ , n+2 \quad for \quad i = 2,4,6,...,n \end{cases}$$

which gives a graceful labeling of $G(A)$.

Conversely, for the given binary matrix $A = (a_{ij})$ its bigraph $G(A)$ represents a spider with $n+1$ legs, of which one leg is of length one and the other n legs are of length two. This spider $G(A)$ can be gracefully labeled by assigning the vertices v_{i1} and v_{i2} $(1 \leq i, j \leq n)$, which are 1 distant and 2 distant apart from the central vertex v_0 as given by the above two cases . In the same manner, the digraph $D(A)$ constructed from A is gracefully labeled by assigning v_0 as 0 and all the other n vertices $v_{11}, v_{21}, ..., v_{n1}$ as $1, 2, 3..., n$ respectively. Hence the proof. □

EXAMPLE 4.3.15. *Let* $A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Then the graceful directed graph $D(A)$ and the graceful bigraph $G(A)$ associated with this binary matrix A are given in Figure 4.5.

If D is a digraph, its corresponding reversed digraph $-D$ can be obtained from D by replacing each arc (u, v) by its reversed arc (v, u). It is known that if D is graceful then $-D$ is also graceful with the same vertex labels. If A is the adjacency matrix of a digraph $D(A)$ then, A^T is the adjacency matrix of a digraph $-D(A)$. Hence we observe the following.

OBSERVATION 4.3.16. *If for some $n \times n$ binary matrix $A = (a_{ij})$ the digraph $D(A)$ is graceful iff the bigraph $B(A)$ is graceful, then the same result also holds for binary matrix A^T, where $A^T = (a_{ji})$.*

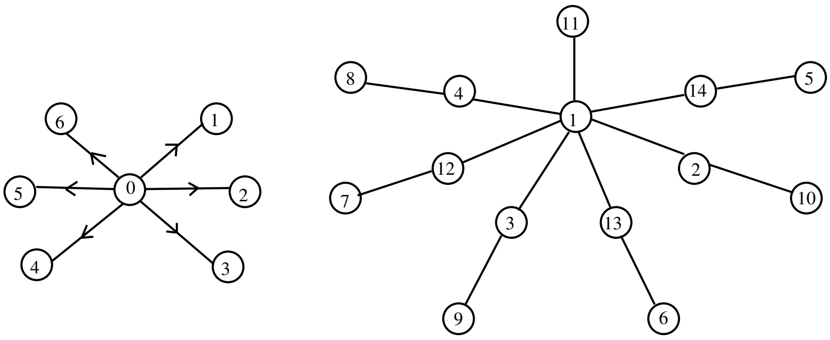

Figure 4.5: A graceful $\vec{K_{1,6}}$ and a graceful spider associated with A

Chapter 5

SEQUENTIAL LABELINGS OF DIGRAPHS

In this chapter, we study sequential labeling of digraphs using some known algebraic structures as discussed in case of graceful digraphs. Further, we give a sequential labeling of some clases of digraphs.

5.1 Introduction

In this chapter, we focus on 1-sequential or sequential labeling of digraphs.

The following are some examples of sequential labeling of digraphs.

EXAMPLE 5.1.1. *Figure 5.1 represents an orientation of K_4 which is sequential.*

EXAMPLE 5.1.2. *Figure 5.2 represents a sequential symmetric path on* 4 *vertices.*

5.2 Preliminary observations

The following are the few observations on sequential labeling of digraphs.

- A sequential graph always gives rise to a sequential digraph.

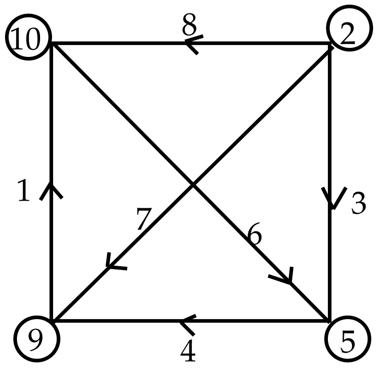

Figure 5.1: A sequential digraph $\vec{K_4}$

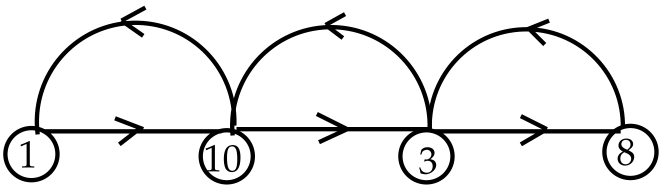

Figure 5.2: A sequential symmetric path

EXAMPLE 5.2.1. *Figure 5.3 represents a sequential W_5 and its trivial orientation which is sequential.*

- An non-sequential graph may underlie a sequential directed one.

EXAMPLE 5.2.2. *It is known that, K_n is 1-sequential iff $n \leq 3$, but we have an orientation of K_4 shown in Figure 5.3 which is sequential.*

- There are digraphs which are not graceful but sequential.

EXAMPLE 5.2.3. *We know that unidirectional path on n odd vertices is not graceful, but $\overrightarrow{P_3}$ is sequential with vertex labels $(1, 5, 2)$.*

Also, we observe the following.

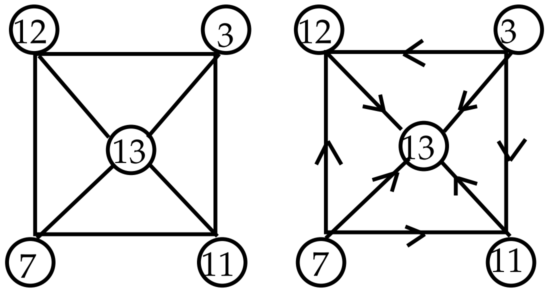

Figure 5.3: A sequential W_5 and a sequential $\vec{W}_5$

- If G is sequentially labelled, then $\overleftrightarrow{G}$ need not be sequential with the same vertex labels.
- If f is a sequential labeling for D, then it is not a sequential labeling for $-D$.

5.3 Relation between graceful and sequential digraphs with n vertices and $n-1$ arcs

The following result gives a relation between graceful and sequential digraphs with n vertices and $n-1$ arcs. Also, this result connects sequential labelings to near complete mappings and cyclic group models.

THEOREM 5.3.1. *A digraph D with n vertices and $n-1$ arcs is graceful iff D is sequential via a function f' such that $f'(v)$ is odd for each $v \in V(D)$.*

Proof. Let f be a graceful labeling of a digraph D with n vertices $v_0, v_1, ..., v_{n-1}$. Then f' defined by $f'(v) = 2f(v) + 1$ for each $v \in V(D)$ extends to a sequential labeling of $V(D) \cup E(D)$ with $f'(V(D)) = \{1, 3, 5, ..., 2n-1\}$ and $f'(E(D)) = \{2, 4, 6, ..., 2n-2\}$. That is, if v_iv_j is an arc in graceful digraph D, then $(f')(v_i) = 2f(v_i) + 1$, $(f')(v_j) = 2f(v_j) + 1$. Hence $f'(v_iv_j) = (2f(v_j) + 1) - (2f(v_i) + 1) = 2(f(v_j) - f(v_i))$ are all even and represents the arc labels of a sequential digraph D.

Conversely, let f' be a sequential labeling with $f'(v)$ odd for each $v \in V(D)$. Then one can observe that f defined by $f(v) = \frac{1}{2}(f'(v) - 1)$ is a graceful labeling of D. $\square$

NOTE 5.3.2. *From the above theorem one can notice that every graceful directed tree is sequential.*

Hence the following are trivial.

1. Unidirectional path $\vec{P_n}$ (n even) is sequential.

2. The alternating path $\vec{P_n}$ on n vertices is sequential.

3. The directed star $\overrightarrow{K_{1,n}}$ is sequential.

From Theorem 5.3.1 one can also say that, the sequential labeling of union of unicycles and paths exists iff there exits a near complete mapping.

EXAMPLE 5.3.3. *A* $(k, 1)$ *near complete mapping of* Z_{14} *for* $k = \{3, 4, 5 : 2\}$ *is* (1 2 4) (6 10 8 11) (3 9 5 13 12) [0 7] *which provides a graceful labeling for the unidirectional components of* $\overrightarrow{C}_3 \cup \overrightarrow{C}_4 \cup \overrightarrow{C}_5 \cup \overrightarrow{P}_2$. *Then* (3 5 9) (13 21 17 23) (7 19 11 27 25) [0 15] *w.r.t.* Z_{28} *gives a sequential labeling for the unidirectional components of* $\overrightarrow{C}_3 \cup \overrightarrow{C}_4 \cup \overrightarrow{C}_5 \cup \overrightarrow{P}_2$.

NOTE 5.3.4. *Disjoint union of unicycles and paths is graceful with a graceful labeling* f *iff it is sequential via* $f(v)$ *odd.*

Also with the same argument the property of graceful digraphs given for cyclic multiplicative groups in Theorem 1.3.24 holds good for sequential labeling of such digraphs. To provide sequential labeling just replace $a^{f(v)}$ by $a^{2f(v)+1}$.

EXAMPLE 5.3.5. *Using Theorem 1.3.24, for Table 1.1 on* N_8*, we can construct the sequential digraph as given in Figure 5.4.*

Hence Theorem 5.3.1 relates sequential digraphs with near complete mappings and cyclic neofields as discussed in case of graceful digraphs (Bloom and Hsu, 1985).

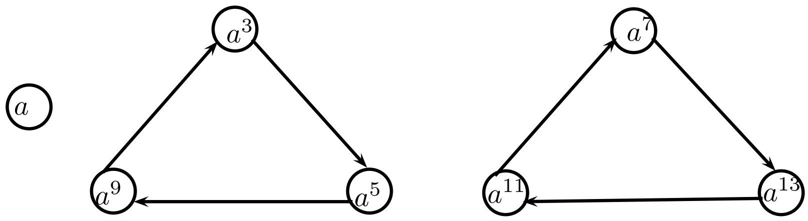

Figure 5.4: A sequential digraph using cyclic multiplicative group

In general, the notation $(D \vec{+} K_m^c)$ specifies the digraph obtained by directing arcs from each of m isolated nodes to each of the nodes of a digraph D.

For any digraph obtained by this construction the following holds:

THEOREM 5.3.6. *(Bloom and Hsu, 1985) If D is graceful digraph with n nodes and $n-1$ arcs, then $(D \vec{+} K_m^c)$ is graceful for every finite m.*

If the notation $(D \vec{+} K_m^c)'$ specifies the digraph obtained by directing arcs to each of m isolated points from each of the nodes of a digraph D, the following holds:

THEOREM 5.3.7. *(Bloom and Hsu, 1985) If D is a graceful digraph with n nodes and $n-1$ arcs, then $(D \vec{+} K_m^c)^1$ is graceful.*

Using Theorem 5.3.1 and the above known results we prove the following.

COROLLARY 5.3.8. *If D a is sequential digraph with n vertices and $n-1$ arcs via a function f' such that $f'(u)$ is odd for each $u \in V(D)$, then $(D \vec{+} K_m^c)$ is graceful for every finite m.*

Proof. Suppose D is sequential digraph via a function f' such that $f'(u)$ is odd for each $u \in V(D)$, with n vertices $u_1, u_2, ..., u_n$. Denote the vertices of K_m^c by $v_1, v_2, .., v_m$, then label the $n+m$ vertices of the digraph $(D \vec{+} K_m^c)$ by ψ as follows.
$\psi(u_i) = (m+1)(\frac{f'(v)-1}{2}) \;\; i = 1, ..., n$

$\psi(v_i) = i \quad i = 1, 2, ..., m.$
which gives a graceful labeling of $(D \vec{+} K^c_m)$. $\square$

In the same manner the following holds.

COROLLARY 5.3.9. *If D is sequential digraph with n vertices and $n-1$ arcs via a function f' such that $f'(u)$ is odd for each $u \in V(D)$, then $(D \vec{+} K^c_m)'$ is graceful for every finite m.*

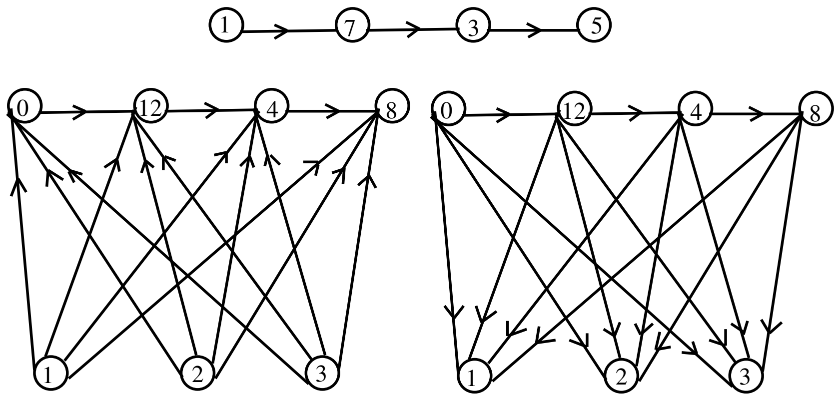

Figure 5.5: A sequntial $\vec{P}_4$ and a graceful $(D \vec{+} K^c_3)$, $(D \vec{+} K^c_3)^1$

OBSERVATION 5.3.10. *It is known that D is sequential iff $D+v$ is graceful via $f(v) = 0$. Also, D is graceful implies its reversed digraph $-D$ graceful with the same vertex labeling. We observe that although D is sequential implies $D+v$ graceful, $-(D+v)$ is graceful does not imply $-D$ is sequential i.e., D is sequential does not imply $-D$ is sequential with the same vertex labeling.*

Hence this observation leads us to the following theorem.

THEOREM 5.3.11. *A digraph D with n vertices and $n-1$ arcs is sequential via $f(x)$ odd for each $x \in V(D)$ iff $(D+v)'$ (and $D+v$) is graceful with $f(v) = 0$.*

Proof. (This result is already proved for $D+v$ in (Shivarajkumar, 2013).) Suppose that D with n vertices and $n-1$ arcs is sequential via $f(x)$ odd for each $x \in V(D)$. Let $v_1, v_2, ..., v_n$ be the n vertices of D which takes odd values from the set $\{1, 3, 5, ..., 2n-1\}$. Then, for the digraph $(D+v)^{'}$ with $f(v) = 0$ the n directed arcs (directed arcs from vertices of D to vertex v) take labels as the inverse element of the vertex labels of D, w.r.t. Z_{2n}. Hence gives a graceful labeling of $(D+v)^{'}$.
Conversely, if $(D+v)^{'}$ with $f(v) = 0$ is graceful, then removal of vertex v with label 0 removes all the inverse elements of n vertices of $D-v$ and gives a sequential labeling of D. $\square$

EXAMPLE 5.3.12. *The following figure shows a sequential $\vec{K_{1,4}}$ and a graceful $(\vec{K_{1,4}}+v)^{1}$ with $f(v) = 0$.*

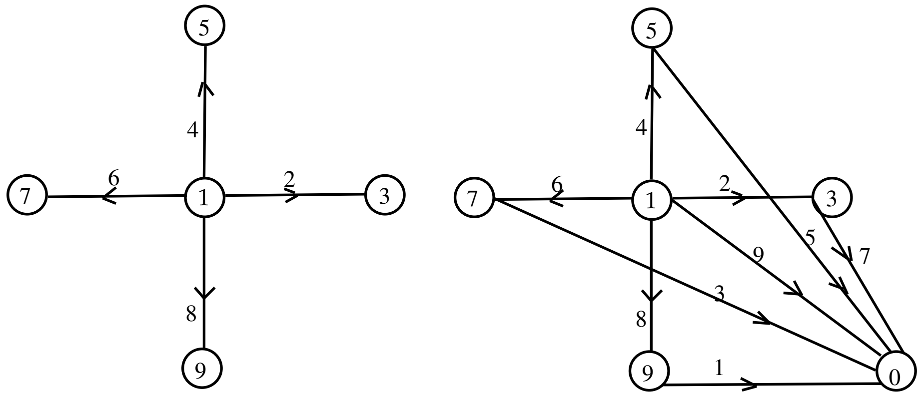

Figure 5.6: A sequntial $\vec{K_{1,4}}$ and a graceful $(\vec{K_{1,4}}+v)^{1}$

From these arguments one can say that, if D is sequential, then $-D$ need not be sequential with the same vertex labeling.

Hence the following theorem gives us an idea for both D and $-D$ to be sequential.

THEOREM 5.3.13. *A digraph D with n vertices and $n-1$ arcs is sequential via $f(v)$ odd for each $v \in V(D)$ iff $-D$ is sequential with the same vertex labeling.*

Proof. Let D be a sequential digraph with n vertices and $n-1$ arcs via $f(v)$ odd. Suppose $v_i v_j$ is an arc in D, then $f(v_j) - f(v_i) \equiv \ l \ mod(2n)$ will be the arc labels in D. Hence in $-D$, $v_j v_i$ represents the arc with $f(v_i) - f(v_j) \equiv \ -l \ mod(2n)$, which are the inverse elements of $n-1$ arc labels of D, w.r.t. Z_{2n}. Since $2n$ is even and all vertex labels are odd, this makes all the $n-1$ arcs in both D and $-D$ to be even. Hence the proof. □

COROLLARY 5.3.14. *A digraph D with n vertices and $n-1$ arcs is graceful iff $-D$ is sequential via a function $f(v)$ is odd for each $v \epsilon V(D)$.*

COROLLARY 5.3.15. *If a digraph D and $-D$ are sequential via a function $f(v)$ odd for each $v \epsilon V(D)$, then D is graceful.*

5.4 A sequential super-digraph construction

To extend the class of known sequential ditrees, it is natural to seek methods for building larger sequential ditrees from smaller sequential labeled ones.

Figure 5.7 shows such a construction in which an arc connects two (sequential) directed stars $\vec{K_{1,n}}$ and $\vec{K_{1,m}}$ to form the illustrated sequentially relabeled digraph $\vec{K_{1,n}}$ $\{u \vec{\ } v\}$ $\vec{K_{1,m}}$.
The notation $\vec{K_{1,n}}$ $\{u \vec{\ } v\}$ $\vec{K_{1,m}}$ specifies the digraph obtained by adding an directed arc $\vec{uv}$ which connects the two directed stars $\vec{K_{1,n}}$ and $\vec{K_{1,m}}$, where u is the central vertex of directed star $\vec{K_{1,n}}$ and v is the central vertex of directed star $\vec{K_{1,m}}$.

PROPOSITION 5.4.1. *If $\vec{K_{1,n}}$ (the in-degree of central vertex u is zero) is a directed star with $(n+1)$ vertices having sequential labeling, then $\vec{K_{1,n}}$ $\{u \vec{\ } v\}$ $\vec{K_{1,m}}$ is sequential* $\forall\ n, m \geq 0$.

Proof. Let θ be a sequential labeling of $\vec{K_{1,n}}$, a digraph with $n+1$ vertices $u = u_0, u_1, ..., u_n$ and n arcs.
Let the vertices of $\vec{K_{1,m}}$ be $v = v_0, v_1, ..., v_n$.
Label the $n+m+2$ vertices of the ditree $D = \vec{K_{1,n}}$ $\{u \vec{\ } v\}$ $\vec{K_{1,m}}$ by ψ as follows:

1) $\psi(u_i) = j$, where for $i = 0, 1, ..., n$: $j = 1, 3, 5, ..., 2n+1$ resp.
2) $\psi(v_0) = 2(m+n)+3$ and $\psi(v_i) = 2n+j$, where for $i = 1, ..., n$: $j = 2, 4, ..., 2m$ resp.
Hence a sequential labeling of the digraph D is obtained. $\square$

Figure 5.7 illustrates a sequential labeling of $\vec{K_{1,n}}\ \{u\vec{\ }v\}\ \vec{K_{1,m}}$.

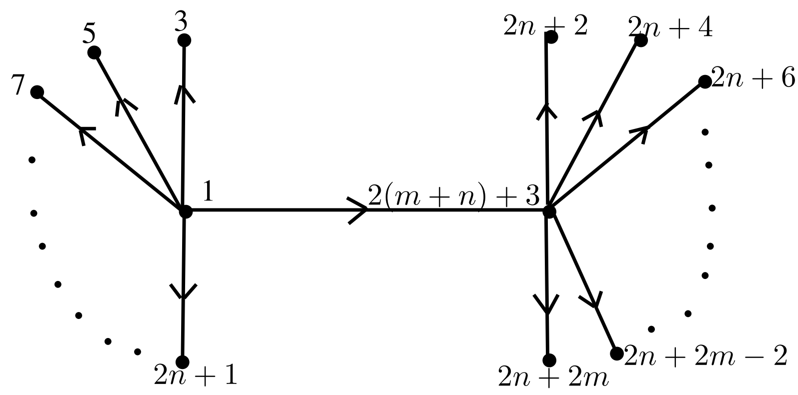

Figure 5.7: A sequential $\vec{K_{1,n}}\ \{u\vec{\ }v\}\ \vec{K_{1,m}}$

PROPOSITION 5.4.2. *If T is a sequential ditree with $f(v)$ odd for each $v \in V(T)$, then disjoint union of 2 copies of T is sequential.*

Proof. Let T be a ditree with $f(v)$ odd for each $v \in V(T)$. Then $f(V(T)) = \{1, 3, 5, ..., 2n-1\}$ and $f(E(T)) = \{2, 4, 6, ..., 2n-2\}$ are the arc labels of T w.r.t. Z_{2n}.
The ditree $T \cup T$ has $2n$ vertices and $2n-2$ arcs. Label the $2n$ vertices of ditree $T \cup T$ as given below.
Consider the ditree T with $f(v)$ odd, then label the n vertices $v_1', v_2', ..., v_n'$ of another copy of T with $4n-1-f(v_i)$ respectively $(1 \le i \le n)$. This gives inverse elements of $f(v_i)$ $(1 \le i \le n)$ w.r.t. Z_{4n-1} as the vertex labels of another copy of T. These values are all even as the difference between any two odd integers is always even. These $2n$ vertex labels of the ditree $T \cup T$ makes the $2n-2$ arcs as all distinct, hence yields a sequential labeling of the ditree $T \cup T$. $\square$

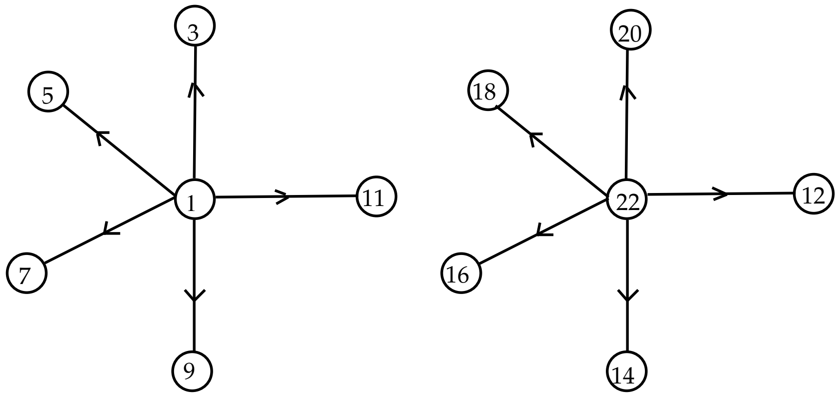

Figure 5.8: A sequential $\vec{K_{1,5}} \cup \vec{K_{1,5}}$

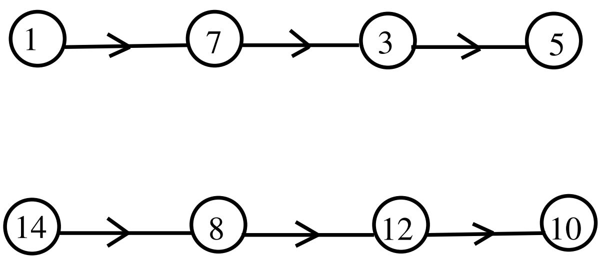

Figure 5.9: A sequential $\vec{P_4} \cup \vec{P_4}$

We define $T^* = (V^*, E^*)$ as a ditree having two copies of a ditree T with an added arc. That is, $V^* = V(T_1) \cup V(T_2)$, $E^* = E(T_1) \cup E(T_2) \cup \{\vec{uv}\}$, with $u \in V(T_1)$, $v \in V(T_2)$, where T_1, T_2 are the two copies of a ditree T.

We observe that, if T is a ditree with $f(x)$ odd for each $x \in V(T)$, then for certain $u \in V(T_1)$ and $v \in V(T_2)$, he ditree T^* is sequential.

EXAMPLE 5.4.3. *Consider a sequential ditree $T = \vec{P_4}$ as given in Figure 5.5. Then Figure 5.10 represents a sequential ditree T^*.*

OBSERVATION 5.4.4. *For the above example, joining an arc from vertex label 7 to 15, 5 to 13 and 3 to 11 also gives different ditree T^* with sequential labeling.*

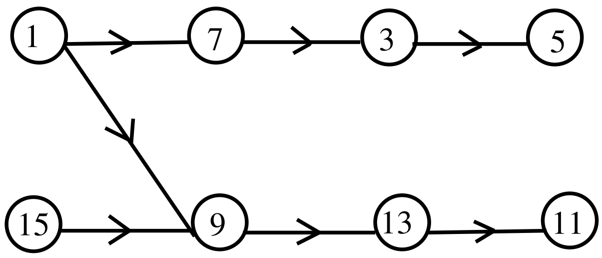

Figure 5.10: A sequential T^*

NOTE 5.4.5. *T^* has $2n$ vertices and $2n-1$ arcs, label the $2n$ vetrices as given in Proposition 5.4.2, but w.r.t Z_{4n}. Identify an arc $\{u \vec{\ } v\}$ such that $f(v) - f(u) = 2n$ w.r.t. Z_{4n}. Then one can obtain a sequential labeing of a ditree T^*.*

5.5 Some more class of sequential digraphs

In this section we will discuss the sequential labeling problems of some more class of digraphs.
It is known that, if G is graceful, then symmetric G is graceful with the same vertex labeling. But, if G is a sequential graph, then symmetric G is not sequential with the same vertex labeling.
For example, we know that the path P_n on n (even) vertices is sequential, but $\overleftrightarrow{P}_n$ is not sequential with the same vertex labeling.
Hence we give the following result.

PROPOSITION 5.5.1. *Symmetric P_n on n (even) vertices is sequential.*

Proof. Symmetric P_n has n vertices and $2(n-1)$ arcs. Label the n vertices $v_1, v_2, ..., v_n$ of symmetric P_n as $1, 3n-2, 3, 3n-4, 5, 3n-6, 7, 3n-8, ..., n-1, 2n$ respectively w.r.t. (mod $(3n-1)$. These vertex labels gives a sequential labeling of symmetric P_n. □

We define, $(n)\vec{P_3}$ as a directed graph comprising $n(>1)$ copies of $\vec{P_3}$ joined at exactly one vertex.

THEOREM 5.5.2. *The digraph $(n)\vec{P_3}$ for any $n(>1)$ is sequential.*

Proof. The digraph $(n)\vec{P_3}$ has $2n+1$ vertices and $2n$ arcs. Let v_0 be the central vertex of the digraph $(n)\vec{P_3}$. Start denoting v_i^1 as the vertices from the left of the central vertex which are at distance 1 from v_0 and v_i^2 as the vertices from the left of the central vertex which are at distance 2 from v_0 $(1 \leq i \leq n)$.
Let $f : V \longrightarrow \{1, 2, ..., 4n+1\}$.
Take $f(v_0) = 4n+1$ and
$f(v_i^1) = i + j,\ for\ i = 1, 2, ..., n:\ j = 0, 1, 2, ..., n-1$ resp.
$f(v_i^2) = 3n + i,\ for\ i = 1, 2, ..., n$ which gives a sequential labeling of the digraph $(n)\vec{P_3}$.

Figure 5.11 gives a sequential labeling of $(n)\vec{P_3}$. □

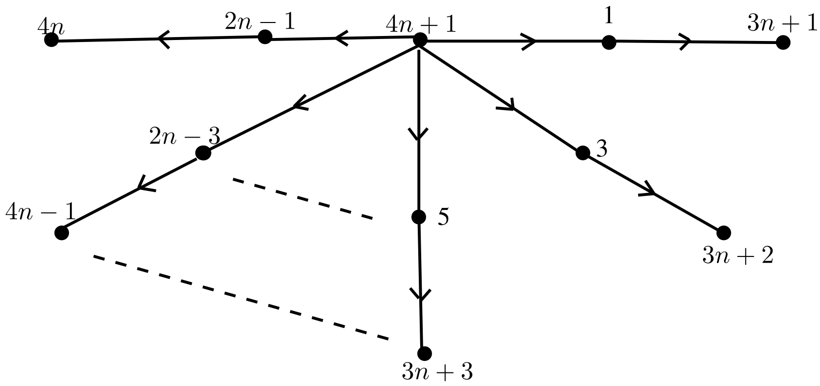

Figure 5.11: A sequential $(n)\vec{P_3}$

NOTE 5.5.3. *If D is a digraph with p vertices and q arcs, let $f : V(D) \longrightarrow \{1, 2, ..., p+q\}$ give a sequential labeling of D. Then the function $f' : V(D) \longrightarrow \{1, 2, ..., p+q\}$ given by $f'(v) = (p+q+1) - f(v)$ also gives a sequential labeling of D.*

EXAMPLE 5.5.4. *The following figure illustrates the above note.*

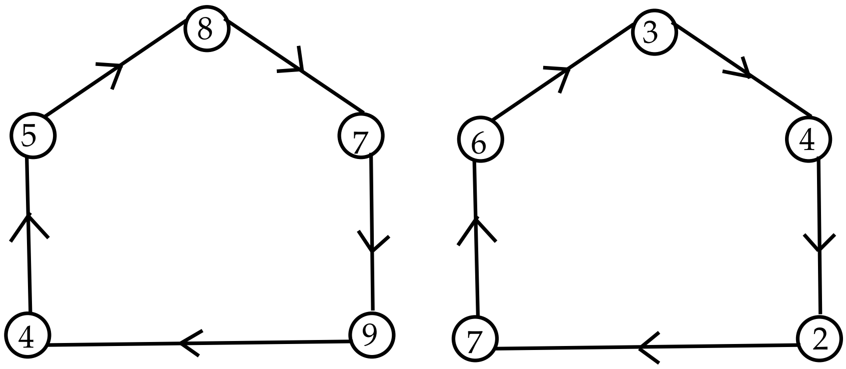

Figure 5.12: Sequential labelings of $\vec{C}_5$

Chapter 6

CONCLUSION AND SCOPE FOR FUTURE RESEARCH

This thesis, provides a glance at the labeling problems of digraphs which are closely related to some of the algebraic structures. The use of modular arithmetic is the one which connects them with all other areas of mathematics.

A (v, k, λ)-difference set is used to construct a graceful digraph with k vertices and $v-1$ arcs. In particular for v odd, the gracefulness of tournaments and graceful symmetric digraphs has been obtained. As a future work, one can also find the distinct graceful labelings, obtain from each CDR for $\mathcal{A}$ or can check the complexity of this problem. A term Zero-sequencing of Z_{2m+1} $m > 2$ is defined and its existence has been proved. Using this existence the gracefulness of the directed cycles DC_m and $\overrightarrow{C_m}(r)$ are given. Also, the gracefulness of some classes of directed cycles was proved, using the relation between complete mappings and partition theory.
A construction of bigraphs from digraphs (vice-versa) was given using an adjacency matrix. This construction was related in terms of graceful labeling. One can also prove these results for some more classes of digraphs and bigraphs.
Further, one can use the special cases of subset sum problems to obtain many classes of graceful and sequential digraphs.

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PUBLICATIONS

List of Publications /Communications Based on Thesis:

1. S. M. Hegde, Kumudakshi, *Construction of graceful digraphs using algebraic structures, J. Discrete Math. Sci. Cryptogr. 19(2016) 103-116.* (Scopus Indexed)

2. S. M. Hegde, Kumudakshi, *Construction of Graceful Directed Graphs* , *J. Comb. Info. Sys. Sci. 40(2015) 252-263.*

3. S. M. Hegde, Kumudakshi, *Graceful digraphs and complete mappings, Electronic Notes in Discrete Mathematics 48(2015) 151-156.* (Scopus Indexed)

4. S. M. Hegde, Kumudakshi, *Further Results On Graceful Directed Graphs, Electronic Notes in Discrete Mathematics 53(2016).*(Scopus Indexed) (to appear)

5. S. M. Hegde, Kumudakshi, *Labelings of digraphs using algebraic structures*(communicated)

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- S. M. Hegde and Kumudakshi, Graceful digraphs and complete mappings, at *Eighth International Workshop on Graph Labelings* at Kalasalingam University, Tamil Nadu during $3^{rd} - 6^{th}$ December, 2014.
- S. M. Hegde and Kumudakshi, Construction of Graceful Digraphs , at $23^{r}d$ *IMSCT* at N.I.T.K, Surathkal during $18^{th} - 21^{st}$ December, 2014.
- S. M. Hegde and Kumudakshi, Further results on graceful directed graphs, at *International Conference on Graph Theory and its Applications* at Amrita University, Coimbatore during $16^{th} - 19^{th}$ December, 2015.