REGULARIZATION METHODS FOR NONLINEAR ILL-POSED HAMMERSTEIN TYPE OPERATOR EQUATIONS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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Arise, Awake, stop not till the goal is reached !!!

-Swami Vivekananda

Dedicated to my beloved parents

DECLARATION

I hereby ***declare*** that the Research Thesis entitled **REGULARIZATION METHODS FOR NONLINEAR ILL-POSED HAMMERSTEIN TYPE OPERATOR EQUATIONS** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirement for the award of the Degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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Date:

CERTIFICATE

This is to *certify* that the Research Thesis entitled **REGULARIZATION METHODS FOR NONLINEAR ILL-POSED HAMMERSTEIN TYPE OPERATOR EQUATIONS** submitted by **Shobha M E** (MA10F03) as the record of the research work carried out by her, is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Research Guide

Prof. Santhosh George

Chairman-DRPC

(Signature with Date and Seal)

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Abstract

This thesis is devoted for obtaining a stable approximate solution for non-linear ill-posed Hammerstein type operator equations $KF(x) = f$. Here $K : X \to Y$ is a bounded linear operator, $F : X \to X$ is a non-linear operator, X and Y are Hilbert spaces. It is assumed throughout that the available data is f^δ with $\|f - f^\delta\| \leq \delta$. Many problems from computational sciences and other disciplines can be brought in a form similar to equation $KF(x) = y$ using mathematical modelling (Engl *et al*. (1990), Scherzer, Engl and Anderssen (1993), Scherzer (1989)). The solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls.

We aim at approximately solving the non-linear ill-posed Hammerstein type operator equations $KF(x) = f$ using a combination of Tikhonov regularization with Newton-type Method in Hilbert spaces and in Hilbert Scales. Also we consider a combination of Tikhonov regularization with Dynamical System Method in Hilbert spaces. Precisely in the methods discussed in this thesis we considered two cases of the operator F : in the first case it is assumed that $F'(.)^{-1}$ exist ($F'(.)$ denotes the Fr$\acute{e}$chet derivative of F) and in the second case it is assumed that $F'(.)^{-1}$ does not exist but F is a monotone operator. The choice of regularization parameter plays an important role in the convergence of regularization method. We use the adaptive scheme suggested by Pereverzev and Schock (2005) for the selection of regularization parameter. The error bounds obtained are of optimal order with respect to a general source condition. Algorithms to implement the method is suggested and the computational results provided endorse the reliability and effectiveness of our methods.

Keywords: Ill-posed operator equations, Hammerstein Operators, Regularization methods, Tikhonov regularization, Monotone Operators, Newton-type method, Hilbert Scales, Dynamical System Method.

Mathematics Subject Classification:47J06, 47H30, 65J20, 47H07, 49M15, 70G60

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Chapter 1

INTRODUCTION

1.1 GENERAL INTRODUCTION

Inverse problems are the problems that consist of finding an unknown property of an object, or a medium from the observation of a response of this object or medium to a probing signal. The theory of inverse problems yields a theoretical basis for remote sensing and that makes the inverse problems more important. The necessity in studying the inverse problems stems from one of the main problems in applied mathematics, gaining reliable computing results with due allowance for errors that inevitably occur in setting co-efficients and parameters of a mathematical model used to perform computations.

A common belief of many mathematicians (see Alber and Ryazantseva (2006), Section 3) in the past was that well-posedness is a necessary condition for the problems to be mathematically or physically meaningful. This raised doubts about whether or not there is any need for methods for solving ill-posed problems (i.e., problems that are not well-posed). The tremendous development of science and technology of the last decades led, more often than not, to practical problems which are ill-posed by their nature. Solving such problems became a necessity and thus, inventing methods for that purpose became a field of research in the intersection of theoretical mathematics with applied science.

1.2 NOTATIONS AND BASIC RESULTS

Let X and Y denote Hilbert spaces over real or complex field and $BL(X,Y)$ denote the space of all bounded linear operators from X to Y. If $Y = X$, then we denote $BL(X,Y)$ by $BL(X)$. We use the notation D(K) to denote the domain of K. If $K \in BL(X,Y)$, then its adjoint, denoted by K^* is a bounded linear operator from Y to X defined by $\langle Kx, y\rangle = \langle x, K^*y\rangle \; \forall x \in X$ and $y \in Y$.

Let $R(K) := \{Kx : x \in X\}$ and $N(K) := \{x \in X : Kx = 0\}$ be the range and null space of K respectively. Further for a subspace S of X, its closure is denoted by $\overline{S}$ and its orthogonal complement denoted by $S^\perp$ is defined as $S^\perp = \{u \in X; \langle x, u\rangle = 0, \forall x \in S\}$.

Throughout this thesis $\gamma,\ \tilde{\gamma},\ \gamma_\rho,\ \tilde{\gamma}_\rho,\ \rho,\ \varepsilon_h,\ \tau_h,\ \tau_0,\ \varepsilon_0,\ r,\ \tilde{r}$ are generic constants which may take different values at different occasions

PROPOSITION 1.2.1 *(Nair (2008), Proposition 11.4) If* $K \in BL(X,Y)$ *then* $R(K)^\perp = N(K^*)$, $N(K)^\perp = \overline{R(K^*)}$, $R(K^*)^\perp = N(K)$ *and* $N(K^*)^\perp = \overline{R(K)}$

We call K a positive self-adjoint operator if $K = K^*$ and $\langle Kx, x\rangle \geq 0, \forall x \in X$. The spectrum and spectral radius of an operator $K \in BL(X)$ are denoted by $\sigma(K)$ and $r_\sigma(K)$ respectively i.e.,

$$\sigma(K) = \{\lambda \in C : K - \lambda I \; does\ not\ have\ a\ bounded\ inverse\}$$

where I is the identity operator on X and

$$r_\sigma(K) = sup\{|\lambda| : \lambda \in \sigma(K)\}.$$

It is well known that $r_\sigma(K) \leq \|K\|$ and $\sigma(K)$ is a compact subset of the scalar field. If K is a non-zero self-adjoint operator (i.e., $K = K^*$), then $\sigma(K)$ is a non-empty subset of real numbers and $r_\sigma(K) = \|K\|$. If K is positive self-adjoint operator, then $\sigma(K)$ is a subset of set of non-negative reals and if $K \in BL(X)$ is compact, then $\sigma(K)$ is a countable set with zero as the only possible limit point.

Let F be an operator mapping a Hilbert space X into a Hilbert space Y. If there exists a bounded linear operator $L : X \to Y$ such that

$$\lim_{\|h\|\to 0} \frac{\|F(x_0 + h) - F(x_0) - L(h)\|}{\|h\|} = 0,$$

then F is said to be a Fr$\acute{e}$chet-differentiable at x_0 and the bounded linear operator $F'(x_0) := L$ is called the first Fr$\acute{e}$chet-derivative of F at x_0.

Let $F : D(F) \subseteq X \to X$ be an operator where X is a real Hilbert space. Then F is said to be monotone if $\langle F(x) - F(y), x - y\rangle \geq 0, \forall x, y \in D(F)$.

1.3 ILL-POSEDNESS OF EQUATIONS

The class of ill-posed problems was first identified by the French mathematician Jacques Hadamard (1865-1963) in 1902. It is a common opinion that ill-posed problems often belong to the field of ”very evolved ” mathematics, something very difficult to understand and rarely met (Petrov and Sizikov (2005)). But this notion is certainly wrong as ill-posed problems are encountered very frequently and an adequate care has to be taken regarding their properties and difficulties. A historical review of ill-posed problems can be found in Petrov (2001).

1.3.1 Classical definition of well-posedness

The problem of solving

$$F(x) = y, \tag{1.3.1}$$

where $F : D(F) \subseteq X \to Y$ is well-posed problem in the sense of Hadamard (see Bonilla (2002), page 12) if:

(a) A solution of (1.3.1) exists (i.e.,operator domain $R(F) = Y$).

(b) The solution is unique (solution x is uniquely determined by the element y, i.e., the inverse operator F^{-1} exists)

(c) The solution depends continuously on the given data (F^{-1} is a continuous operator).

If any of the above conditions is violated, then (1.3.1) is called an ill-posed equation. Equation (1.3.1) is linear ill-posed equation if the operator F is linear and if F is non-linear it is called non-linear ill-posed equation.

Given below are some examples of ill-posed problems. The first two are examples of linear ill-posed problem and the next two are that of non-linear ill-posed problems.

1.4 EXAMPLES OF ILL-POSED PROBLEMS

EXAMPLE 1.4.1 Differentiation (Engl *et al*. (2000))

Differentiation could be viewed as an inverse problem of solving the operator equation

$$Kx = y$$

where $K : C[0,1] \to C^1[0,1]$ *is defined as*

$$(Kx)(t) := \int_0^t x(s)ds, t \in [0,1].$$

This problem is unstable as can be seen from the following argument. Suppose we have a sequence of perturbed data given by

$$y_n(t) := y(t) + \frac{\sin nt}{\sqrt{n}}, t \in [0,1]$$

for $n \in N$. *Then for each* $n \in N$, *we have*

$$y_n'(t) = y'(t) + \sqrt{n}\cos nt, t \in [0,1].$$

Now

$$\|y - y_n\|_2 = \sqrt{\frac{1}{2n} - \frac{\sin 2n}{4n^2}} \to 0 \; as \; n \to \infty,$$

but

$$\|y' - y_n'\|_2 = \sqrt{\frac{n}{2} + \frac{\sin 2n}{4}} \to \infty \; as \; n \to \infty.$$

Thus the solution does not depend continuously on the available data and hence the problem is ill-posed.

EXAMPLE 1.4.2 Simplified tomography (see Groetsch (1984))

Consider a two dimensional object contained within a circle of radius R. *The object is illuminated with a radiation of density* I_0. *As the radiation beams pass through the object* *it absorbs some radiation. Assume that the radiation absorption co-efficient $f(x, y)$ of the object varies from point to point of the object. The absorption co-efficient satisfies the law*

$$\frac{dI}{dy} = -fI$$

where I is the intensity of the radiation. By taking the above equation as the definition of the absorption co-efficient, we have

$$I_x = I_0 \exp(-\int_{-y(x)}^{y(x)} f(x, y)dy)$$

where $y = \sqrt{R^2 - x^2}$. Let $p(x) = \ln(\frac{I_0}{I_x})$, i.e.,

$$p(x) = \int_{-y(x)}^{y(x)} f(x, y)dy.$$

Suppose that f is circularly symmetric, i.e., $f(x, y) = f(r)$ with $r = \sqrt{x^2 + y^2}$, then

$$p(x) = \int_x^R \frac{2r}{\sqrt{r^2 - x^2}} f(r)dr. \tag{1.4.2}$$

The inverse problem is to find the absorption co-efficient f satisfying the equation (1.4.2).

EXAMPLE 1.4.3 Non-Linear singular integral equation (see Buong (1998))

Consider the non-linear singular integral equation in the form

$$\int_0^t (t - s)^{-\lambda} x(s)ds + F(x(t)) = f_0(t), 0 < \lambda < 1, \tag{1.4.3}$$

where $f_0 \in L^2[0, 1]$ and the non-linear function F(t) satisfies the following conditions:

- ◇ $|F(t)| \leq a_1 + a_2|t|, a_1, a_2 > 0$
- ◇ $F(t_1) \leq F(t_2) \Leftrightarrow t_1 \leq t_2,$
- ◇ *F is differentiable.*

Thus, F is a monotone operator from $X = L^2[0, 1]$ into $Y = L^2[0, 1]$. In addition, assume that F is a compact operator. Then the equation (1.4.3) is an ill-posed problem, because the operator K defined by

$$Kx(t) = \int_0^t (t - s)^{-\lambda} x(s)ds,$$

is also compact.

EXAMPLE 1.4.4 The backwards heat equation problem (Ramm (2005))

Consider the backwards heat equation problem:

$$u_t = u_{xx}, t \geq 0, x \in [0, \pi];$$

$$u(0,t) = u(\pi, t) = 0, u(x,T) = v(x).$$

Given $v(x)$, *one wants to find* $u(x,0) := w(x)$.

By separation of variables one finds $u(x,t) = \Sigma_{n=1}^{\infty} u_n(t)\sin(nx), u_n(t) = e^{-n^2(t-T)}v_n, v_n = \frac{2}{\pi}\int_0^{\pi} v(x)\sin(nx)dx$. *Therefore,* $w(x) = \Sigma_{n=1}^{\infty} e^{n^2(T)} v_n \sin(nx)$, *provided this series converges, in* $L^2[0,\pi]$, *that is, provided that*

$$\Sigma_{n=1}^{\infty} e^{2n^2(T)}|v_n|^2 < \infty. \qquad (1.4.4)$$

This cannot happen unless v_n *decays sufficiently fast. Therefore the backwards heat equation problem is ill-posed: it is not solvable for a given* $v(x)$ *unless (1.4.4) holds, and small perturbations of the data* v *in* $L^2[0,\pi]$*-norm may lead to arbitrary large perturbations of the function* $w(x)$, *but also may lead to a function* v *for which the solution* $u(x,t)$ *does not exist for* $t < T$.

In practical problems the operator F and the data y of (1.3.1) are not precisely known. Without the knowledge of the continuous dependence of the approximate solution on the intrinsic errors involved, a direct numerical resolution of (1.3.1) is not possible. Attempts to avoid this difficulty led investigators to the new theory and conceptually new methods, viz-a-viz the regularization methods, for obtaining stable solution of ill-posed problems.

1.5 REGULARIZATION METHODS

1.5.1 Generalized Inverse

If $y \notin R(F)$, then (1.3.1) has no solution and hence the equation (1.3.1) is ill-posed. In such a case we may broaden the notion of a solution in a meaningful sense. For $F \in BL(X,Y)$ and $y \in Y$, an element $u \in X$ is said to be a least square solution of (1.3.1) if

$$\|F(u) - y\| = \inf\{\|F(x) - y\| : x \in X\}.$$

Note that if F is not one-one then the least square solution u, if exists, is not unique since $u + v$ is also a least square solution for every $v \in N(F)$. For $y \in R(F) + R(F)^{\perp}$, the unique least-square solution of minimal norm of (1.3.1) is called the generalized solution or pseudo solution of (1.3.1). For $F \in BL(X,Y)$, the map $F^\dagger$ which associates each $y \in D(F^\dagger) := R(F) + R(F)^\perp$, to the generalized solution of (1.3.1) is called the generalized inverse of F. We also see that if $y \in R(F)$, and F is injective, then the generalized solution of (1.3.1) is the solution of (1.3.1). If F is bijective, then it follows that $F^\dagger = F^{-1}$.

THEOREM 1.5.1 *(Nair (2009), Theorem 4.4) Let $F \in BL(X,Y)$. Then $F^\dagger : D(F^\dagger) \to X$ is a closed densely defined linear operator and $F^\dagger$ is bounded if and only if $R(F)$ is closed.*

If F is nonlinear monotone and continuous, then consider the set $N := \{x : F(x) = y\}$. Note that N is closed and convex if F is monotone and continuous (see, e.g., Ramm (2007)) and hence has a unique element of minimal norm, denoted by $\hat{x}$ such that $F(\hat{x}) = y$. So if F is nonlinear, monotone and continuous, then instead of the unique least-square solution of minimal norm we consider the unique element of minimal norm of N as the minimal norm solution of (1.3.1).

REMARK 1.5.2 *Theorem 1.5.1 shows that the problem of finding the generalized solution of (1.3.1) is also ill-posed, i.e., $F^\dagger$ is discontinuous if $R(F)$ is not closed. This observation is important since a wide class of operators of practical importance, especially compact operators of infinite rank falls into this category (Groetsch (1993)). Further in application the data y may not be available exactly.*

Let $y^\delta \in Y$ be the available noisy data with

$$\|y - y^\delta\| \leq \delta. \tag{1.5.5}$$

If $F^\dagger$ is discontinuous then for y^δ close to y, the generalized solution $F^\dagger y^\delta$, even when it is defined need not be close to $F^\dagger y$. To manage this situation the so called "regularization procedures" have to be employed and obtain approximations for $F^\dagger y$.

1.5.2 Regularization principle

The process of obtaining a stable approximate solution to an ill-posed operator equation is called a regularization method. In the regularization procedure (see Engl *et al*. (2000),page 56) the ill-posed equation is replaced by a family of well-posed equations based on a regularization parameter $\alpha > 0$.

A family of operators $\{R_\alpha : 0 < \alpha \leq \alpha_0\}$ is called a regularization method for the problem (1.3.1) with y in range of F, if there exists a parameter choice rule $\alpha = \alpha(\delta, y^\delta)$ such that $\lim\sup_{\delta>0}\{\|R_{\alpha(\delta,y^\delta)}y^\delta - x\| : y^\delta \in Y, \|y - y^\delta\| \leq \delta\} = 0.$

A regularization procedure can be classified as continuous regularization and iterative regularization based on the kind of parameters involved in the procedure. Tikhonov regularization and Lavrentiev regularization are few of the continuous regularization procedures, while Landweber iteration is one of the iterative regularization method.

We give a brief note on Tikhonov regularization and Lavrentiev regularization for linear ill-posed problems.

1.5.3 Tikhonov regularization

Tikhonov regularization (Groetsch (1984), Tikhonov (1963), Tikhonov and Arsenin (1977)) named after Andrey Tikhonov, is the most well-known regularization method for ill-posed problems. In this method the solution x_α^δ of the minimization problem $\min_{x\in X}\{\|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2\}$ is used to approximate $\hat{x}$ where $\alpha > 0$ is called the regularization parameter. Observe that x_α^δ is the unique solution of the well-posed equation

$$(F^*F + \alpha I)x_\alpha^\delta = F^*y^\delta$$

where F^* is the adjoint of the operator F.

1.5.4 Lavrentiev regularization

If $X = Y$ and F is a positive self-adjoint operator on X, then one may consider a simpler regularization method (George (2006b)) to solve equation (1.3.1), where the family of vectors $w_\alpha^\delta, \alpha > 0$, satisfying

$$(F + \alpha I)w_\alpha^\delta = y^\delta \tag{1.5.6}$$

is considered, to obtain approximations for $\hat{x}$. Note that for positive self-adjoint operator F, the ordinary Tikhonov regularization applied to (1.3.1) results in a more complicated equation $(F^2 + \alpha I)x_\alpha^\delta = Fy^\delta$ than (1.5.6). Moreover, it is known (George (2006b)) that the approximation obtained by regularization procedure (1.5.6) has better convergence properties than the approximation obtained by Tikhonov regularization. The above regularization procedure which gives the family of vectors w_α^δ in (1.5.6) is called Lavrentiev regularization or Simplified regularization of (1.3.1)(see Groetsch and Guacaneme (1987)).

1.5.5 Iterative regularization method

Iterative regularization methods are used for approximately solving $F(x) = y$ when F is a non-linear operator. Recall (Mahale and Nair (2009)) that an iterative method with iterations defined by

$$x_{k+1}^\delta = \Phi(x_0^\delta, x_1^\delta, \cdots, x_k^\delta; y^\delta),$$

where $x_0^\delta := x_0 \in D(F)$ is a known initial approximation of $\hat{x}$, for a known function Φ together with a stopping rule which determines a stopping index $k_\delta \in \mathbb{N}$ is called an iterative regularization method if

$$\|x_{k_\delta}^\delta - \hat{x}\| \to 0 \quad \text{as} \quad \delta \to 0.$$

The Levenberg-Marquardt method (Hanke (2010), Hochbruck and Honig (2010), Jin (2010), Pornasawad and Bockmann (2010), Bockmann *et al*. (2011)) and iteratively regularized Guass-Newton Method (IRGNA)(Bakushinskii (1992), Blaschke *et al*. (1997)) are some of the well-known iterative regularization methods.

1.5.6 Dynamical System Method

Ramm (2005), considered a method called Dynamical System Method (DSM) for solving nonlinear equation $F(u) = 0$. The DSM consists of finding a nonlinear locally Lipschitz operator $\Phi(u, t)$, such that the Cauchy problem:

$$u'(t) = \Phi(u, t), \qquad u(0) = u_0 \tag{1.5.7}$$

has the following three properties:

$$\exists u(t) \; \forall t \geq 0, \tag{1.5.8}$$

such that $\exists u(\infty)$ and $F(u(\infty)) = 0$. i.e.,

(a) (1.5.7) is globally uniquely solvable;

(b) its unique solution has a limit at infinity;

(c) and this limit solves $F(u) = 0$.

1.5.7 Regularized Projection method

Even though, a stable solution of linear ill-posed problem (1.3.1) can be obtained via regularization methods, for numerical calculations, one has to look for an implementable method i.e., a method for which one can realize a solution in a finite dimensional space. A natural practical approach in this direction is the least-square projection method, i,e., to find the minimum-norm solution of $Fx = y$ in a finite dimensional subspace of X. That is, given a sequence $V_1 \subset V_2 \subset V_3 \subset \cdots$ of finite-dimensional subspace of X such that $\overline{U_{n\in\mathbb{N}} V_n} = X$, let x_n be the least-square solution of minimal norm in the space V_n (see Engl and Neubauer (1985)). Obviously $x_n = F_n^\dagger y$ where $F_n := FP_n$ and P_n is the orthogonal projector onto V_n. It is known (Engl *et al*. (2000)) that x_n is a stable approximation of $x^\dagger$, but without additional assumptions it cannot be guaranteed that x_n converges to $x^\dagger$ (See Seidman (1980)).

1.6 CHOICE OF REGULARIZATION PARAMETER

When we consider the rate of convergence of a regularization method (R_α, α) one can think of the rate of convergence of

$$\|R_\alpha y - \hat{x}\| \to 0, \ as \ \alpha \to 0, \tag{1.6.9}$$

or of the rate of convergence of

$$\|R_\alpha y^\delta - \hat{x}\| \to 0, \ as \ \delta \to 0. \tag{1.6.10}$$

Since

$$\|R_\alpha y^\delta - \hat{x}\| \leq \|R_\alpha y^\delta - R_\alpha y\| + \|R_\alpha y - \hat{x}\|,$$

the rate of convergence depends on the choice of the regularization parameter. So the most important procedure in regularization method is the selection of regularization parameter. A choice $\alpha = \alpha_\delta$ of the regularization parameter may be made in either an a priori or a posteriori way. An extensive discussion of ”aposteriori” choice has been done in regularization theory (Gfrerer (1987), Mathe and Pereverzev (2002)).

Suppose there exist a function φ on $[0, \infty)$ such that

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))\nu, \tag{1.6.11}$$

where x_0 is an initial guess, $\hat{x}$ is the solution of (1.3.1), $F'(\hat{x})$ is the Frechet derivative of F at $\hat{x}$ and $\|\hat{x} - R_\alpha y\| \leq \varphi(\alpha)$, then φ is called a source function and (1.6.11) the source condition.

A parameter choice strategy $\alpha = \alpha_\delta$ is said to be of optimal order (yields an optimal convergence rate) for a $y \in Y$ if $\psi_y(\delta) = \varphi(\tilde{\psi}_y(\delta))$ as $\delta \to 0$ where

$$\psi_y(\delta) := \sup\{\|R_\alpha y^\delta - \hat{x}\| : \|y - y^\delta\| \leq \delta\}$$

$$\tilde{\psi}_y(\delta) := \sup\{\inf\{\|R_\beta y^\delta - \hat{x}\| : \beta > 0\} : \|y - y^\delta\| \leq \delta\}.$$

Pereverzev and Schock (2005), considered an adaptive selection of the parameter which does not involve even the regularization method in an explicit manner. Let us briefly discuss this adaptive method in a general context of approximating an element $\hat{x} \in X$ by elements from a set $\{x_\alpha^\delta : \alpha > 0, \delta > 0\}$.

Suppose $\hat{x} \in X$ is to be approximated by using elements x_α^δ for $\alpha > 0, \delta > 0$. Assume that there exist increasing functions $\varphi(t)$ and $\psi(t)$ for $t > 0$ such that

$$\lim_{t \to o} \varphi(t) = 0 = \lim_{t \to o} \psi(t),$$

and

$$\|\hat{x} - x_\alpha^\delta\| \leq \varphi(t) + \frac{\delta}{\psi(t)}$$

for all $\alpha > 0, \delta > 0$. Here, the function φ may be associated with the unknown element $\hat{x}$, whereas the function ψ may be related to the method involved in obtaining x_α^δ. Note that the quantity $\varphi(\alpha) + \frac{\delta}{\psi(\alpha)}$ attains its minimum for the choice $\alpha := \alpha_\delta$ such that $\varphi(\alpha_\delta) = \frac{\delta}{\psi(\alpha_\delta)}$, that is for

$$\alpha_\delta = (\varphi\psi)^{-1}(\delta)$$

and in that case

$$\|\hat{x} - x^\delta_{\alpha_\delta}\| \leq 2\varphi(\alpha_\delta).$$

The above choice of the parameter is a priori in the sense that it depends on the unknown functions φ and ψ.

In an ”aposteriori” choice, one finds a parameter α_δ without making use of the unknown source function φ such that one obtains an error estimate of the form

$$\|\hat{x} - x^\delta_{\alpha_\delta}\| \leq c\varphi(\alpha_\delta).$$

for some $c > 0$ with $\alpha_\delta = (\varphi\psi)^{-1}(\delta)$. The procedure considered by Pereverzev and Schock (2005) starts with a finite number of positive real numbers, α_0, α_1, α_2, . . . ,α_N, such that

$$\alpha_0 < \alpha_1 < \alpha_2 < ... < \alpha_N.$$

The following theorem is essentially a reformulation of a theorem proved in Pereverzev and Schock (2005).

THEOREM 1.6.1 *(George and Nair (2008), Theorem 4.3) Assume that there exists* $i \in \{0, 1, 2, \cdots, N\}$ *such that* $\varphi(\alpha_i) \leq \frac{\delta}{\psi(\alpha_i)}$ *and for some* $\mu > 1$,

$$\psi(\alpha_i) \leq \mu\psi(\alpha_{i-1}), \quad \forall i \in \{0, 1, 2, \cdots, N\}.$$

Let

$$l := max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\psi(\alpha_i)}\} < N,$$

$$k := max\{i : \|x^\delta_{\alpha_i} - x^\delta_{\alpha_j}\| \leq 4\frac{\delta}{\psi(\alpha_j)}, \quad \forall j = 0, 1, \cdots, i - 1.\}.$$

Then $l \leq k$ *and*

$$\|\hat{x} - x^\delta_{\alpha_k}\| \leq 6\mu\varphi(\alpha_\delta), \quad \alpha_\delta := (\varphi\psi)^{-1}(\delta)$$

1.7 HILBERT SCALES

In order to improve the error estimates available in Tikhonov regularization of linear ill-posed problem, Natterer (1984) carried out error analysis in the frame work of Hilbert scales, subsequently many authors extended, modified and generalized Natterer's work to obtain error bounds for linear and non-linear ill-posed problems (see Neubauer (2000), Jin and Tautenhahn (2011b), Tautenhahn (1996), Lu *et al*. (2010)).

Let $L : D(L) \subset X \to X$, be a linear, unbounded, self-adjoint, densely defined and strictly positive operator on X. Let X_t be the completion of $D := \bigcap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|x\|_t$ induced by the inner product

$$\langle u, v\rangle := \langle L^t u, L^t u\rangle, u, v \in D,$$

then $(X_s)_{s\in R}$ is called the Hilbert scale induced by L (see Engl *et al*. (2000), page 211). In chapter 7, we consider the problem of solving an ill-posed Hammerstein type operator equation in the setting of Hilbert scales.

1.8 HAMMERSTEIN OPERATORS

Let a function $k(t, s, u)$ be defined for $t \in [a, b], s \in [c, d]$ and $-\infty < u < \infty$. Then the non-linear integral operator

$$Ax(t) = \int_c^d k(t, s, x(s))ds \tag{1.8.12}$$

is called Uryson integral operator and the function k(t,s,u) is called its kernel. If the kernel k has the special form $k(t, s, u) = k(t, s)f(s, u)$, then (1.8.12) are called Hammerstein Operators (cf. Krasnoselskii *et al*. (1976), Page 375).

Note that each Hammerstein Operator admits a representation of the form $A = KF$ where K is a linear integral operator defined by

$$Kx(t) = \int_c^d k(t, s)x(s)ds$$

and F is a non-linear superposition operator (cf. Krasnoselskii *et al*. (1976))

$$Fx(s) = f(s, x(s)).$$

Hence the study of a Hammerstein operator can be reduced to the study of the linear operator K and the non-linear operator F. An equation of the form

$$(KF)x(t) = y(t) \tag{1.8.13}$$

is called a Hammerstein type operator equation (George (2006a), George and Nair (2008), George and Kunhanandan (2009)).

1.8.1 Examples of Hammerstein type operator equations

EXAMPLE 1.8.1 *(see Engl et al. (2000), Page 260) Consider the integral equation*

$$\int_0^t (t-s)x^3(s)ds = y(t).$$

The above equation can be written in the form of (1.8.13), where

$$K : L^2[0,1] \to L^2[0,1]$$

is defined by $Kx(t) = \int_0^t (t-s)x(s)ds$ *and* $F : D(F) = H^1[0,1] \to L^2[0,1]$ *is defined by* $F(x(s)) = x^3(s)$.

EXAMPLE 1.8.2 *Non-Linear Hammerstein integral equation (see Engl et al. (2000))*

Consider $F(x) = y$ *where* $F : D = L^2[0,1] \to L^2[0,1]$ *defined by*

$$F(x)(t) := \int_0^1 k(s,t)u(s,x(s))ds = y(t),$$

is injective with a non-degenerate kernel $k(.,.) \in L^2([0,1]\times[0,1])$ *and* $u : [0,1]\times R \to R$ *satisfies*

$$|u(t,s)| \leq a(t) + b|s| t \in [0,1], s \in R$$

for some $a \in L^2[0,1]$ *and* $b > 0$, *it can be seen that* F *is compact and continuous on* $L^2[0,1]$*(see Joshi and Bose (2008)).*

1.9 OUTLINE OF THE THESIS

The subject matter of the thesis is regularization of nonlinear ill-posed Hammerstein type operator equations $KF(x) = f$. It is assumed that the available data is f^δ such that $\|f - f^\delta\| \leq \delta$. We try to solve approximately $KF(x) = f$, by splitting the equation into linear equation

$$Kz = f \tag{1.9.14}$$

and non-linear equation

$$F(x) = z. \tag{1.9.15}$$

By doing this we try to simplify the procedure by specifying a regularization strategy (Tikhonov regularization) for linear equation (1.9.14) and an iterative method for non-linear part (1.9.15). The thesis is arranged in eight chapters.

In Chapter 2, for solving $KF(x) = f$, we consider a method which is a combination of Tikhonov regularization for solving (1.9.14) and Two Step Newton Method for solving (1.9.15). The Tikhonov regularized solution of (1.9.14) is given by

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*(f^\delta - KF(x_0)) + F(x_0). \tag{1.9.16}$$

We solve (1.9.15), for two cases of operator F. In the first case where $F'(x_0)$ is boundedly invertible, the iterative method is defined as

$$y_{n,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_0)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta),$$

$$x_{n+1,\alpha}^\delta = y_{n,\alpha}^\delta - F'(x_0)^{-1}(F(y_{n,\alpha}^\delta) - z_\alpha^\delta),$$

where $x_{0,\alpha}^\delta = x_0$, is the initial guess for the solution $\hat{x}$ of $KF(x) = f$. And in the second case where $F'(x_0)$ in non-invertible but F is a monotone operator, we define the iterative method as

$$\tilde{y}_{n,\alpha}^\delta = \tilde{x}_{n,\alpha}^\delta - R(\tilde{x}_{0,\alpha}^\delta)^{-1}[F(\tilde{x}_{n,\alpha}^\delta) - z_\alpha^\delta + \frac{\alpha}{c}(\tilde{x}_{n,\alpha}^\delta - \tilde{x}_{0,\alpha}^\delta)]$$

$$\tilde{x}_{n+1,\alpha}^\delta = \tilde{y}_{n,\alpha}^\delta - R(\tilde{x}_{0,\alpha}^\delta)^{-1}[F(\tilde{y}_{n,\alpha}^\delta) - z_\alpha^\delta + \frac{\alpha}{c}(\tilde{y}_{n,\alpha}^\delta - \tilde{x}_{0,\alpha}^\delta)]$$

where $\tilde{x}^{\delta}_{0,\alpha} := x_0$ is the initial guess and $R(x_0) := F'(x_0) + \frac{\alpha}{c}I$, with $c \leq \alpha$. We make use of the adaptive scheme suggested by Pereverzev and Schock (2005) for choosing the regularization parameter α, depending on the noisy data f^δ and the error δ. We obtain order optimal error bounds under general source condition and with the proposed method we get linear convergence.

Chapter 3 deals with the finite dimensional realization of the method considered in Chapter 2. The algorithm for the proposed method is presented followed by two numerical examples which confirm the efficiency of our approach.

Chapter 4 is the modified form of Newton's method dealt in Chapter 2 and 3. The Two Step Newton method for the case where $F'(u)^{-1}$ exists, for all $u \in D(F)$ is as follows:

$$v^{\delta}_{n,\alpha} = u^{\delta}_{n,\alpha} - F'(u^{\delta}_{n,\alpha})^{-1}(F(u^{\delta}_{n,\alpha}) - z^{\delta}_{\alpha}),$$

$$u^{\delta}_{n+1,\alpha} = v^{\delta}_{n,\alpha} - F'(u^{\delta}_{n,\alpha})^{-1}(F(v^{\delta}_{n,\alpha}) - z^{\delta}_{\alpha}).$$

where $u^{\delta}_{0,\alpha} := x_0 \in X$ is the initial guess for the solution $\hat{x}$ of $KF(x) = f$. The modified iterative method where $F'(u)^{-1}$ does not exist but F is monotone is defined as

$$\tilde{v}^{\delta}_{n,\alpha} = \tilde{u}^{\delta}_{n,\alpha} - R(\tilde{u}^{\delta}_{n,\alpha})^{-1}[F(\tilde{u}^{\delta}_{n,\alpha}) - z^{\delta}_{\alpha} + \frac{\alpha}{c}(\tilde{u}^{\delta}_{n,\alpha} - x_0)],$$

$$\tilde{u}^{\delta}_{n+1,\alpha} = \tilde{v}^{\delta}_{n,\alpha} - R(\tilde{u}^{\delta}_{n,\alpha})^{-1}[F(\tilde{v}^{\delta}_{n,\alpha}) - z^{\delta}_{\alpha} + \frac{\alpha}{c}(\tilde{v}^{\delta}_{n,\alpha} - x_0)]$$

where $\tilde{u}_{0,\alpha} := x_0$ and $R(x) := F'(u) + \frac{\alpha}{c}I,\ c \leq \alpha$. We also discuss the finite dimensional realization of the above defined method. In this Chapter, the Fr$\acute{e}$chet derivative of F at all points $u_n,\ n \geq 0$ is taken into account unlike the method in Chapter 2 and 3, where the Fr$\acute{e}$chet derivative of F is considered only at initial guess. This approach leads to cubic convergence compared to linear and quadratic convergence obtained by George and Nair (2008) and George and Kunhanandan (2009) respectively. Also the derived error estimates using general source condition and adaptive choice method of Pereverzev and Schock (2005) are of optimal order. We give the algorithm required to implement the method and also the numerical examples to test the reliability of our approach.

In Chapter 5, we study the Modified form of the method considered in Chapter 4. The aim is to improve the convergence rates obtained in the previous Chapters. Infact we obatined semi-local quartic convergence. Also the projection scheme of the method and numerical examples are presented.

In Chapter 6, the problem of approximately solving the non-linear Hammerstein operator equation $KF(x) = f$ is dealt in the setting of Hilbert Scales. The proposed method in this chapter is also a combination of Tikhonov regularization and Newton Method. Two cases of operator F are discussed. For the case where $F'(x_0)^{-1}$ exists and is bounded, the iterative scheme is given as

$$y^\delta_{n,\alpha,s} = x^\delta_{n,\alpha,s} - F'(x_0)^{-1}[F(x^\delta_{n,\alpha,s}) - z^\delta_{\alpha,s}],$$
$$x^\delta_{n+1,\alpha,s} = y^\delta_{n,\alpha,s} - F'(x_0)^{-1}[F(y^\delta_{n,\alpha,s}) - z^\delta_{\alpha,s}],$$

where $x^\delta_{0,\alpha,s} := x_0$, is the initial approximation for the solution $\hat{x}$ of $KF(x) = f$ and

$$z^\delta_{\alpha,s} := F(x_0) + (L^{-s}K^*K + \alpha I)^{-1}L^{-s}K^*(f^\delta - KF(x_0))$$

is the Tikhonov regularized solution of linear equation $Kz = f$. Here and below L is a linear unbounded self-adjoint, densely defined and strictly positive operator in X. The second case when $F'(x_0)^{-1}$ does not exist but F is monotone, we define the iterative scheme as

$$\tilde{y}^\delta_{n,\alpha,s} = \tilde{x}^\delta_{n,\alpha,s} - (F'(x_0) + \frac{\alpha}{c}L^{s/2})^{-1}[F(\tilde{x}^\delta_{n,\alpha,s}) - z^\delta_{\alpha,s} + \frac{\alpha}{c}L^{s/2}(\tilde{x}^\delta_{n,\alpha,s} - x_0)],$$
$$\tilde{x}^\delta_{n+1,\alpha} = \tilde{y}^\delta_{n,\alpha,s} - (F'(x_0) + \frac{\alpha}{c}L^{s/2})^{-1}[F(\tilde{y}^\delta_{n,\alpha,s}) - z^\delta_{\alpha,s} + \frac{\alpha}{c}L^{s/2}(\tilde{y}^\delta_{n,\alpha,s} - x_0)],$$

where $\tilde{x}^\delta_{0,\alpha,s} := x_0$, and $0 < c \leq \alpha$. Adaptive scheme of Perverzev and Schock is used for selection of regularization parameter α and error estimates derived are of optimal order.

In Chapter 7, we report on a method which comprises of Tikhonov regularization and Dynamical System Method (DSM)(Ramm (2007), Ramm (2005)) for approximately solving $KF(x) = f$. We apply the DSM for two cases of operator F(as in previous Chapters). Here we study both the iterative and continuous scheme of DSM and present the error analysis using the adaptive choice considered by Perverzev and Schock. The error estimates obtained are found to be of optimal order.

In Chapter 8, we end the thesis with some concluding remarks and also give the scope for future work.

Chapter 2

TWO STEP NEWTON-TIKHONOV METHOD

In this Chapter we present a combination of modified Newton method and Tikhonov regularization for obtaining a stable approximate solution for nonlinear ill-posed Hammerstein type operator equations $KF(x) = f$. It is assumed that the available data is f^δ with $\|f - f^\delta\| \leq \delta$, $K : Z \to Y$ is a bounded linear operator and $F : X \to Z$ is a non-linear operator where X, Y, Z are Hilbert spaces. Precisely two cases of F are considered, in the first case it is assumed that $F'(x_0)^{-1}$ exist ($F'(x_0)$ is the Fr$\acute{e}$chet derivative of F at an initial guess x_0) and in the second case it is assumed that $F'(x_0)^{-1}$ doesnot exist but F is a monotone operator. The error bounds derived under a general source condition are of optimal order. And the regularization parameter is chosen according to the adaptive scheme of Perverzev and Schock (2005).

2.1 INTRODUCTION

The study of inverse (ill-posed) problems is an active area of research both theoretically and numerically as these problems arise from important physical and engineering applications (see Engl (1993), Neubauer (1988), Ramm (2005), Natterer (2001)). It can be quite challenging to solve such problems because of their ill-posed nature. Many of these problems can be characterized abstractly as

$$A(x) = f$$

where f denotes the data, A an abstract (ill-posed) operator and x the unknown solution. However, in practice, because of modelling, experimental and computational errors, f is only available as an approximation f^δ. Consequently, it is necessary to solve

$$A(x^\delta) = f^\delta$$

instead of

$$A(x) = f,$$

and, for given classes of operators A, examine how the errors $x^\delta - x$ depend on $f^\delta - f$.

Tikhonov's regularization (e.g., Engl *et al*. (2000)) method has been used extensively to stabilize the approximate solution of nonlinear ill-posed problems. In recent years, increased emphasis has been placed on iterative regularization procedures (Kaltenbacher *et al*. (2008), George and Nair (1997)) for the approximate solution of such problems.

This Chapter is devoted for the study of non-linear ill-posed Hammerstein type operator equations by the use of iterative regularization procedures. A method is proposed for which local-linear convergence is established theoretically and validated numerically. Recall that George (2006a), George (2006b), George and Nair (2008), George and Kunhanandan (2009), an equation of the form

$$(KF)x = f \tag{2.1.1}$$

is called a non-linear ill-posed Hammerstein type operator equation. Here $F : D(F) \subseteq X \to Z$, is a nonlinear operator, $K : Z \to Y$ is a bounded linear operator and X, Z, Y are Hilbert spaces with corresponding inner product $\langle ., . \rangle$ and norm $\|.\|$ respectively. A typical example of a Hammerstein type operator is the nonlinear integral operator

$$(Ax)(t) := \int_0^1 k(s,t) f(s, x(s)) ds$$

where $k(s,t) \in L^2([0,1] \times [0,1]),\ \ x \in L^2[0,1]$ and $t \in [0,1]$.

The above integral operator A admits a representation of the form $A = KF$ where $K : L^2[0,1] \to L^2[0,1]$ is a linear integral operator with kernel $k(t,s)$ defined as

$$Kx(t) = \int_0^1 k(t,s)x(s)ds$$

and $F : D(F) \subseteq L^2[0,1] \to L^2[0,1]$ is a nonlinear superposition operator (cf. Krasnoselskii *et al*. (1976)) defined as

$$Fx(s) = f(s, x(s)). \tag{2.1.2}$$

George and his collaborators (George (2006a), George (2006b), George and Nair (2008), George and Kunhanandan (2009)), studied ill-posed Hammerstein type equation extensively under some assumptions on the Fr$\acute{e}$chet derivative of F. Precisely, in George (2006a), George and Nair (2008), it is assumed that $F'(x_0)^{-1}$ exists and in George and Kunhanandan (2009) it is assumed that $F'(x)^{-1}$ exists for all $x \in B_r(x_0)$ (Here $B_r(x_0)$ stands for ball of radius r around x_0).

Throughout this thesis it is assumed that the available data is f^δ with

$$\|f - f^\delta\| \leq \delta \tag{2.1.3}$$

and hence one has to consider the equation

$$(KF)x = f^\delta \tag{2.1.4}$$

instead of (2.1.1). Observe that the solution x of (2.1.4) can be obtained by solving

$$Kz = f^\delta \tag{2.1.5}$$

for z and then solving the non-linear problem

$$F(x) = z. \tag{2.1.6}$$

For solving (2.1.6), George and Kunhanandan (2009) considered the sequence defined iteratively by

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - F'(x^\delta_{n,\alpha})^{-1}(F(x^\delta_{n,\alpha}) - z^\delta_\alpha)$$

where $x^\delta_{0,\alpha} := x_0$,

$$z^\delta_\alpha = (K^*K + \alpha I)^{-1}K^*(f^\delta - KF(x_0)) + F(x_0) \tag{2.1.7}$$

and obtained local quadratic convergence.

Recall that a sequence (x_n) in X with $\lim x_n = x^*$ is said to be convergent of order $p > 1$, if there exist positive reals a, b, such that for all $n \in N$

$$\|x_n - x^*\| \leq ae^{-bp^n}. \tag{2.1.8}$$

If the sequence (x_n) has the property that $\|x_n - x^*\| \leq aq^n$, $0 < q < 1$, then (x_n) is said to be linearly convergent. For an extensive discussion of convergence rate see Kelley (1995).

George and Nair (2008), studied the modified Lavrentiev regularization

$$z_\alpha^\delta = (K + \alpha I)^{-1}(f^\delta - KF(x_0))$$

for obtaining an approximate solution of (2.1.5) when K is a positive self-adjoint operator and considered the modified Newton's iterations,

$$x_{n,\alpha}^\delta = x_{n-1,\alpha}^\delta - F'(x_0)^{-1}(F(x_{n-1,\alpha}^\delta) - F(x_0) - z_\alpha^\delta)$$

for solving (2.1.6). In fact in George and Nair (2008) and George and Kunhanandan (2009), a solution $\hat{x}$ of (2.1.1) is called an x_0-minimum norm solution if it satisfies

$$\|F(\hat{x}) - F(x_0)\| := \min\{\|F(x) - F(x_0)\| : KF(x) = f, x \in D(F)\}. \tag{2.1.9}$$

We also assume throughout that the solution $\hat{x}$ satisfies (2.1.9). In all these papers (George (2006a), George (2006b), George and Nair (2008), George and Kunhanandan (2009)), it is assumed that the ill-posedness of (2.1.1) is due to the nonclosedness of the range of linear operator K.

Recently, Argyros and Hilout (2010) studied the convergence analysis of Directional Two Step Newton Method in a Hilbert space for approximating a zero x^* of a differentiable function F defined on a convex subset D of a Hilbert space X, with values in $\mathbb{R}$. Motivated by this method we construct an iterative regularization method which is a combination of Two Step Newton method and Tikhonov regularization for approximating the solution $\hat{x}$ of (2.1.1) where we consider two cases of operator F:

The IFD Class (Invertible Fr$\acute{e}$chet Derivative) $F'(x_0)^{-1}$ exist and is a bounded operator, i.e., (2.1.6) is regular. Here $F'(x_0)$ denote the Fr$\acute{e}$chet derivative of F at an initial guess x_0. Consequently, in this situation, the ill-posedness of (2.1.1) is essentially due to the nonclosedness of the range of the linear operator K (see Ramm (2005), page 26).

EXAMPLE 2.1.1 *Let the function f in (2.1.2) be differentiable with respect to the second variable. Then, it follows that the operator F in (2.1.2) is Fr$\acute{e}$chet differentiable with*

$$[F'(x)u](t) = \partial_2 f(t, x(t))u(t), \qquad t \in [0, 1],$$

where $\partial_2 f(t, s)$ represents the partial derivative of f with respect to the second variable. If, in addition, the existence of a constant $\kappa_1 > 0$ is assumed such that, for all $x \in B_r(x_0)$ and for all $t \in [0, 1]$, $\partial_2 f(t, x(t)) \geq \kappa_1$, then $F'(u)^{-1}$ exist and is a bounded operator for all $u \in B_r(x_0)$.

The MFD Class (Monotone Fr$\acute{e}$chet Derivative) F is a monotone operator (Semenova (2010), Tautenhahn (1998))(i.e., $\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in D(F)$) and $F'(x_0)^{-1}$ does not exists. Consequently, in this situation, the ill-posedness of (2.1.1) is due to the ill-posedness of F as well as the nonclosedness of the range of the linear operator K.

EXAMPLE 2.1.2 *(Nair and Ravishankar (2008), Example 6.1) Let $F : L^2[0, 1] \to L^2[0, 1]$ be defined by*

$$F(x)(t) = K(x)(t) + f(t), \qquad x, f \in L^2[0, 1], \;\; t \in [0, 1]$$

where $K : L^2[0, 1] \to L^2[0, 1]$ is a compact linear operator such that range of K denoted by $R(K)$ is not closed and $\langle Kh, h\rangle \geq 0$ for $h \in L^2[0, 1]$. Then, $F(x) = y$ is ill-posed as K is a compact operator with non-closed range. The Fr$\acute{e}$chet derivative $F'(x)$ of F is given by

$$F'(x)h = Kh, \qquad \forall x, h \in L^2[0, 1].$$

Now, since $\langle Kh, h\rangle \geq 0$ for all $h \in L^2[0, 1]$, F is monotone. Further $F'(u)^{-1}$ does not exists for any $u \in L^2[0, 1]$. Consequently, the operator KF, with K and F as defined above is an example of the MFD Class.

One of the advantages of (approximately) solving (2.1.5) and (2.1.6) to obtain an approximate solution for (2.1.4) is that, one can use any regularization method for linear ill-posed equations, for solving (2.1.5) and any method for solving (2.1.6). In fact in this chapter we consider Tikhonov regularization for approximately solving (2.1.5) and we consider a modified two step Newton method for solving (2.1.6). Note that the regularization parameter α is chosen according to the adaptive method considered by Pereverzev and Schock (2005) for the linear ill-posed operator equations (2.1.5) and the same parameter α is used for solving the non-linear operator equation (2.1.6), so the choice of the regularization parameter is not depending on the non-linear operator F, this is another advantage over treating (2.1.4) as a single non-linear operator equation.

This chapter is organized as follows. Preparatory results are given in Section 2.2 and Section 2.3 comprises of the Two Step Newton-Tikhonov Method (TSNTM) for case I (IFD Class) and case II(MFD Class) with the error analysis.

2.2 PREPARATORY RESULTS

In this section we consider Tikhonov regularized solution z_α^δ defined in (2.1.7) and obtain an a priori and an a posteriori error estimate for $\|F(\hat{x}) - z_\alpha^\delta\|$. The following assumption is required to obtain the error estimate .

ASSUMPTION 2.2.1 *There exists a continuous, strictly monotonically increasing function* $\varphi : (0, a] \to (0, \infty)$ *with* $a \geq \|K^2\|$ *satisfying;*

- $\underset{\lambda\to 0}{lim} \varphi(\lambda) = 0$

- $$\underset{\lambda \geq 0}{sup} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha), \qquad \forall \alpha \in (0, a]$$

  and

- *there exists* $v \in X, \|v\| \leq 1$ *such that*

$$F(\hat{x}) - F(x_0) = \varphi(K^*K)v.$$

THEOREM 2.2.2 *(cf. George and Kunhanandan (2009), section 3) Let* z_α^δ *be as in (2.1.7) and Assumption 2.2.1 holds. Then*

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq c_\phi(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}). \tag{2.2.1}$$

Proof. Let z_α^δ be as in (2.1.7). We observe that

$$\begin{aligned}\|F(\hat{x}) - z_\alpha^\delta\| &\leq \|F(\hat{x}) - z_\alpha\| + \|z_\alpha - z_\alpha^\delta\| \\ &\leq \|F(\hat{x}) - z_\alpha\| + \frac{\delta}{\sqrt{\alpha}} \end{aligned} \tag{2.2.2}$$

and

$$\begin{aligned} F(\hat{x}) - z_\alpha &= F(\hat{x}) - F(x_0) - (K^* + \alpha I)^{-1}K^*K[F(\hat{x}) - F(x_0)] \\ &= [I - (K^*K + \alpha I)^{-1}K^*K](F(\hat{x}) - F(x_0)) \\ &= \alpha(K^*K + \alpha I)^{-1}(F(\hat{x}) - F(x_0)). \end{aligned}$$

So by Assumption 2.2.1

$$\begin{aligned}\|F(\hat{x}) - z_\alpha\| &\leq \|\alpha(K^*K + \alpha I)^{-1}\varphi(K^*K)v\| \\ &\leq \underset{0<\lambda\leq\|K\|^2}{sup} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha}\|v\| \leq \varphi(\alpha). \end{aligned} \tag{2.2.3}$$

Therefore by (2.2.2) and (2.2.3), we have

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}.$$

2.2.1 A priori choice of the parameter

Note that the estimate $\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}$ in (2.2.1) is of optimal order for the choice $\alpha := \alpha_\delta$ which satisfies $\varphi(\alpha_\delta) = \frac{\delta}{\sqrt{\alpha_\delta}}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$. Then we have $\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$ and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)). \tag{2.2.4}$$

So the relation (2.2.1) leads to $\|F(\hat{x}) - z_\alpha^\delta\| \leq 2\psi^{-1}(\delta)$.

2.2.2 An adaptive choice of the parameter

The error estimate in the above Theorem has optimal order with respect to δ. Unfortunately, an a priori parameter choice (2.2.4) cannot be used in practice since the smoothness properties of the unknown solution $\hat{x}$ reflected in the function φ are generally unknown. There exist many parameter choice strategies in the literature, for example see

Bakushinsky and Smirnova (2005), George and Nair (1993), Raus (1984), George and Nair (1998), Groetsch and Guacaneme (1987), Guacaneme (1990), and Tautenhahn (2002).

Pereverzev and Schock (2005) considered an adaptive selection of the parameter which does not involve even the regularization method in an explicit manner. In this method the regularization parameter α_i are selected from some finite set $D_M := \{\alpha_i = \alpha_0\mu^{2i}, i = 0, 1, 2, ..., M\}, \mu > 1$ and the corresponding regularized solution, say $z^\delta_{\alpha_i}$ are studied on-line. George and Nair (2008), George and Kunhanandan (2009), considered the adaptive method of Pereverzev and Schock (2005) for selecting the regularization parameter for approximately solving Hammerstein-type operator equations. The selection of numerical value k for the parameter α according to the adaptive choice is performed using the rule;

$$k := \max\{i : \alpha_i \in D^+_M\} \tag{2.2.5}$$

where $D^+_M = \{\alpha_i \in D_M : \|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, j = 0, 1, 2, ..., i-1\}$. Let

$$l := \max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\} < N. \tag{2.2.6}$$

We will be using the following theorem from George and Kunhanandan (2009) for our error analysis.

THEOREM 2.2.3 *(cf. George and Kunhanandan (2009), Theorem 4.3) Let l be as in (2.2.6), k be as in (2.2.5) and $z^\delta_{\alpha_k}$ be as in (2.1.7) with $\alpha = \alpha_k$. Then $l \leq k$ and*

$$\|F(\hat{x}) - z^\delta_{\alpha_k}\| \leq (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta).$$

Proof. Observe that, to prove $l \leq k$, it is enough to prove that, for $i = 1, 2, \cdots, N$

$$\varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}} \Longrightarrow \|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, \forall j = 0, 1, 2, \cdots, i.$$

For $j \leq i$,

$$\begin{aligned}
\|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| &\leq \|z^\delta_{\alpha_i} - F(\hat{x})\| + \|F(\hat{x}) - z^\delta_{\alpha_j}\| \\
&\leq [\varphi(\alpha_i) + \frac{\delta}{\sqrt{\alpha_i}}] + [\varphi(\alpha_j) + \frac{\delta}{\sqrt{\alpha_j}}] \\
&\leq \frac{2\delta}{\sqrt{\alpha_i}} + \frac{2\delta}{\sqrt{\alpha_j}} \\
&\leq \frac{4\delta}{\sqrt{\alpha_j}}.
\end{aligned}$$

This proves the relation $l \leq k$. Now since $\sqrt{\alpha_{l+m}} = \mu^m \sqrt{\alpha_l}$, by using triangle inequality successively, we obtain

$$\begin{aligned}
\|F(\hat{x}) - z^\delta_{\alpha_k}\| &\leq \|F(\hat{x}) - z^\delta_{\alpha_l}\| + \sum_{j=l+1}^{k} \frac{4\delta}{\sqrt{\alpha_{j-1}}} \\
&\leq \|F(\hat{x}) - z^\delta_{\alpha_l}\| + \sum_{m=0}^{k-l-1} \frac{4\delta}{\sqrt{\alpha_l}\mu^m} \\
&\leq \|F(\hat{x}) - z^\delta_{\alpha_l}\| + (\frac{\mu}{\mu-1})\frac{4\delta}{\sqrt{\alpha_l}}.
\end{aligned}$$

Therefore by Assumption 2.3.1 and (2.2.6) we have

$$\begin{aligned}
\|F(\hat{x}) - z^\delta_{\alpha_k}\| &\leq c_\phi[\varphi(\alpha_l) + \frac{\delta}{\sqrt{\alpha_l}}] + (\frac{\mu}{\mu-1})\frac{4\delta}{\sqrt{\alpha_l}} \\
&\leq (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta).
\end{aligned}$$

The last step follows from the inequality $\sqrt{\alpha_\delta} \leq \sqrt{\alpha_{l+1}} \leq \mu\sqrt{\alpha_l}$ and $\frac{\delta}{\sqrt{\alpha_\delta}} = \psi^{-1}(\delta)$. This completes the proof.

2.3 CONVERGENCE ANALYSIS

2.3.1 TSNTM for IFD Class

In this subsection, for an initial guess $x_0 \in X$, we consider the sequence y^δ_{n,α_k} and x^δ_{n,α_k} defined iteratively by

$$y^\delta_{n,\alpha_k} = x^\delta_{n,\alpha_k} - F'(x_0)^{-1}(F(x^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k}) \tag{2.3.1}$$

and

$$x^\delta_{n+1,\alpha_k} = y^\delta_{n,\alpha_k} - F'(x_0)^{-1}(F(y^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k}), \tag{2.3.2}$$

where $x^\delta_{0,\alpha_k} = x_0$, for obtaining an approximation for $x^\delta_{\alpha_k}$ the solution of $F(x) = z^\delta_{\alpha_k}$. We will be using the following parameters;

$$M \geq \|F'(x_0)\|;$$

$$\beta := \|F'(x_0)^{-1}\|;$$

$$k_0 < \frac{1}{4}\min\{1, \frac{1}{\beta}\};$$

$$\delta_0 < \frac{\sqrt{\alpha_0}}{4k_0\beta};$$

$$\rho := \frac{1}{M}(\frac{1}{4k_0\beta} - \frac{\delta_0}{\sqrt{\alpha_0}});$$

$$\gamma_\rho := \beta[M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}];$$

and

$$e^\delta_{n,\alpha_k} := \|y^\delta_{n,\alpha_k} - x^\delta_{n,\alpha_k}\|, \qquad \forall n \geq 0. \tag{2.3.3}$$

For convenience, we use the notation x_n, y_n and e_n for x^δ_{n,α_k}, y^δ_{n,α_k} and e^δ_{n,α_k} respectively.

Further we define

$$q := k_0 r, \quad r \in (r_1, r_2) \tag{2.3.4}$$

where

$$r_1 = \frac{1 - \sqrt{1 - 4k_0\gamma_\rho}}{2k_0}$$

and

$$r_2 = min\{\frac{1}{k_0}, \frac{1 + \sqrt{1 - 4k_0\gamma_\rho}}{2k_0}\}.$$

Note that r is well defined because $\gamma_\rho \leq \frac{1}{4k_0}$. Further we use the relation $e_0 \leq \gamma_\rho$ for proving our results, which can be seen as follows;

$$\begin{aligned}
e_0 = \|y_0 - x_0\| &= \|F'(x_0)^{-1}(F(x_0) - z^\delta_{\alpha_k})\| \\
&\leq \|F'(x_0)^{-1}\|\|(F(x_0) - z^\delta_{\alpha_k})\| \\
&\leq \beta\|F(x_0) - z_{\alpha_k} + z_{\alpha_k} - z^\delta_{\alpha_k}\| \\
&\leq \beta[\|F(x_0) - F(\hat{x})\| + \|z_{\alpha_k} - z^\delta_{\alpha_k}\|] \\
&\leq \beta[M\rho + \frac{\delta}{\sqrt{\alpha}}] \\
&\leq \beta[M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}] \\
&= \gamma_\rho.
\end{aligned}$$

We need the following Assumption for the convergence of iterative method and to obtain the error estimate.

ASSUMPTION 2.3.1 *(cf. Semenova (2010), Assumption 3 (A3)) There exist a constant* $k_0 > 0,\ r > 0$ *such that for every* $x, u \in B(x_0, r) \cup B(\hat{x}, r) \subseteq D(F)$ *and* $v \in X$ *there exists an element* $\Phi(x, u, v) \in X$ *such that*

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v),\ \ \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|.$$

THEOREM 2.3.2 *Let* $e_n,\ q < 1$ *be as in (2.3.3), (2.3.4) respectively and* $\{x_n\}, \{y_n\}$ *be as in (2.3.2), (2.3.1) respectively with* $\delta \in (0, \delta_0]$. *Then by Assumption 2.3.1 and Theorem 2.2.3* $x_n, y_n \in B_r(x_0)$ *and the following estimates hold for all* $n \geq 0$.

(a) $\|x_{n+1} - y_n\| \leq q\|y_n - x_n\|$;

(b) $\|y_{n+1} - x_{n+1}\| \leq q^2\|y_n - x_n\|$;

(c) $e_n \leq q^{2n}\gamma_\rho, \qquad \forall n \geq 0.$

Proof. Suppose $x_n, y_n \in B_r(x_0)$. Then

$$\begin{aligned} x_{n+1} - y_n &= y_n - x_n - F'(x_0)^{-1}(F(y_n) - F(x_n)) \\ &= F'(x_0)^{-1}[F'(x_0)(y_n - x_n) - (F(y_n) - F(x_n))] \\ &= F'(x_0)^{-1}\int_0^1 [F'(x_0) - F'(x_n + t(y_n - x_n))](y_n - x_n)dt \end{aligned}$$

and hence by Assumption 2.3.1, we have

$$\|x_{n+1} - y_n\| \ \leq\ k_0 r\|y_n - x_n\| \leq q\|y_n - x_n\|.$$

This proves (a). To prove (b) we observe that

$$\begin{aligned} e_{n+1} = \|y_{n+1} - x_{n+1}\| &= \|x_{n+1} - y_n - F'(x_0)^{-1}(F(x_{n+1}) - F(y_n))\| \\ &= \|F'(x_0)^{-1}\int_0^1 [F'(x_0) - F'(y_n + t(x_{n+1} - y_n)] \\ &\quad dt(x_{n+1} - y_n)\| \\ &\leq k_0 r\|y_n - x_{n+1}\| \\ &\leq q^2\|x_n - y_n\|. \end{aligned}$$

The last but one step follows from Assumption 2.3.1 and the last step follows from (a). This completes the proof of (b), and (c) follows from (b). Now we shall show that $x_n, y_n \in B_r(x_0)$ by induction. For $n = 1,$ by (a), we have

$$\begin{aligned} \|x_1 - y_0\| &\leq \frac{k_0}{2}\|y_0 - x_0\|^2 \\ &\leq k_0 r e_0. \end{aligned} \tag{2.3.5}$$

So by triangular inequality and (2.3.5)

$$\begin{aligned} \|x_1 - x_0\| &\leq \|x_1 - y_0\| + \|y_0 - x_0\| \\ &\leq (1+q)e_0 \\ &\leq \frac{e_0}{1-q} \leq \frac{\gamma_\rho}{1-q} \\ &\leq r, \end{aligned} \tag{2.3.6}$$

i.e., $x_1 \in B_r(x_0)$. Observe that by (b), we have

$$\|y_1 - x_1\| \leq q^2 e_0. \tag{2.3.7}$$

Therefore by (2.3.6), (2.3.7) and triangle inequality,

$$\begin{aligned} \|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| \\ &\leq (1+q+q^2)e_0 \\ &\leq \frac{e_0}{1-q} \leq \frac{\gamma_\rho}{1-q} \\ &\leq r, \end{aligned}$$

i.e., $y_1 \in B_r(x_0)$. Suppose $x_m, y_m \in B_r(x_0)$, for some $m \geq 0$. Then

$$\begin{aligned}
\|x_{m+1} - x_0\| &\leq \|x_{m+1} - x_m\| + \|x_m - x_{m-1}\| + \cdots + \|x_1 - x_0\| \\
&\leq (q+1)e_m + (q+1)e_{m-1} + \cdots + (q+1)e_0 \\
&\leq (q+1)(e_m + e_{m-1} + \cdots + e_0) \\
&\leq (q+1)(q^{2m} + q^{2(m-1)} + \cdots + 1)e_0 \\
&\leq (q+1)\frac{1 - (q^{2m+1})}{1 - q^2} e_0 \\
&\leq \frac{e_0}{1-q} \leq \frac{\gamma_\rho}{1-q} \\
&\leq r,
\end{aligned}$$

i.e., $x_{m+1} \in B_r(x_0)$ and

$$\begin{aligned}
\|y_{m+1} - x_0\| &\leq \|y_{m+1} - x_{m+1}\| + \|x_{m+1} - x_0\| \\
&\leq q^2 e_m + (q+1)e_m + (q+1)e_{m-1} + \cdots + (q+1)e_0 \\
&\leq (q^2 + q + 1)e_m + (q+1)e_{m-1} + \cdots + (q+1)e_0 \\
&\leq (q^{2(m+1)} + \cdots + q^3 + q^2 + q + 1)e_0 \\
&\leq \frac{e_0}{1-q} \leq \frac{\gamma_\rho}{1-q} \\
&\leq r,
\end{aligned}$$

i.e., $y_{m+1} \in B_r(x_0)$. Thus by induction $x_n, y_n \in B_r(x_0)$, for all $n \geq 0$. This completes the proof of the Theorem.

The main result of this section is the following Theorem.

THEOREM 2.3.3 *Let $\{x_n\}$ and $\{y_n\}$ be as in (2.3.2) and (2.3.1) respectively and assumptions of Theorem 2.3.2 hold. Then (x_n) is a Cauchy sequence in $B_r(x_0)$ and converges to $x_{\alpha_k}^\delta \in \overline{B_r(x_0)}$. Further $F(x_{\alpha_k}^\delta) = z_{\alpha_k}^\delta$ and*

$$\|x_n - x_{\alpha_k}^\delta\| \leq C_1 q^{2n}$$

where $C_1 = \frac{\gamma_\rho}{1-q}$.

Proof. Using the relation (b) and (c) of Theorem 2.3.2, we obtain

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq \sum_{i=0}^{i=m-1}\|x_{n+i+1}-x_{n+i}\| \\ &\leq \sum_{i=0}^{i=m-1}(1+q)e_{n+i} \\ &\leq \sum_{i=0}^{i=m-1}(1+q)q^{2(n+i)}e_0 \\ &= (1+q)q^{2n}e_0+(1+q)q^{2(n+1)}e_0+.....+(1+q)q^{2(n+m)}e_0 \\ &\leq (1+q)q^{2n}(1+q^2+q^{2(2)}+\cdots+q^{2m})e_0 \\ &\leq q^{2n}[\frac{1-(q^2)^{m+1}}{1-q}]\gamma_\rho \\ &\leq C_1q^{2n}.\end{aligned}$$

Thus (x_n) is a Cauchy sequence in $B_r(x_0)$ and hence it converges, say to $x_{\alpha_k}^\delta \in \overline{B_r(x_0)}$. Observe that

$$\begin{aligned}\|F(x_n)-z_{\alpha_k}^\delta\| &= \|F'(x_0)(x_n-y_n)\| \\ &\leq \|F'(x_0)\|\|x_n-y_n\| \\ &\leq Me_n \leq Mq^{2n}\gamma_\rho. \qquad (2.3.8)\end{aligned}$$

Now by letting $n\to\infty$ in (2.3.8) we obtain $F(x_{\alpha_k}^\delta)=z_{\alpha_k}^\delta$. This completes the proof.

Hereafter we assume that $\|\hat{x}-x_0\|<\rho\leq r$.

THEOREM 2.3.4 *Suppose that $k_0r<1$ and the hypothesis of Assumption 2.3.1 holds. Then*

$$\|\hat{x}-x_{\alpha_k}^\delta\| \leq \frac{\beta}{1-k_0r}\|F(\hat{x})-z_{\alpha_k}^\delta\|.$$

Proof. Note that $k_0r<1$ and by Assumption 2.3.1, we have

$$\begin{aligned}\|\hat{x}-x_{\alpha_k}^\delta\| &\leq \|\hat{x}-x_{\alpha_k}^\delta+F'(x_0)^{-1}[F(x_{\alpha_k}^\delta)-F(\hat{x})+F(\hat{x})-z_{\alpha_k}^\delta]\| \\ &\leq \|F'(x_0)^{-1}[F'(x_0)(\hat{x}-x_{\alpha_k}^\delta)+F(x_{\alpha_k}^\delta)-F(\hat{x})]\| \\ &\quad +\|F'(x_0)^{-1}(F(\hat{x})-z_{\alpha_k}^\delta)\| \\ &\leq k_0\|x_0-\hat{x}-t(x_{\alpha_k}^\delta-\hat{x})\|\|\hat{x}-x_{\alpha_k}^\delta\|+\beta\|F(\hat{x})-z_{\alpha_k}^\delta\| \\ &\leq k_0r\|\hat{x}-x_{\alpha_k}^\delta\|+\beta\|F(\hat{x})-z_{\alpha_k}^\delta\|.\end{aligned}$$

This completes the proof. The following Theorem is a consequence of Theorem 2.3.3 and Theorem 2.3.4.

THEOREM 2.3.5 *Let x_n be as in (2.3.2). Suppose the hypotheses of Theorem 2.3.3 and Theorem 2.3.4 hold. Then*

$$\|\hat{x} - x_n\| \leq C_1 q^{2n} + \frac{\beta}{1 - k_0 r}\|F(\hat{x}) - z_{\alpha_k}^{\delta}\|$$

where C_1 is as in Theorem 2.3.3.

Observe that from section 2.2, $l \leq k$ and $\alpha_\delta \leq \alpha_{l+1} \leq \mu\alpha_l$, we have

$$\frac{\delta}{\sqrt{\alpha_k}} \leq \frac{\delta}{\sqrt{\alpha_l}} \leq \mu\frac{\delta}{\sqrt{\alpha_\delta}} = \mu\varphi(\alpha_\delta) = \mu\psi^{-1}(\delta).$$

This leads to the following theorem,

THEOREM 2.3.6 *Let x_n be as in (2.3.2), assumptions in Theorem 2.2.3 and Theorem 2.3.5 hold. Let*

$$n_k := \min\{n : q^{2n} \leq \frac{\delta}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - x_{n_k}\| = O(\psi^{-1}(\delta)).$$

2.3.2 TSNTM for MFD Class

In this subsection we assume that X is a real Hilbert space. Then the iterative method for MFD class is defined as:

$$\tilde{y}_{n,\alpha_k}^{\delta} = \tilde{x}_{n,\alpha_k}^{\delta} - R(\tilde{x}_{0,\alpha_k}^{\delta})^{-1}[F(\tilde{x}_{n,\alpha_k}^{\delta}) - z_{\alpha_k}^{\delta} + \frac{\alpha_k}{c}(\tilde{x}_{n,\alpha_k}^{\delta} - \tilde{x}_0)] \tag{2.3.9}$$

and

$$\tilde{x}_{n+1,\alpha_k}^{\delta} = \tilde{y}_{n,\alpha_k}^{\delta} - R(\tilde{x}_{0,\alpha_k}^{\delta})^{-1}[F(\tilde{y}_{n,\alpha_k}^{\delta}) - z_{\alpha_k}^{\delta} + \frac{\alpha_k}{c}(\tilde{y}_{n,\alpha_k}^{\delta} - \tilde{x}_0)] \tag{2.3.10}$$

where $\tilde{x}_{0,\alpha_k}^{\delta} := x_0$ is the initial guess and $R(x_0) := F'(x_0) + \frac{\alpha_k}{c}I$, with $c \leq \alpha_k < 1$. First we prove that $\tilde{x}_{n,\alpha_k}$ converges to the zero x_{c,α_k}^{δ} of

$$F(x) + \frac{\alpha_k}{c}(x - x_0) = z_{\alpha_k}^{\delta} \tag{2.3.11}$$

and then we prove that x_{c,α_k}^{δ} is an approximation for $\hat{x}$.

Let

$$\tilde{e}^{\delta}_{n,\alpha_k} := \|\tilde{y}^{\delta}_{n,\alpha_k} - \tilde{x}^{\delta}_{n,\alpha_k}\|, \qquad \forall n \geq 0. \tag{2.3.12}$$

For the sake of simplicity, we use the notation $\tilde{x}_n$, $\tilde{y}_n$ and $\tilde{e}_n$ for $\tilde{x}^{\delta}_{n,\alpha_k}$, $\tilde{y}^{\delta}_{n,\alpha_k}$ and $\tilde{e}^{\delta}_{n,\alpha_k}$ respectively.

Hereafter we assume that $\|\hat{x} - x_0\| < \rho \leq \tilde{r}$ where

$$\rho < \frac{1}{M}(1 - \frac{\delta_0}{\sqrt{\alpha_0}})$$

with $\delta_0 < \sqrt{\alpha_0}$ and $\tilde{r} \in (\tilde{r_1}, \tilde{r_2})$ where

$$\tilde{r_1} = \frac{1 - \sqrt{1 - 4k_0\tilde{\gamma_\rho}}}{2k_0}$$

and

$$\tilde{r_2} = min\{\frac{1}{k_0}, \frac{1 + \sqrt{1 - 4k_0\tilde{\gamma_\rho}}}{2k_0}\}.$$

Let

$$\tilde{\gamma}_\rho := M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}.$$

and

$$q_1 = k_0\tilde{r}. \tag{2.3.13}$$

THEOREM 2.3.7 *Let $\tilde{e}_n$ and $q_1 < 1$ be defined as in equation (2.3.12) and (2.3.13) respectively, $\tilde{x}_n$ and $\tilde{y}_n$ be as in (2.3.10) and (2.3.9) respectively with $\delta \in (0, \delta_0]$ and suppose Assumption 2.3.1 holds. Then we have the following:*

(a) $\|\tilde{x}_n - \tilde{y}_{n-1}\| \leq q_1\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;

(b) $\|\tilde{y}_n - \tilde{x}_n\| \leq q_1^2\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;

(c) $\tilde{e}_n \leq q_1^{2n}\tilde{\gamma}_\rho, \qquad \forall n \geq 0.$

Proof. Suppose $\tilde{x}_n, \tilde{y}_n \in B_{\tilde{r}}(x_0)$, then

$$\begin{aligned}
\tilde{x}_n - \tilde{y}_{n-1} &= \tilde{y}_{n-1} - \tilde{x}_{n-1} - R(x_0)^{-1}(F(\tilde{y}_{n-1}) - F(\tilde{x}_{n-1}) \\
&\quad + \frac{\alpha_k}{c}(\tilde{y}_{n-1} - \tilde{x}_{n-1})) \\
&= R(x_0)^{-1}[R(x_0)(\tilde{y}_{n-1} - \tilde{x}_{n-1}) \\
&\quad -(F(\tilde{y}_{n-1}) - F(\tilde{x}_{n-1})) - \frac{\alpha_k}{c}(\tilde{y}_{n-1} - \tilde{x}_{n-1})] \\
&= R(x_0)^{-1}\int_0^1 [F'(x_0) - (F(\tilde{y}_{n-1}) - F(\tilde{x}_{n-1}))] \\
&\quad \times(\tilde{y}_{n-1} - \tilde{x}_{n-1})dt.
\end{aligned}$$

Now since $\|R(x_0)^{-1}F'(x_0)\| \leq 1$, the proof of (a) follows as in Theorem 2.3.2. Again observe that

$$\begin{aligned}
\tilde{e}_n &\leq \|\tilde{x}_n - \tilde{y}_{n-1} - R(x_0)^{-1}(F(\tilde{x}_n) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{x}_n - x_0))\| \\
&\quad +\|R(x_0)^{-1}(F(\tilde{y}_{n-1}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{y}_{n-1} - x_0))\| \\
&\leq \|R(x_0)^{-1}[R(x_0)(\tilde{x}_n - \tilde{y}_{n-1}) - (F(\tilde{x}_n) - F(\tilde{y}_{n-1})) - \frac{\alpha_k}{c}(\tilde{x}_n - \tilde{y}_{n-1})]\| \\
&\leq \|R(x_0)^{-1}\int_0^1 [F'(x_0) - (F(\tilde{x}_n) - F(\tilde{y}_{n-1}))]dt(\tilde{x}_n - \tilde{y}_{n-1})\|.
\end{aligned}$$

So the remaining part of the proof is analogous to the proof of Theorem 2.3.2.

THEOREM 2.3.8 *Let $\tilde{y}_n$ and $\tilde{x}_n$ be as in (2.3.9) and (2.3.10) respectively and assumptions of Theorem 2.3.7 holds. Then $(\tilde{x}_n)$ is a Cauchy sequence in $B_{\tilde{r}}(x_0)$ and converges to $x^\delta_{c,\alpha_k} \in \overline{B_{\tilde{r}}(x_0)}$. Further $F(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^\delta_{c,\alpha_k} - x_0) = z^\delta_{\alpha_k}$ and*

$$\|\tilde{x}_n - x^\delta_{c,\alpha_k}\| \leq \tilde{C}_1 q_1^{2n}$$

where $\tilde{C}_1 = \frac{\tilde{\gamma}_\rho}{1-q_1}$.

Proof. Analogous to the proof of Theorem 2.3.3, one can prove that $(\tilde{x}_n)$ is a Cauchy sequence in $B_{\tilde{r}}(x_0)$ and hence it converges, say to $x^\delta_{c,\alpha_k} \in \overline{B_{\tilde{r}}(x_0)}$ and

$$\begin{aligned} \|F(\tilde{x}_n) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{x}_n - x_0)\| &= \|R(\tilde{x}_0)(\tilde{x}_n - \tilde{y}_n)\| \\ &\leq \|R(\tilde{x}_0)\|\|\tilde{x}_n - \tilde{y}_n\| \\ &\leq (\|F'(x_0)\| + \frac{\alpha_k}{c})\tilde{e}_n \\ &\leq (\|F'(x_0)\| + \frac{\alpha_k}{c})q_1^{2n}\tilde{e}_0 \\ &\leq (\|F'(x_0)\| + \frac{\alpha_k}{c})q_1^{2n}\tilde{\gamma}_\rho. \end{aligned} \quad (2.3.14)$$

Now by letting $n \to \infty$ in (2.3.14) we obtain $F(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^\delta_{c,\alpha_k} - x_0) = z^\delta_{\alpha_k}$. This completes the proof.

The following assumptions are needed in addition to the earlier assumptions for our convergence analysis.

ASSUMPTION 2.3.9 *There exists a continuous, strictly monotonically increasing function $\varphi_1 : (0, b] \to (0, \infty)$ with $b \geq \|F'(x_0)\|$ satisfying;*

- $\lim_{\lambda \to 0} \varphi_1(\lambda) = 0,$
- $$\sup_{\lambda \geq 0} \frac{\alpha\varphi_1(\lambda)}{\lambda + \alpha} \leq \varphi_1(\alpha) \qquad \forall \lambda \in (0, b]$$

  and
- *there exists $v \in X$ with $\|v\| \leq 1$ (cf. Mahale and Nair (2009)) such that*

$$x_0 - \hat{x} = \varphi_1(F'(x_0))v.$$

ASSUMPTION 2.3.10 *For each $x \in B_{\tilde{r}}(x_0)$ there exists a bounded linear operator $G(x, x_0)$ (see Ramm et al. (2003)) such that*

$$F'(x) = F'(x_0)G(x, x_0)$$

with $\|G(x, x_0)\| \leq k_2$.

Let $k_2 < \frac{1-k_0\tilde{r}}{1-c}$ with $\tilde{r} < \frac{1}{k_0}$ and for the sake of simplicity assume that $\varphi_1(\alpha) \leq \varphi(\alpha)$ for $\alpha > 0$.

THEOREM 2.3.11 *Suppose x^δ_{c,α_k} is the solution of (2.3.11) and Assumption 2.3.1, 2.3.9 and 2.3.10 hold. Then*

$$\|\hat{x} - x^\delta_{c,\alpha_k}\| = O(\psi^{-1}(\delta)).$$

Proof. Note that $c(F(x^\delta_{c,\alpha_k}) - z^\delta_{\alpha_k}) + \alpha_k(x^\delta_{c,\alpha_k} - x_0) = 0$, so

$$\begin{aligned}(F'(x_0)+\alpha_k I)(x^\delta_{c,\alpha_k} - \hat{x}) &= (F'(x_0)+\alpha_k I)(x^\delta_{c,\alpha_k} - \hat{x}) \\ &\quad -c(F(x^\delta_{c,\alpha_k}) - z^\delta_{\alpha_k}) - \alpha_k(x^\delta_{c,\alpha_k} - x_0) \\ &= \alpha_k(x_0 - \hat{x}) - c(F(\hat{x}) - z^\delta_{\alpha_k}) + F'(x_0)(x^\delta_{c,\alpha_k} - \hat{x}) \\ &\quad -c[F(x^\delta_{c,\alpha_k}) - F(\hat{x})].\end{aligned}$$

Thus

$$\begin{aligned}\|x^\delta_{c,\alpha_k} - \hat{x}\| &\leq \|\alpha_k(F'(x_0)+\alpha_k I)^{-1}(x_0 - \hat{x})\| + \|(F'(x_0)+\alpha_k I)^{-1} \\ &\quad c(F(\hat{x}) - z^\delta_{\alpha_k})\| + \|(F'(x_0)+\alpha_k I)^{-1}[F'(x_0)(x^\delta_{c,\alpha_k} - \hat{x}) \\ &\quad -c(F(x^\delta_{c,\alpha_k}) - F(\hat{x}))]\| \\ &\leq \|\alpha_k(F'(x_0)+\alpha_k I)^{-1}(x_0 - \hat{x})\| + \|F(\hat{x}) - z^\delta_{\alpha_k}\| + \Gamma \qquad (2.3.15)\end{aligned}$$

where $\Gamma := \|(F'(x_0)+\alpha_k I)^{-1}\int_0^1 [F'(x_0) - cF'(\hat{x} + t(x^\delta_{c,\alpha_k} - \hat{x})](x^\delta_{c,\alpha_k} - \hat{x})dt\|$. So by Assumption 2.3.10, we obtain

$$\begin{aligned}\Gamma &\leq \|(F'(x_0)+\alpha_k I)^{-1}\int_0^1 [F'(x_0) - F'(\hat{x} + t(x^\delta_{c,\alpha_k} - \hat{x}))] \\ &\quad \times(x^\delta_{c,\alpha_k} - \hat{x})dt\| + (1-c)\|(F'(x_0)+\alpha_k I)^{-1}F'(x_0) \\ &\quad \times \int_0^1 G(\hat{x} + t(x^\delta_{c,\alpha_k} - \hat{x}), x_0)(x^\delta_{c,\alpha_k} - \hat{x})dt\| \\ &\leq k_0\tilde{r}\|x^\delta_{c,\alpha_k} - \hat{x}\| + (1-c)k_2\|x^\delta_{c,\alpha_k} - \hat{x}\| \qquad (2.3.16)\end{aligned}$$

and hence by (2.3.15) and (2.3.16) we have

$$\begin{aligned}\|x^\delta_{c,\alpha_k} - \hat{x}\| &\leq \frac{\|\alpha_k(F'(x_0)+\alpha_k I)^{-1}(x_0 - \hat{x})\| + \|F(\hat{x}) - z^\delta_{\alpha_k}\|}{1-(1-c)k_2 - k_0\tilde{r}} \\ &\leq \frac{\varphi_1(\alpha_k) + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1-(1-c)k_2 - k_0\tilde{r}}. \qquad (2.3.17)\end{aligned}$$

That completes the proof of the theorem.

The following Theorem is a consequence of Theorem 2.3.8 and Theorem 2.3.11.

THEOREM 2.3.12 *Let $\tilde{x}_n$ be as in (2.3.10), assumptions in Theorem 2.3.8 and Theorem 2.3.11 hold. Then*

$$\|\hat{x} - \tilde{x}_n\| \leq \tilde{C}_1 q_1^{2n} + O(\psi^{-1}(\delta))$$

where $\tilde{C}_1$ is as in Theorem 2.3.8.

THEOREM 2.3.13 *Let* $\tilde{x}_n$ *be as in (2.3.2), assumptions in Theorem 2.2.3, Theorem 2.3.8 and Theorem 2.3.11 hold. Let*

$$n_k := \min\{n : q_1^{2n} \leq \frac{\delta}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - \tilde{x}_{n_k}\| = O(\psi^{-1}(\delta)).$$

Chapter 3

DISCRETIZED TWO STEP NEWTON-TIKHONOV METHOD

An iteratively regularized projection scheme for the ill-posed Hammerstein type operator equation $KF(x) = f$ has been considered. The proposed method is the finite dimensional realization of the method considered in Chapter 2. Precisely, the method is a combination of discretized Tikhonov regularization and modified Newton's method. The analysis in finite dimensional setting is carried out for both IFD and MFD Class. Adaptive choice of the parameter suggested by Perverzev and Schock(2005) is employed in this chapter also for selecting the regularization parameter α. An algorithm and numerical examples are given to test the reliability of the method.

3.1 INTRODUCTION

For an implementable method for solving (2.1.1) needs numerical calculations in finite dimensional spaces. One of the approaches in this regard is through discretization (see Engl *et al*. (2000), Page 63). Here the regularization is achieved by a finite dimensional approximation alone. Regularization of ill-posed problems by projection methods can be found in literature, for e.g in Groetsch and Neubauer (1988), Kaltenbacher *et al*. (2008), Krisch (1996), Perverzev and Probdorf (2000).

This Chapter is concerned with the finite dimensional realization of a method considered in Chapter 2 for (nonlinear) Hammerstein-type equation (2.1.1).

The organization of this Chapter is as follows: Preparatory results are given in Section 3.2. Section 3.3 comprises the proposed iterative method for IFD Class and MFD Class in finite dimensional setting. Section 3.4 deals with the algorithm for implementing the proposed method. Numerical examples are given in Section 3.5.

3.2 PRELIMINARIES

Let $V_1 \subseteq V_2 \subseteq V_3 \subseteq .....$ be a sequence of finite-dimensional subspaces of X with $\overline{U_{n\in\mathbb{N}}V_n} = X$ and $P_h,\ (h = \frac{1}{n})$ is the orthogonal projector of X onto V_n. Let

$$\varepsilon_h := \|K(I - P_h)\|,$$

$$\tau_h := \|F'(x)(I - P_h)\|, \quad \forall x \in D(F).$$

Let $\{b_h : h > 0\}$ is such that $\underset{h\to 0}{lim} \frac{\|(I-P_h)x_0\|}{b_h} = 0,\ \underset{h\to 0}{lim} \frac{\|(I-P_h)F(x_0)\|}{b_h} = 0$ and $\underset{h\to 0}{lim} b_h = 0$. We assume that $\varepsilon_h \to 0$ and $\tau_h \to 0$ as $h \to 0$. The above assumption is satisfied if, $P_h \to I$ pointwise and if K and $F'(x)$ are compact operators. Further we assume that $\varepsilon_h < \varepsilon_0$, $\tau_h \leq \tau_0$, $b_h \leq b_0$.

The discretized Tikhonov regularization method for solving equation $Kz = f^\delta$ consists of solving the equation

$$(P_hK^*KP_h + \alpha P_h)(z_\alpha^{h,\delta} - P_hF(x_0)) = P_hK^*[f^\delta - KF(x_0)] \tag{3.2.1}$$

for $z_\alpha^{h,\delta}$.

Throughout the Chapter we assume that F possess a uniformly bounded Fr$\acute{e}$chet derivative for all $x \in D(F)$ i.e., $\|F'(x)\| \leq M$, for some $M > 0$.

THEOREM 3.2.1 *Suppose Assumption 2.2.1 holds. Let $z_\alpha^{h,\delta}$ be as in (3.2.1) and $b_h \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$. Then*

$$\|F(\hat{x}) - z_\alpha^{h,\delta}\| \leq C(\varphi(\alpha) + (\frac{\delta + \varepsilon_h}{\sqrt{\alpha}})), \tag{3.2.2}$$

where $C = \frac{1}{2}\max\{M\rho, 1\} + 1.$

Proof. Let $z_\alpha = (K^*K + \alpha I)^{-1}K^*(f - KF(x_0)) + F(x_0)$. Then

$$\begin{aligned}
\|z_\alpha - z_\alpha^h\| &= \|(K^*K+\alpha I)^{-1}K^*(f-KF(x_0)) - (P_hK^*KP_h \\
&\quad +\alpha I)^{-1}P_hK^*(f-KF(x_0)) + F(x_0) - P_hF(x_0)\| \\
&\leq \|(P_hK^*KP_h + \alpha P_h)^{-1}P_hK^*(KP_h - K)(K^*K \\
&\quad +\alpha I)^{-1}K^*K[F(\hat{x}) - F(x_0)]\| + \|(I-P_h)F(x_0)\| \\
&\leq \|F(\hat{x}) - F(x_0)\|\frac{\varepsilon_h}{2\sqrt{\alpha}} + b_h \\
&\leq \|\int_0^1 F'(x_0 + t(\hat{x}-x_0))(\hat{x}-x_0)dt\|\frac{\varepsilon_h}{2\sqrt{\alpha}} + b_h \\
&\leq M\rho\frac{\varepsilon_h}{2\sqrt{\alpha}} + b_h
\end{aligned} \tag{3.2.3}$$

and

$$\begin{aligned}
\|z_\alpha^h - z_\alpha^{h,\delta}\| &= \|(P_hK^*KP_h + \alpha I)^{-1}P_hK^*(f - f^\delta)\| \\
&\leq \frac{\delta}{2\sqrt{\alpha}}.
\end{aligned} \tag{3.2.4}$$

Now the result follows from (3.2.3), (3.2.4), (2.2.3) and the following triangle inequality;

$$\|F(\hat{x}) - z_\alpha^{h,\delta}\| \leq \|F(\hat{x}) - z_\alpha\| + \|z_\alpha - z_\alpha^h\| + \|z_\alpha^h - z_\alpha^{h,\delta}\|.$$

3.2.1 A priori choice of the parameter

Note that the estimate $\varphi(\alpha) + \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$ in (3.2.2) is of optimal order for the choice $\alpha := \alpha(\delta, h)$ which satisfies $\varphi(\alpha(\delta,h)) = \frac{\delta+\varepsilon_h}{\sqrt{\alpha(\delta,h)}}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq a$. Then we have $\delta + \varepsilon_h = \sqrt{\alpha(\delta,h)}\varphi(\alpha(\delta,h)) = \psi(\varphi(\alpha(\delta,h)))$ and

$$\alpha(\delta,h) = \varphi^{-1}(\psi^{-1}(\delta+\varepsilon_h)).$$

So the relation (3.2.2) leads to $\|F(\hat{x}) - z_\alpha^{h,\delta}\| \leq 2C\psi^{-1}(\delta+\varepsilon_h)$.

3.2.2 An adaptive choice of the parameter

Let

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\}$$

be the set of possible values of the parameter α.

Let

$$l := \max\{i : \varphi(\alpha_i) \leq \frac{\delta + \varepsilon_h}{\sqrt{\alpha_i}}\} < N, \tag{3.2.5}$$

$$k = \max\{i : \alpha_i \in D_N^+\} \tag{3.2.6}$$

where $D_N^+ = \{\alpha_i \in D_N : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \leq \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}, j = 0, 1, 2, ..., i-1\}$.

THEOREM 3.2.2 *(cf. George and Kunhanandan (2009), Theorem 2.5) Let l be as in (3.2.5), k be as in (3.2.6) and $z_{\alpha_k}^{h,\delta}$ be as in (3.2.1) with $\alpha = \alpha_k$. Then $l \leq k$ and*

$$\|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\| \leq C(2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta + \varepsilon_h),$$

where C is as in Theorem 3.2.1.

3.3 CONVERGENCE ANALYSIS

3.3.1 DTSNTM for IFD Class

Let

$$\|F'(x_0)^{-1}\| := \beta_1. \tag{3.3.1}$$

The discretized iterative scheme of (2.3.1) and (2.3.2) for approximately solving (2.1.6) with $z_{\alpha_k}^{h,\delta}$ in place of z is defined as:

$$y_{n,\alpha_k}^{h,\delta} = x_{n,\alpha_k}^{h,\delta} - P_h F'(x_{0,\alpha_k}^{h,\delta})^{-1} P_h(F(x_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta}), \tag{3.3.2}$$

$$x_{n+1,\alpha_k}^{h,\delta} = y_{n,\alpha_k}^{h,\delta} - P_h F'(x_{0,\alpha_k}^{h,\delta})^{-1} P_h(F(y_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta}) \tag{3.3.3}$$

where $x_{0,\alpha_k}^{h,\delta} := P_h x_0$ and $z_{\alpha_k}^{h,\delta}$ is as defined in (3.2.1).

Note: Observe that if $b_0 < \frac{1}{k_0}$ then $F'(P_h x_0)^{-1}$ exists and is bounded. This can be seen as follows:

$$\begin{aligned} \|F'(P_h x_0)^{-1}\| &= \sup_{\|v\|\leq 1} \|[I + F'(x_0)^{-1}(F'(P_h x_0) - F'(x_0))]^{-1} F'(x_0)^{-1} v\| \\ &\leq \sup_{\|v\|\leq 1} \frac{\|F'(x_0)^{-1}\|}{1 - \|F'(x_0)^{-1}(F'(P_h x_0) - F'(x_0))v\|} \end{aligned} \tag{3.3.4}$$

Using Assumption 2.3.1, we get

$$\|F'(x_0)^{-1}(F'(P_h x_0) - F'(x_0))v\| \leq k_0 b_0. \tag{3.3.5}$$

And hence by (3.3.1), (3.3.4) and (3.3.5) we have, $\|F'(P_h x_0)^{-1}\| \leq \frac{\beta_1}{1-k_0 b_0}$.

Thus without loss of generality we assume that

$$\|F'(P_h x_0)^{-1}\| \leq \beta, \tag{3.3.6}$$

for some $\beta > 0$.

LEMMA 3.3.1 *Let $b_0 < \frac{1}{k_0}$ and (3.3.6) hold. Then*

$$\|P_h F'(P_h x_0)^{-1} P_h F'(P_h x_0)\| \leq 1 + \beta\tau_0.$$

Proof. One can see that

$$\begin{aligned}
\|P_h F'(x_{0,\alpha_k}^{h,\delta})^{-1} P_h F'(P_h x_0)\| &= \sup_{\|v\|\leq 1} \|P_h F'(P_h x_0)^{-1} P_h F'(P_h x_0) v\| \\
&\leq \sup_{\|v\|\leq 1} \|P_h F'(P_h x_0)^{-1} P_h F'(P_h x_0) \\
&\quad \times (P_h + I - P_h)v\| \\
&\leq \sup_{\|v\|\leq 1} \|[P_h F'(P_h x_0)^{-1} P_h F'(P_h x_0) P_h]v\| + \\
&\quad \sup_{\|v\|\leq 1} \|P_h F'(P_h x_0)^{-1} P_h F'(P_h x_0)(I - P_h)v\| \\
&\leq 1 + \beta\tau_h \leq 1 + \beta\tau_0.
\end{aligned}$$

Let

$$e_{n,\alpha_k}^{h,\delta} := \|y_{n,\alpha_k}^{h,\delta} - x_{n,\alpha_k}^{h,\delta}\|, \qquad \forall n \geq 0. \tag{3.3.7}$$

For our further analysis, we assume that,

$$k_0 < \frac{1}{4\beta(1+\beta\tau_0)}$$

and

$$\delta_0 + \varepsilon_0 < \frac{1}{4\beta k_0(1+\beta\tau_0)(M+1+C_{M\rho})}\sqrt{\alpha_0}$$

where $C_{M\rho} = \frac{1}{2}\max\{M\rho, 1\}$.

Let $\|\hat{x} - x_0\| \leq \rho$ with

$$\rho < \frac{1}{M}[\frac{1}{4\beta k_0(1+\beta\tau_0)} - (M+1+C_{M\rho})\frac{\delta_0+\varepsilon_0}{\sqrt{\alpha_0}}],$$

$$\gamma_\rho := \beta[M\rho + (M+1+C_{M\rho})(\frac{\varepsilon_0+\delta_0}{\sqrt{\alpha_0}})] \leq \frac{1}{4k_0(1+\beta\tau_0)} \tag{3.3.8}$$

and let

$$q_p := (1+\beta\tau_0)k_0 r, \;\; r \in (r_1, r_2) \tag{3.3.9}$$

where

$$r_1 = \frac{1 - \sqrt{1 - 4k_0(1+\beta\tau_0)\gamma_\rho}}{2k_0(1+\beta\tau_0)}$$

and

$$r_2 = min\{\frac{1}{k_0(1+\beta\tau_0)}, \frac{1 + \sqrt{1 - 4k_0(1+\beta\tau_0)\gamma_\rho}}{2k_0(1+\beta\tau_0)}\}.$$

Note that by (3.3.8), r is well defined and $q_p < 1$.

LEMMA 3.3.2 *Let $z^{h,\delta}_{\alpha_k}$ and $e^{h,\delta}_{0,\alpha_k}$ be as defined in (3.2.1) and (3.3.7) respectively. Suppose (3.3.6) holds and $b_h \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}$, then $e^{h,\delta}_{0,\alpha_k} \leq \gamma_\rho$.*

Proof. Observe that

$$\begin{aligned} e^{h,\delta}_{0,\alpha_k} &= \|y^{h,\delta}_{0,\alpha_k} - P_h x_0\| \\ &= \|P_h F'(P_h x_0)^{-1} P_h (F(P_h x_0) - z^{h,\delta}_{\alpha_k})\| \\ &\leq \|P_h F'(P_h x_0)^{-1} P_h\| \|F(P_h x_0) - z^{h,\delta}_{\alpha_k}\| \\ &\leq \beta\|F(P_h x_0) - z^h_{\alpha_k} + z^h_{\alpha_k} - z^{h,\delta}_{\alpha_k}\| \\ &\leq \beta(\|F(P_h x_0) - z^h_{\alpha_k}\| + \|z^h_{\alpha_k} - z^{h,\delta}_{\alpha_k}\|) \end{aligned} \tag{3.3.10}$$

and

$$\begin{aligned} \|F(P_h x_0) - z^h_{\alpha_k}\| &\leq \|F(P_h x_0) - F(x_0)\| + \|F(x_0) - z_{\alpha_k}\| + \|z_{\alpha_k} - z^h_{\alpha_k}\| \\ &\leq \|\int_0^1 F'(x_0 + t(P_h x_0 - x_0))(P_h x_0 - x_0)dt\| \\ &\quad + \|(K^*K + \alpha_k I)^{-1} K^*K(F(\hat{x}) - F(x_0))\| + \|z_{\alpha_k} - z^h_{\alpha_k}\| \\ &\leq M b_h + \|F(\hat{x}) - F(x_0)\| + \|z_{\alpha_k} - z^h_{\alpha_k}\| \\ &\leq M b_h + M\rho + \|z_{\alpha_k} - z^h_{\alpha_k}\|. \end{aligned} \tag{3.3.11}$$

Therefore by (3.3.10), (3.3.11), (3.2.3) and (3.2.4) we have

$$\begin{aligned} e^{h,\delta}_{0,\alpha_k} &\leq \beta[(M+1)b_h + (1+\frac{\varepsilon_h}{2\sqrt{\alpha_k}})M\rho + \frac{\delta}{2\sqrt{\alpha_k}}] \\ &\leq \beta[(M+1)\frac{\varepsilon_h+\delta}{\sqrt{\alpha_k}} + M\rho + \frac{M\rho\varepsilon_h}{2\sqrt{\alpha_k}} + \frac{\delta}{\sqrt{\alpha_k}}] \\ &\leq \beta[M\rho + (M+1+C_{M\rho})(\frac{\varepsilon_0+\delta_0}{\sqrt{\alpha_0}})] \\ &\leq \gamma_\rho. \end{aligned}$$

THEOREM 3.3.3 *Let* $e^{h,\delta}_{n,\alpha_k}$, q_p *be as in (3.3.7), (3.3.9) respectively. Let* $\{y^{h,\delta}_{n,\alpha_k}\}$, $\{x^{h,\delta}_{n,\alpha_k}\}$ *be as in (3.3.2), (3.3.3) respectively with* $\delta \in (0,\delta_0]$, *and* $\varepsilon_h \in (0,\varepsilon_0]$. *Then under the assumptions of Theorem 3.2.2 and Lemma 3.3.1, the following hold for all* $n \geq 0$.

(a) $\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\| \leq (1+q_p)\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\|$;

(b) $\|y^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{n,\alpha_k}\| \leq q_p^2\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\|$;

(c) $e^{h,\delta}_{n,\alpha_k} \leq q_p^{2n}\gamma_\rho$ *and*

(d) $x^{h,\delta}_{n,\alpha_k},\ y^{h,\delta}_{n,\alpha_k} \in B_r(P_hx_0), \quad \forall n \geq 0.$

Proof. Suppose $x^{h,\delta}_{n,\alpha_k},\ \ y^{h,\delta}_{n,\alpha_k} \in B_r(P_hx_0)$, then

$$\begin{aligned} x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k} &= y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k} - P_hF'(P_hx_0)^{-1}P_h \\ &\quad \times(F(y^{h,\delta}_{n-1,\alpha_k}) - F(x^{h,\delta}_{n-1,\alpha_k})) \\ &= P_hF'(P_hx_0)^{-1}[P_hF'(P_hx_0)(y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}) \\ &\quad -P_h(F(y^{h,\delta}_{n-1,\alpha_k}) - F(x^{h,\delta}_{n-1,\alpha_k}))] \\ &= P_hF'(P_hx_0)^{-1}P_h\int_0^1[F'(P_hx_0) - F'(x^{h,\delta}_{n-1,\alpha_k} \\ &\quad +t(y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}))](y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k})dt \end{aligned}$$

and hence by Assumption 2.3.1 and Lemma 3.3.1, we have

$$\begin{aligned} \|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\| &\leq (1+\beta\tau_0)\|\int_0^1\Phi(P_hx_0, x^{h,\delta}_{n-1,\alpha_k} + t(y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}), \\ &\quad y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k})dt\| \\ &\leq (1+\beta\tau_0)k_0r\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\|. \end{aligned} \tag{3.3.12}$$

Now we obtain (a) from (3.3.12) and the triangle inequality;

$$\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\| \leq \|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\| + \|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\|.$$

To prove (b) we observe that

$$\begin{aligned} e^{h,\delta}_{n,\alpha_k} = \|y^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{n,\alpha_k}\| &= \|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k} - P_hF'(P_hx_0)^{-1}P_h(F(x^{h,\delta}_{n,\alpha_k}) \\ &\quad -F(y^{h,\delta}_{n-1,\alpha_k}))\| \\ &= \|P_hF'(P_hx_0)^{-1}[P_hF'(P_hx_0)(x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}) \\ &\quad -P_h(F(x^{h,\delta}_{n,\alpha_k}) - F(y^{h,\delta}_{n-1,\alpha_k}))]\| \\ &\leq (1+\beta\tau_0)k_0r\|x^{h,\delta}_{n,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\|. \end{aligned} \tag{3.3.13}$$

Hence from (3.3.12), (3.3.13) and (a) we have

$$\begin{aligned} e^{h,\delta}_{n,\alpha_k} &\leq ((1+\beta\tau_0)k_0r)^2\|x^{h,\delta}_{n-1,\alpha_k} - y^{h,\delta}_{n-1,\alpha_k}\| \\ &\leq q_p^2 e^{h,\delta}_{n-1,\alpha_k}. \end{aligned}$$

This completes the proof of (b). Since $e^{h,\delta}_{0,\alpha_k} \leq \gamma_\rho$, (c) follows from (b). Now by induction, as in Chapter 2 one can prove that $x^{h,\delta}_{n,\alpha_k}, y^{h,\delta}_{n,\alpha_k} \in B_r(P_hx_0), \ \forall n \geq 0$. This completes the proof of the Theorem.

THEOREM 3.3.4 *Let $y^{h,\delta}_{n,\alpha_k}$ and $x^{h,\delta}_{n,\alpha_k}$ be as in (3.3.2) and (3.3.3) respectively. If Theorem 3.3.3 holds, then $(x^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_r(P_hx_0)$ and converges to $x^{h,\delta}_{\alpha_k} \in \overline{B_r(P_hx_0)}$. Further $P_hF(x^{h,\delta}_{\alpha_k}) = z^{h,\delta}_{\alpha_k}$ and*

$$\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{\alpha_k}\| \leq C_2 q_p^{2n}$$

where $C_2 = \frac{\gamma_\rho}{1-q_p}$.

Proof. Analogous to the proof of Theorem 2.3.3 in Chapter 2 one can show that $x^{h,\delta}_{n,\alpha_k}$ is a Cauchy sequence in $B_r(P_hx_0)$ converging to $x^{h,\delta}_{\alpha_k} \in \overline{B_r(P_hx_0)}$ and

$$\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{\alpha_k}\| \leq C_2 q_p^{2n}.$$

Further observe that,

$$\begin{aligned}\|P_h(F(x_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta})\| &= \|P_hF'(P_hx_0)(x_{n,\alpha_k}^{h,\delta} - y_{n,\alpha_k}^{h,\delta})\| \\ &\leq \|F'(P_hx_0)\|\|x_{n,\alpha_k}^{h,\delta} - y_{n,\alpha_k}^{h,\delta}\| \\ &\leq Me_{n,\alpha_k}^{h,\delta} \leq Mq_p^{2n}\gamma_\rho. \end{aligned} \tag{3.3.14}$$

Now by letting $n \to \infty$ in (3.3.14) we obtain $P_hF(x_{\alpha_k}^{h,\delta}) = z_{\alpha_k}^{h,\delta}$. This completes the proof.

Next we assume that

$$\|\hat{x} - x_0\| < \rho \leq r.$$

THEOREM 3.3.5 *Suppose the hypothesis of Assumption 2.2.1 and 2.3.1 hold. Then*

$$\|\hat{x} - x_{\alpha_k}^{h,\delta}\| \leq \frac{\beta}{(1-q_p)}\|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\|.$$

Proof. One can see that

$$\begin{aligned}\|\hat{x} - x_{\alpha_k}^{h,\delta}\| &= \|\hat{x} - x_{\alpha_k}^{h,\delta} + P_hF'(P_hx_0)^{-1}P_h[F(x_{\alpha_k}^{h,\delta}) - F(\hat{x}) + F(\hat{x}) - z_{\alpha_k}^{h,\delta}]\| \\ &\leq \|P_hF'(P_hx_0)^{-1}[P_hF'(P_hx_0)(\hat{x} - x_{\alpha_k}^{h,\delta}) + P_h(F(x_{\alpha_k}^{h,\delta}) \\ &\quad -F(\hat{x}))]\| + \|P_hF'(P_hx_0)^{-1}P_h(F(\hat{x}) - z_{\alpha_k}^{h,\delta})\| \\ &\leq \|P_hF'(P_hx_0)^{-1}P_h\int_0^1[F'(P_hx_0) - F'(\hat{x} + t(x_{\alpha_k}^{h,\delta} - \hat{x}))] \\ &\quad \times(\hat{x} - x_{\alpha_k}^{h,\delta})dt\| + \|P_hF'(P_hx_0)^{-1}P_h(F(\hat{x}) - z_{\alpha_k}^{h,\delta})\| \\ &\leq (1+\beta\tau_0)k_0r\|\hat{x} - x_{\alpha_k}^{h,\delta}\| + \beta\|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\|.\end{aligned}$$

The last step follows from Assumption 2.3.1, Lemma 3.3.1 and the relation $\|P_hx_0 - \hat{x} - t(x_{\alpha_k}^{h,\delta} - \hat{x})\| \leq r$. This completes the proof.

The following theorem is a consequence of Theorem 3.3.4 and Theorem 3.3.5.

THEOREM 3.3.6 *Let $x_{n,\alpha_k}^{h,\delta}$ be as in (3.3.3), assumptions in Theorem 3.3.4 and Theorem 3.3.5 hold. Then*

$$\|\hat{x} - x_{n,\alpha_k}^{h,\delta}\| \leq C_2q_p^{2n} + \frac{\beta}{(1-q_p)}\|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\|$$

where C_2 is as in Theorem 3.3.4.

Now since $l \leq k$ and $\alpha_\delta \leq \alpha_{l+1} \leq \mu\alpha_l$ we have

$$\frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_l}} \leq \mu\frac{\delta+\varepsilon_h}{\sqrt{\alpha_\delta}} = \mu\varphi(\alpha(\delta,h)) = \mu\psi^{-1}(\delta+\varepsilon_h).$$

This leads to the following theorem,

THEOREM 3.3.7 *Let $x^{h,\delta}_{n,\alpha_k}$ be as in (3.3.3), assumptions in Theorem 3.3.6 hold. Let*

$$n_k := \min\{n : q_p^{2n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - x^{h,\delta}_{n_k,\alpha_k}\| = O(\psi^{-1}(\delta+\varepsilon_h)).$$

3.3.2 DTSNTM for MFD Class

In this subsection we consider the discretized form of (2.3.9) and (2.3.10) as;

$$\tilde{y}^{h,\delta}_{n,\alpha_k} = \tilde{x}^{h,\delta}_{n,\alpha_k} - R(\tilde{x}^{h,\delta}_{0,\alpha_k})^{-1}P_h[F(\tilde{x}^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{0,\alpha_k})] \tag{3.3.15}$$

and

$$\tilde{x}^{h,\delta}_{n+1,\alpha_k} = \tilde{y}^{h,\delta}_{n,\alpha_k} - R(\tilde{x}^{h,\delta}_{0,\alpha_k})^{-1}P_h[F(\tilde{y}^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{y}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{0,\alpha_k})], \tag{3.3.16}$$

where $R(\tilde{x}^{h,\delta}_{0,\alpha_k}) := P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k})P_h + \frac{\alpha_k}{c}P_h$, $\tilde{x}^{h,\delta}_{0,\alpha_k} := P_hx_0$ and $c \leq \alpha_k$.

First we consider the iterative scheme defined by (3.3.15) and (3.3.16) for approximating the zero $x^{h,\delta}_{c,\alpha_k}$ of

$$P_h(F(x) + \frac{\alpha_k}{c}(x - x_0)) = P_hz^{h,\delta}_{\alpha_k} \tag{3.3.17}$$

and then show that $x^{h,\delta}_{c,\alpha_k}$ is an approximation to the solution $\hat{x}$ of (2.1.1).

Note that with the above notation

$$\begin{aligned}
\|R(\tilde{x}^{h,\delta}_{0,\alpha_k})^{-1}P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k})\| &= \|(P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k})P_h + \frac{\alpha_k}{c}P_h)^{-1}P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k})\| \\
&= \|(P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k})P_h + \frac{\alpha_k}{c}P_h)^{-1}P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k}) \\
&\quad [P_h + I - P_h]\| \\
&\leq \|(P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k})P_h + \frac{\alpha_k}{c}P_h)^{-1}P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k})P_h\| \\
&\quad + \|(P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k})P_h + \frac{\alpha_k}{c}P_h)^{-1}P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k}) \\
&\quad (I - P_h)\| \\
&\leq 1 + \frac{\|P_hF'(\tilde{x}^{h,\delta}_{0,\alpha_k})(I - P_h)\|}{\frac{\alpha_k}{c}} \\
&\leq 1 + \tau_h \leq 1 + \tau_0. \qquad (3.3.18)
\end{aligned}$$

Let

$$\tilde{e}^{h,\delta}_{n,\alpha_k} := \|\tilde{y}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{n,\alpha_k}\|, \qquad \forall n \geq 0 \qquad (3.3.19)$$

and let $\delta_0 + \varepsilon_0 < (\frac{2}{2M+3})\sqrt{\alpha_0}$ and $\|\hat{x} - x_0\| \leq \rho$, with

$$\rho < \frac{1}{M}(1 - (\frac{3}{2} + M)\frac{\delta_0 + \varepsilon_0}{\sqrt{\alpha_0}})$$

and

$$\tilde{\gamma}_\rho := M\rho + (\frac{3}{2} + M)(\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}}).$$

Further let

$$\tilde{\gamma}_\rho < \frac{1}{4k_0(1+\tau_0)},$$

$$\tilde{r}_1 = \frac{1 - \sqrt{1 - 4(1+\tau_0)k_0\tilde{\gamma}_\rho}}{2(1+\tau_0)k_0}$$

and

$$\tilde{r}_2 = min\{\frac{1}{(1+\tau_0)k_0}, \frac{1 + \sqrt{1 - 4(1+\tau_0)k_0\tilde{\gamma}_\rho}}{2(1+\tau_0)k_0}\}.$$

For $\tilde{r} \in (\tilde{r}_1, \tilde{r}_2)$, let

$$\tilde{q_p} = (1+\tau_0)k_0\tilde{r}, \qquad (3.3.20)$$

then $\tilde{q_p} < 1$.

LEMMA 3.3.8 *Let $z^{h,\delta}_{\alpha_k}$ and $\tilde{e}^{h,\delta}_{0,\alpha_k}$ be as defined in (3.2.1) and (3.3.19) respectively. Then $\tilde{e}^{h,\delta}_{0,\alpha_k} \le \tilde{\gamma}_\rho$.*

Proof. Observe that

$$\begin{aligned}
\tilde{e}^{h,\delta}_{0,\alpha_k} &= \|\tilde{y}^{h,\delta}_{0,\alpha_k} - P_h x_0\| \\
&= \|(P_h F'(P_h x_0) + \frac{\alpha_k}{c})^{-1} P_h (F(P_h x_0) - z^{h,\delta}_{\alpha_k})\| \\
&\le \|F(P_h x_0) - z^h_{\alpha_k} + z^h_{\alpha_k} - z^{h,\delta}_{\alpha_k}\| \\
&\le \|F(P_h x_0) - z^h_{\alpha_k}\| + \|z^h_{\alpha_k} - z^{h,\delta}_{\alpha_k}\|.
\end{aligned} \tag{3.3.21}$$

Therefore by (3.3.21), (3.3.11), (3.2.3) and (3.2.4) we have

$$\begin{aligned}
\tilde{e}^{h,\delta}_{0,\alpha_k} &\le (M+1)b_h + (1 + \frac{\varepsilon_h}{2\sqrt{\alpha_k}})M\rho + \frac{\delta}{2\sqrt{\alpha_k}} \\
&\le (M+1)\frac{\varepsilon_h + \delta}{\sqrt{\alpha_k}} + M\rho + \frac{1}{2}\max\{M\rho, 1\}\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}} \\
&\le (M+1)\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}} + M\rho + \frac{\varepsilon_0 + \delta_0}{2\sqrt{\alpha_0}} \\
&\le \tilde{\gamma}_\rho.
\end{aligned}$$

THEOREM 3.3.9 *Let $\tilde{e}^{h,\delta}_{n,\alpha_k}$ and $\tilde{q}_p$ be as in equation (3.3.19) and (3.3.20) respectively, $\tilde{y}^{h,\delta}_{n,\alpha_k}$ and $\tilde{x}^{h,\delta}_{n,\alpha_k}$ be as defined in (3.3.15) and (3.3.16) respectively with $\delta \in (0, \delta_0]$ and $\varepsilon_h \in (0, \varepsilon_0]$. Then under the assumptions of Theorem 3.2.2, Lemma 3.3.8 and (3.3.18), $\tilde{x}^{h,\delta}_{n,\alpha_k}, \tilde{y}^{h,\delta}_{n,\alpha_k} \in B_{\tilde{r}}(P_h x_0)$ and the following estimates hold for all $n \ge 0$.*

(a) $\|\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{y}^{h,\delta}_{n-1,\alpha_k}\| \le \tilde{q}_p \|\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}\|$;

(b) $\|\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}\| \le (1 + \tilde{q}_p)\|\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}\|$;

(c) $\|\tilde{y}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{n,\alpha_k}\| \le \tilde{q}_p{}^2 \|\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}\|$;

(d) $\tilde{e}^{h,\delta}_{n,\alpha_k} \le \tilde{q}_p{}^{2n}\gamma_\rho, \quad \forall n \ge 0.$

Proof. Suppose $\tilde{x}^{h,\delta}_{n,\alpha_k}$, $\tilde{y}^{h,\delta}_{n,\alpha_k} \in B_{\tilde{r}}(P_h x_0)$. Then

$$\begin{aligned}\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{y}^{h,\delta}_{n-1,\alpha_k} &= \tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k} - R(\tilde{x}^{h,\delta}_{0,\alpha_k})^{-1} P_h(F(\tilde{y}^{h,\delta}_{n-1,\alpha_k}) \\ &\quad -F(\tilde{x}^{h,\delta}_{n-1,\alpha_k}) + \frac{\alpha_k}{c}(\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k})) \\ &= R(\tilde{x}^{h,\delta}_{0,\alpha_k})^{-1}[R(\tilde{x}^{h,\delta}_{0,\alpha_k})(\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}) \\ &\quad -P_h(F(\tilde{y}^{h,\delta}_{n-1,\alpha_k}) - F(x^{h,\delta}_{n-1,\alpha_k})) - \frac{\alpha_k}{c}(\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k})] \\ &= R(\tilde{x}^{h,\delta}_{0,\alpha_k})^{-1} P_h \int_0^1 [F'(\tilde{x}^{h,\delta}_{0,\alpha_k}) - F'(\tilde{x}^{h,\delta}_{n-1,\alpha_k} \\ &\quad +t(\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}))] P_h(\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}) dt\end{aligned}$$

and hence by Assumption 2.3.1, Lemma 3.3.8 and (3.3.18) we have

$$\begin{aligned}\|\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{y}^{h,\delta}_{n-1,\alpha_k}\| &\leq (1+\tau_0)\| \int_0^1 \Phi(\tilde{x}^{h,\delta}_{0,\alpha_k}, \tilde{x}^{h,\delta}_{n-1,\alpha_k} + t(\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k})), \\ &\quad (\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}) dt\| \\ &\leq (1+\tau_0) k_0 \tilde{r} \|\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}\|.\end{aligned}$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}\| \leq \|\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{y}^{h,\delta}_{n-1,\alpha_k}\| + \|\tilde{y}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{n-1,\alpha_k}\|.$$

Again (c) follows from (a), Assumption 2.3.1 and (3.3.18) and the following expression

$$\tilde{e}^{h,\delta}_{n,\alpha_k} = \|R(\tilde{x}^{h,\delta}_{0,\alpha_k})^{-1} P_h \int_0^1 [F'(\tilde{x}^{h,\delta}_{0,\alpha_k}) - (F'(\tilde{x}^{h,\delta}_{n,\alpha_k} + t(\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{y}^{h,\delta}_{n-1,\alpha_k}))] dt (\tilde{x}^{h,\delta}_{n,\alpha_k} - \tilde{y}^{h,\delta}_{n-1,\alpha_k})\|$$

and (d) follows from (c). The remaining part of the proof is analogous to the proof of Theorem 2.3.2.

THEOREM 3.3.10 *Let $\tilde{y}^{h,\delta}_{n,\alpha_k}$ and $\tilde{x}^{h,\delta}_{n,\alpha_k}$ be as in (3.3.15) and (3.3.16) respectively and assumptions of Theorem 3.3.9 hold. Then $(\tilde{x}^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_{\tilde{r}}(P_h x_0)$ and converges, say to $x^{h,\delta}_{c,\alpha_k} \in \overline{B_{\tilde{r}}(P_h x_0)}$. Further $P_h[F(x^{h,\delta}_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^{h,\delta}_{c,\alpha_k} - x_0)] = z^{h,\delta}_{\alpha_k}$ and $\|\tilde{x}^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{c,\alpha_k}\| \leq \tilde{C}_2 \tilde{q}_p{}^{2n}$ where $\tilde{C}_2 = \frac{\tilde{\gamma}_\rho}{1-\tilde{q}_p}$.*

Proof. Analogous to the proof of Theorem 2.3.3 of Chapter 2 one can show that $(\tilde{x}^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_{\tilde{r}}(P_h x_0)$ and hence it converges, say to $x^{h,\delta}_{c,\alpha_k} \in \overline{B_{\tilde{r}}(P_h x_0)}$. Observe that from (3.3.15)

$$\begin{aligned} \|P_h(F(\tilde{x}^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k}) + \frac{\alpha_k}{c}(\tilde{x}^{h,\delta}_{n,\alpha_k} - P_h x_0)\| &= \|R(\tilde{x}^{h,\delta}_{0,\alpha_k})(\tilde{y}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{n,\alpha_k})\| \\ &\leq \|R(\tilde{x}^{h,\delta}_{0,\alpha_k})\|\|\tilde{y}^{h,\delta}_{n,\alpha_k} - \tilde{x}^{h,\delta}_{n,\alpha_k}\| \\ &\leq (\|P_h F'(\tilde{x}^{h,\delta}_{0,\alpha_k})P_h\| + \frac{\alpha_k}{c})\tilde{e}^{h,\delta}_{n,\alpha_k} \\ &\leq (M + \frac{\alpha_k}{c})\tilde{q}_p{}^{2n}\tilde{\gamma}_\rho. \end{aligned} \quad (3.3.22)$$

Now by letting $n \to \infty$ in (3.3.22) we obtain $P_h F(x^{h,\delta}_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^{h,\delta}_{c,\alpha_k} - P_h x_0) = z^{h,\delta}_{\alpha_k}$. This completes the proof.

REMARK 3.3.11 *Note that* $0 < \tilde{q}_p < 1$ *and hence the sequence* $(\tilde{x}^{h,\delta}_{n,\alpha_k})$ *converges linearly to* $x^{h,\delta}_{c,\alpha_k}$.

Next we use Assumptions 2.3.9 and 2.3.10 as in Chapter 2 to prove our further results in this section.

THEOREM 3.3.12 *Suppose* $x^{h,\delta}_{c,\alpha_k}$ *is the solution of (3.3.17) and in addition if* $\tau_0 < 1$, *then*

$$\|x^{h,\delta}_{c,\alpha_k} - x^{\delta}_{c,\alpha_k}\| \leq \frac{2}{1-\tau_0}(\frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}}). \quad (3.3.23)$$

Proof. Suppose x^{δ}_{c,α_k} and $x^{h,\delta}_{c,\alpha_k}$ are the solutions of (2.3.11) and (3.3.17) respectively, then by (2.3.11) we have,

$$P_h F(x^{\delta}_{c,\alpha_k}) + \frac{\alpha_k}{c}(P_h x^{h,\delta}_{c,\alpha_k} - P_h x_0) = P_h z^{\delta}_{\alpha_k}. \quad (3.3.24)$$

So from (3.3.17) and (3.3.24),

$$P_h[F(x^{h,\delta}_{c,\alpha_k}) - F(x^{\delta}_{c,\alpha_k})] + \frac{\alpha_k}{c}P_h(x^{h,\delta}_{c,\alpha_k} - x^{\delta}_{c,\alpha_k}) = P_h(z^{h,\delta}_{\alpha_k} - z^{\delta}_{\alpha_k}). \quad (3.3.25)$$

Let $M_f = \int_0^1 F'(x^{\delta}_{c,\alpha_k} + t(x^{h,\delta}_{c,\alpha_k} - x^{\delta}_{c,\alpha_k}))dt$. Then by (3.3.25) we have

$$P_h[M_f(x^{h,\delta}_{c,\alpha_k} - x^{\delta}_{c,\alpha_k})] + \frac{\alpha_k}{c}P_h(x^{h,\delta}_{c,\alpha_k} - x^{\delta}_{c,\alpha_k}) = P_h(z^{h,\delta}_{\alpha_k} - z^{\delta}_{\alpha_k})$$

and hence

$$\begin{aligned}\|x^{h,\delta}_{c,\alpha_k} - x^{\delta}_{c,\alpha_k}\| &\leq \|z^{h,\delta}_{\alpha_k} - z^{\delta}_{\alpha_k}\| + \|M_f(P_h - I)\| \|x^{h,\delta}_{c,\alpha_k} - x^{\delta}_{c,\alpha_k}\| \\ &\leq \|z^{h,\delta}_{\alpha_k} - z^{\delta}_{\alpha_k}\| + \tau_0 \|x^{h,\delta}_{c,\alpha_k} - x^{\delta}_{c,\alpha_k}\|.\end{aligned}$$

Thus

$$\begin{aligned}\|x^{h,\delta}_{c,\alpha_k} - x^{\delta}_{c,\alpha_k}\| &\leq \frac{1}{1-\tau_0}\|z^{h,\delta}_{\alpha_k} - z^{\delta}_{\alpha_k}\| \\ &\leq \frac{1}{1-\tau_0}[\|z^{h,\delta}_{\alpha_k} - z^{h}_{\alpha_k}\| + \|z^{h}_{\alpha_k} - z_{\alpha_k}\| \\ &\quad + \|z_{\alpha_k} - z^{\delta}_{\alpha_k}\|]. \qquad (3.3.26)\end{aligned}$$

Now the result follows from (3.2.3), (3.2.4), (3.3.26) and the relation

$$\|z_{\alpha_k} - z^{\delta}_{\alpha_k}\| \leq \frac{\delta}{2\sqrt{\alpha_k}}.$$

The following theorem is a consequence of Theorem 3.3.10, (2.3.17) and Theorem 3.3.12. We assume that $\tilde{r} < \frac{1}{k_0}$ and $k_2 < \frac{1-k_0\tilde{r}}{1-c}$ with $c < 1$.

THEOREM 3.3.13 *Let $\tilde{x}^{h,\delta}_{n,\alpha_k}$ be as in (3.3.16), assumptions in Theorem 3.3.10, Theorem 3.3.12 and (2.3.17) hold with $\tilde{r}$ in place of r. Then*

$$\|\hat{x} - \tilde{x}^{h,\delta}_{n,\alpha_k}\| \leq \tilde{C}_2 \tilde{q_p}^{2n} + \frac{\varphi_1(\alpha_k) + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta+\varepsilon_h)}{1-(1-c)k_2 - k_0\tilde{r}} + \frac{2}{1-\tau_0}(\frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}})$$

where $\tilde{C}_2$ is as in Theorem 3.3.10.

THEOREM 3.3.14 *Let $\tilde{x}^{h,\delta}_{n,\alpha_k}$ be as in (3.3.16) and assumptions in Theorem 3.3.13 hold. Further let $\varphi_1(\alpha_k) \leq \varphi(\alpha_k)$ and*

$$n_k := \min\{n : \tilde{q_p}^{2n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - \tilde{x}^{h,\delta}_{n_k,\alpha_k}\| = O(\psi^{-1}(\delta+\varepsilon_h)).$$

3.4 ALGORITHM

Note that for $i, j \in \{0, 1, 2, \cdots, N\}$

$$z_{\alpha_i}^{h,\delta} - z_{\alpha_j}^{h,\delta} = (\alpha_j - \alpha_i)(P_h K^* K P_h + \alpha_j I)^{-1}(P_h K^* K P_h + \alpha_i I)^{-1} P_h K^*(f^\delta - KF(x_0)).$$

Therefore the adaptive algorithm associated with the choice of the parameter specified in Theorems 3.2.2, 3.3.7 and 3.3.14 involve the following steps.

Part I:

- $\alpha_0 = (M + 1 + M\rho)^2(\delta + \varepsilon_h)^2$, $\mu > 1$
- $\alpha_i = \mu^{2i}\alpha_0$;
- solve for w_i:
$$(P_h K^* K P_h + \alpha_i I)w_i = P_h K^*(f^\delta - KF(x_0)); \tag{3.4.1}$$
- solve for $j < i$, z_{ij}^h : $\quad (P_h K^* K P_h + \alpha_j I)z_{ij} = (\alpha_j - \alpha_i)w_i$;
- if $\|z_{ij}^h\| > \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$, then take $k = i - 1$;
- otherwise, repeat with $i + 1$ in place of i.

Part II:

- choose $n_k = \min\{n : q_p^{2n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$, for IFD Class and for MFD Class choose $n_k = \min\{n : \tilde{q_p}^{2n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$;

Part III:

- solve $x_{n_k,\alpha_k}^{h,\delta}$ using the iteration (3.3.3) for IFD Class and $\tilde{x}_{n_k,\alpha_k}^{h,\delta}$ using the iteration (3.3.16) for MFD Class.

3.4.1 Implementation of the method

We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with dim$V_n = n+1$ and let $P_h = P_{\frac{1}{n}}$ denote the orthogonal projection on X with range $R(P_h) = V_n$. We assume that $\|P_h x - x\| \to 0 \ as \ h \to 0$ for all $x \in X$. Precisely we choose V_n as the space of linear splines $\{v_1, v_2, \cdots, v_{n+1}\}$ in a uniform grid of $n+1$ points in $[0,1]$ as a basis of V_n.

Since $w_i \in V_n$; $w_i = \sum_{i=1}^{n+1} \lambda_i v_i$ for some scalars $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$. It can be seen that w_i is a solution of (3.4.1) if and only if $\bar{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{n+1})^T$ is the unique solution of

$$(M_n + \alpha_i B_n)\bar{\lambda} = \bar{a}$$

where

$$M_n = (\langle Kv_i, Kv_j \rangle), i, j = 1, 2, \cdots, n+1,$$

$$B_n = (\langle v_i, v_j \rangle), i, j = 1, 2, \cdots, n+1$$

and

$$\bar{a} = (\langle P_h K^*(f^\delta - KF(x_0)), v_i \rangle)^T, i = 1, 2, \cdots, n+1.$$

Observe that $z_{ij}^{h,\delta}$ is in V_n and hence $z_{ij}^{h,\delta} = \sum_{m=1}^{n+1} \mu_m^{ij} v_m$ for some scalars $\mu_m^{ij}, m = 1, 2, \cdots, n+1$. One can see that for $j < i$, $z_{ij}^{h,\delta}$ is a solution of

$$(P_h K^* K P_h + \alpha_j I) z_{ij}^{h,\delta} = (\alpha_j - \alpha_i) w_i$$

if and only if $\overline{\mu^{ij}} = (\mu_1^{ij}, \mu_2^{ij}, \cdots, \mu_{n+1}^{ij})^T$ is the unique solution of

$$(M_n + \alpha_j B_n)\overline{\mu^{ij}} = \bar{b}$$

where

$$\bar{b} = (\langle (\alpha_j - \alpha_i) w_i, v_i \rangle)^T.$$

Compute $z_{ij}^{h,\delta}$ till $\|z_{ij}^{h,\delta}\| > \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$ and fix $k = i-1$. Let $n_k = \min\{n : q^{2n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$.

Case I: IFD Class. Since $y_{n_k,\alpha_k}^{h,\delta}, \ x_{n_k,\alpha_k}^{h,\delta} \in V_n$, let $y_{n_k,\alpha_k}^{h,\delta} = \sum_{i=1}^{n+1} \xi_i^n v_i$ and $x_{n_k,\alpha_k}^{h,\delta} = \sum_{i=1}^{n+1} \eta_i^n v_i$, where ξ_i^n and η_i^n are some scalars. Then from (3.3.2) we have

$$P_h F'(P_h x_0)(y_{n_k,\alpha_k}^{h,\delta} - x_{n_k,\alpha_k}^{h,\delta}) = P_h[z_{\alpha_k}^{h,\delta} - F(x_{n_k,\alpha_k}^{h,\delta})]. \quad (3.4.2)$$

Observe that $(y^{h,\delta}_{n_k,\alpha_k} - x^{h,\delta}_{n_k,\alpha_k})$ is a solution of (3.4.2) if and only if $(\overline{\xi^n - \eta^n}) = (\xi^n_1 - \eta^n_1, \xi^n_2 - \eta^n_2, \cdots, \xi^n_{n+1} - \eta^n_{n+1})^T$ is the unique solution of

$$Q_n(\overline{\xi^n - \eta^n}) = B_n[\overline{\lambda} - F_{h1}]$$

where $Q_n = \langle F'(P_h x_0)v_i, v_j\rangle, \quad i,j = 1,2,\cdots,n+1,$

$$F_{h1} = [F(x^{h,\delta}_{n_k,\alpha_k})(t_1), F(x^{h,\delta}_{n_k,\alpha_k})(t_2), \cdots, F(x^{h,\delta}_{n_k,\alpha_k})(t_{n+1})]^T,$$

where $t_1, t_2, \cdots, t_{n+1}$ are the grid points.

Further from (3.3.3) it follows that

$$P_h F'(P_h x_0)(x^{h,\delta}_{n_k+1,\alpha_k} - y^{h,\delta}_{n_k,\alpha_k}) = P_h[z^{h,\delta}_{\alpha_k} - F(y^{h,\delta}_{n_k,\alpha_k})]. \tag{3.4.3}$$

Thus $(x^{h,\delta}_{n_k+1,\alpha_k} - y^{h,\delta}_{n_k,\alpha_k})$ is a solution of (3.4.3) if and only if $(\overline{\eta^{n+1} - \xi^n}) = (\eta^{n+1}_1 - \xi^n_1, \eta^{n+1}_2 - \xi^n_2, \cdots, \eta^{n+1}_{n+1} - \xi^n_{n+1})^T$ is the unique solution of

$$Q_n(\overline{\eta^{n+1} - \xi^n}) = B_n[\overline{\lambda} - F_{h2}]$$

where $F_{h2} = [F(y^{h,\delta}_{n_k,\alpha_k})(t_1), F(y^{h,\delta}_{n_k,\alpha_k})(t_2), \cdots, F(y^{h,\delta}_{n_k,\alpha_k})(t_{n+1})]^T$.

Case II: MFD Class. Since $\tilde{y}^{h,\delta}_{n_k,\alpha_k}$ and $\tilde{x}^{h,\delta}_{n_k,\alpha_k}$ are in V_n; $\tilde{y}^{h,\delta}_{n_k,\alpha_k} = \sum_{i=1}^{n+1} \xi^n_i v_i$ and $\tilde{x}^{h,\delta}_{n_k,\alpha_k} = \sum_{i=1}^{n+1} \eta^n_i v_i$, where ξ^n_i and η^n_i are some scalars for $1 \le i \le n+1$. Then from (3.3.15) we have

$$(P_h F'(\tilde{x}^{h,\delta}_{0,\alpha_k}) + \frac{\alpha_k}{c})(\tilde{y}^{h,\delta}_{n_k,\alpha_k} - \tilde{x}^{h,\delta}_{n_k,\alpha_k}) = P_h[z^{h,\delta}_{\alpha_k} - F(\tilde{x}^{h,\delta}_{n_k,\alpha_k}) + \frac{\alpha_k}{c}(\tilde{x}^{h,\delta}_{0,\alpha_k} - \tilde{x}^{h,\delta}_{n_k,\alpha_k})]. \tag{3.4.4}$$

One can see that $(\tilde{y}^{h,\delta}_{n_k,\alpha_k} - \tilde{x}^{h,\delta}_{n_k,\alpha_k})$ is a solution of (3.4.4) if and only if $(\overline{\xi^n - \eta^n}) = (\xi^n_1 - \eta^n_1, \xi^n_2 - \eta^n_2, \cdots, \xi^n_{n+1} - \eta^n_{n+1})^T$ is the unique solution of

$$(Q_n + \frac{\alpha_k}{c}B_n)(\overline{\xi^n - \eta^n}) = B_n[\overline{\lambda} - F_{h1} + \frac{\alpha_k}{c}(X_0 - \overline{\eta^n})]$$

where $Q_n = \langle F'(x^{h,\delta}_{0,\alpha_k})v_i, v_j\rangle, \quad i,j = 1,2,\cdots,n+1,$

$$F_{h1} = [F(\tilde{x}^{h,\delta}_{n_k,\alpha_k})(t_1), F(\tilde{x}^{h,\delta}_{n_k,\alpha_k})(t_2), \cdots, F(\tilde{x}^{h,\delta}_{n_k,\alpha_k})(t_{n+1})]^T$$

and $X_0 = [x_0(t_1), x_0(t_2), \cdots, x_0(t_{n+1})]^T$ where $t_1, t_2, \cdots, t_{n+1}$ are the grid points.

Further from (3.3.16) it follows that

$$(P_h F'(\tilde{x}^{h,\delta}_{0,\alpha_k})+\frac{\alpha_k}{c})(\tilde{x}^{h,\delta}_{n_k+1,\alpha_k}-\tilde{y}^{h,\delta}_{n_k,\alpha_k}) = P_h[z^{h,\delta}_{\alpha_k}-F(\tilde{y}^{h,\delta}_{n_k,\alpha_k})+\frac{\alpha_k}{c}(\tilde{x}^{h,\delta}_{0,\alpha_k}-\tilde{y}^{h,\delta}_{n_k,\alpha_k})]. \quad (3.4.5)$$

Thus $(\tilde{x}^{h,\delta}_{n_k+1,\alpha_k} - \tilde{y}^{h,\delta}_{n_k,\alpha_k})$ is a solution of (3.4.5) if and only if $(\overline{\eta^{n+1} - \xi^n}) = (\eta_1^{n+1} - \xi_1^n, \eta_2^{n+1} - \xi_2^n, \cdots, \eta_{n+1}^{n+1} - \xi_{n+1}^n)^T$ is the unique solution of

$$(Q_n + \frac{\alpha_k}{c}B_n)(\overline{\eta^{n+1} - \xi^n}) = B_n[\overline{\lambda} - F_{h2} + \frac{\alpha_k}{c}(X_0 - \overline{\xi^n})]$$

where $F_{h2} = [F(\tilde{y}^{h,\delta}_{n_k,\alpha_k})(t_1), F(\tilde{y}^{h,\delta}_{n_k,\alpha_k})(t_2), \cdots, F(\tilde{y}^{h,\delta}_{n_k,\alpha_k})(t_{n+1})]^T$.

3.5 NUMERICAL EXAMPLES

In this section we consider two examples for illustrating the algorithm mentioned in the above section.

EXAMPLE 3.5.1 *(cf. Semenova (2010), section 4.3) In this example for IFD Class we consider the operator* $KF : L^2(0,1) \longrightarrow L^2(0,1)$ *with* $K : L^2(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds$$

where

$$k(t,s) = \begin{cases} (1-t)s, 0 \le s \le t \le 1 \\ (1-s)t, 0 \le t \le s \le 1 \end{cases}$$

and $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$F(u) := u^3.$$

Then the Fréchet derivative of F *is given by*

$$F'(u)w = 3(u)^2 w.$$

In our computation, we take $f(t) = \frac{837t}{6160} - \frac{t^2}{16} - \frac{t^{11}}{110} - \frac{3t^5}{80} - \frac{3t^8}{112}$ *and* $f^\delta = f + \delta$*. Then the exact solution*

$$\hat{x}(t) = 0.5 + t^3.$$

We use

$$x_0(t) = 0.5 + t^3 - \frac{3}{56}(t - t^8)$$

as our initial guess.

n	k	α_k	$\|x_k^h - \hat{x}\|$	$\frac{\|x_k^h - \hat{x}\|}{(\delta+\varepsilon_h)^{1/2}}$
8	4	0.1820	0.5484	1.7273
16	4	0.1065	0.5376	1.6984
32	4	0.1061	0.5301	1.6759
64	4	0.1061	0.5257	1.6624
128	4	0.1061	0.5234	1.6551
256	4	0.1060	0.5222	1.6513
512	4	0.1060	0.5216	1.6493
1024	4	0.1060	0.5213	1.6484

Table 3.1: Iterations and corresponding Error Estimates of Example 3.5.1

We choose $\alpha_0 = (1.3)^2(\delta + \varepsilon_h)^2$, $\mu = 1.3$, $\delta + \varepsilon_h = 0.1$ *the Lipschitz constant* k_0 *equals approximately 0.2134 as in Semenova (2010) and* $r = 1$, $\tau_0 = \frac{1}{64}$, *so that* $q_p = (1+\beta\tau_0)k_0 r = 0.2133$. *The results of the computation are presented in Table 3.1. The plots of the exact solution and the approximate solution obtained is given in Figures 3.1 and 3.2. The last column of the Table 3.1 shows that the error* $\|x_k^h - \hat{x}\|$ *is of order* $(\delta + \varepsilon_h)^{1/2}$.

EXAMPLE 3.5.2 *In this example for MFD class we consider the operator* $KF : L^2(0,1) \longrightarrow L^2(0,1)$ *where* $K : L^2(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds$$

and $F : D(F) \subseteq H^1(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$F(u) := \int_0^1 k(t,s)u^3(s)ds,$$

where

$$k(t,s) = \begin{cases} (1-t)s, 0 \le s \le t \le 1 \\ (1-s)t, 0 \le t \le s \le 1 \end{cases} .$$

Then for all $x(t), y(t) : x(t) > y(t) :$ *(see section 4.3 in Semenova (2010))*

$$\langle F(x) - F(y), x - y\rangle = \int_0^1 \left[\int_0^1 k(t,s)(x^3 - y^3)(s)ds\right](x-y)(t)dt \ge 0.$$

Thus the operator F *is monotone. The Fréchet derivative of* F *is given by*

$$F'(u)w = 3\int_0^1 k(t,s)(u(s))^2 w(s)ds.$$

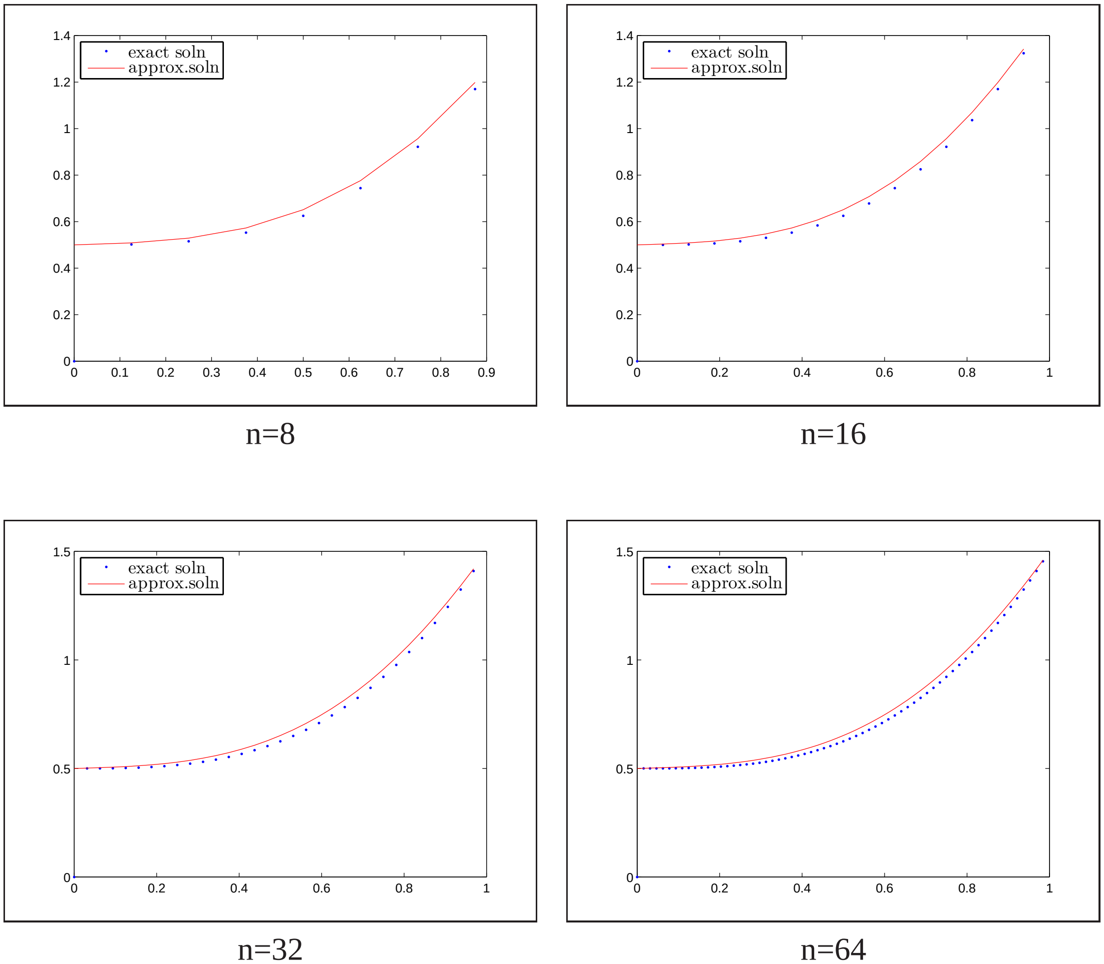


Figure 3.1: Curve of the exact and approximate solutions of Example 3.5.1

So for any $u \in B_r(\hat{x}), \hat{x}(s) \geq k_3 > 0, \forall s \in (0,1)$, *we have*

$$F'(u)w = F'(\hat{x})G(u,\hat{x})w,$$

where $G(u,\hat{x}) = (\frac{u}{\hat{x}})^2$.

In our computation, we take $f(t) = \frac{1}{110}(\frac{t^{13}}{156} - \frac{t^3}{6} + \frac{25t}{156})$ *and* $f^\delta = f + \delta$. *Then the exact solution*

$$\hat{x}(t) = t^3.$$

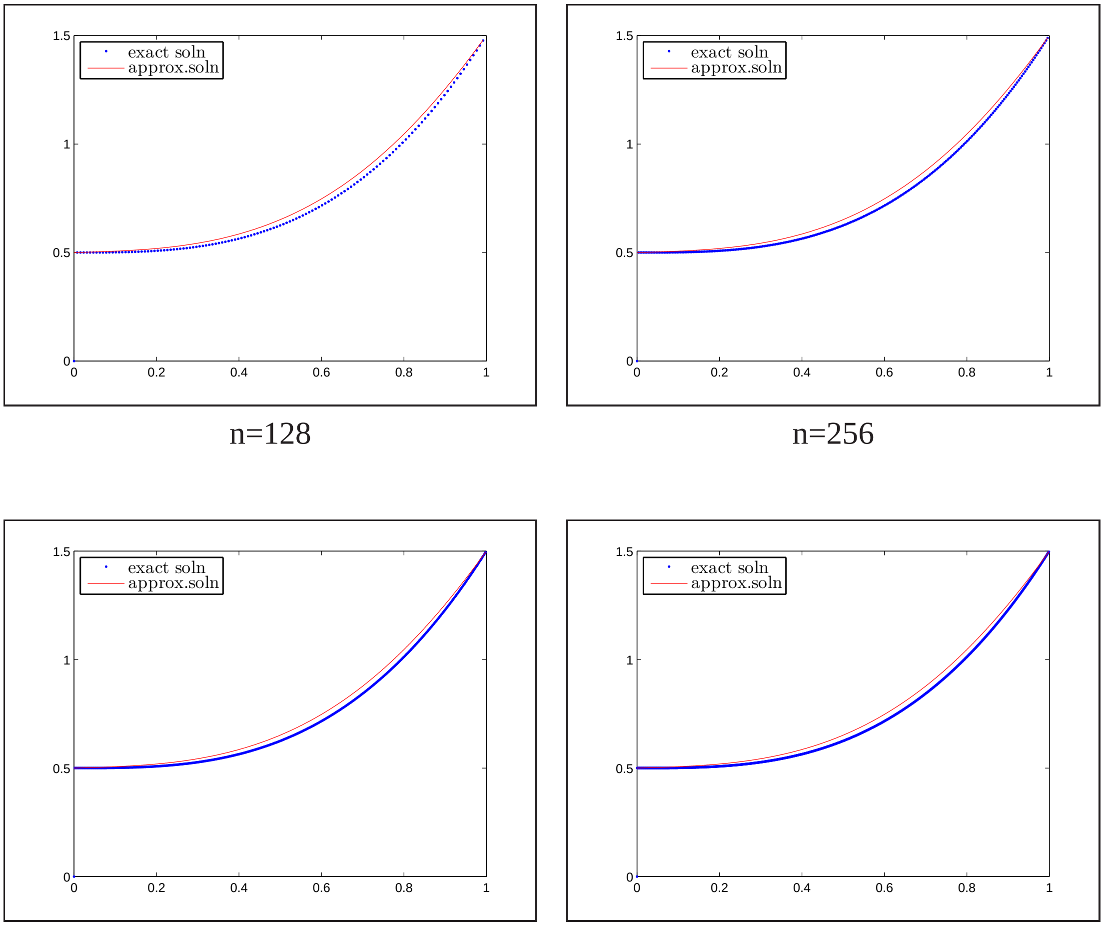
We use

$$x_0(t) = t^3 + \frac{3}{56}(t - t^8)$$

as our initial guess, so that the function $x_0 - \hat{x}$ *satisfies the source condition*

$$x_0 - \hat{x} = \varphi_1(F'(\hat{x}))1$$

where $\varphi_1(\lambda) = \lambda$. *Thus we expect to have an accuracy of order at least* $(\delta + \varepsilon_h)^{1/2}$.



n=128 n=256

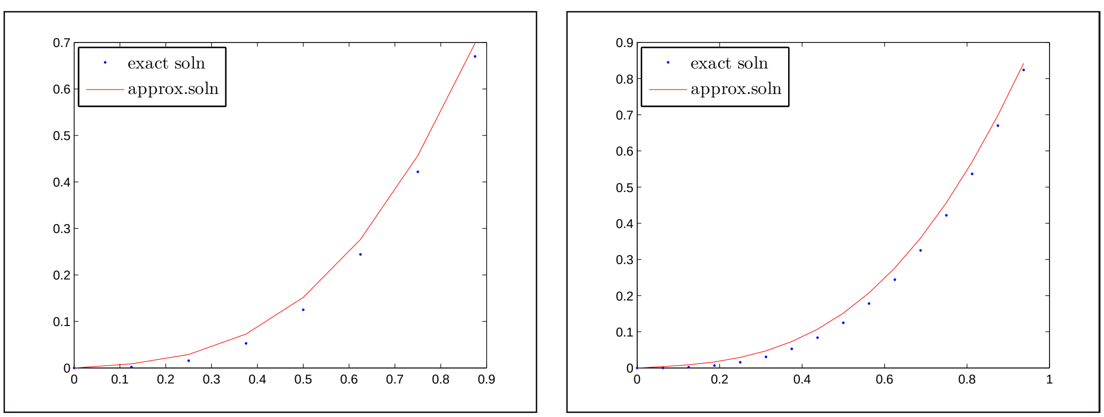
n=512 n=1024

Figure 3.2: Curve of the exact and approximate solutions of Example 3.5.1

We choose $\alpha_0 = (1.3)(\delta + \varepsilon_h)$, $\delta + \varepsilon_h = 0.0667 =: c$ *the Lipschitz constant* k_0 *equals approximately 0.21 as in (Semenova (2010)) and* $\tilde{r} = 1$, *so that* $\tilde{q_p} = (1 + \tau_0)k_0\tilde{r} = 0.21$. *The results of the computation are presented in Table 3.2. The plots of the exact solution and the approximate solution obtained are given in Figures 3.3 and 3.4.*

n	k	α_k	$\|\tilde{x}_k^h - \hat{x}\|$	$\frac{\|\tilde{x}_k^h - \hat{x}\|}{(\delta+\varepsilon_h)^{1/2}}$
8	4	0.0494	0.1881	0.7200
16	4	0.0477	0.1432	0.5531
32	4	0.0473	0.1036	0.4010
64	4	0.0472	0.0726	0.2812
128	4	0.0471	0.0491	0.1900
256	4	0.0471	0.0306	0.1187
512	4	0.0471	0.0140	0.0543
1024	4	0.0471	0.0133	0.0515

Table 3.2: Iterations and corresponding Error Estimates of Example 3.5.2



n=8 n=16

Figure 3.3: Curve of the exact and approximate solutions of Example 3.5.2

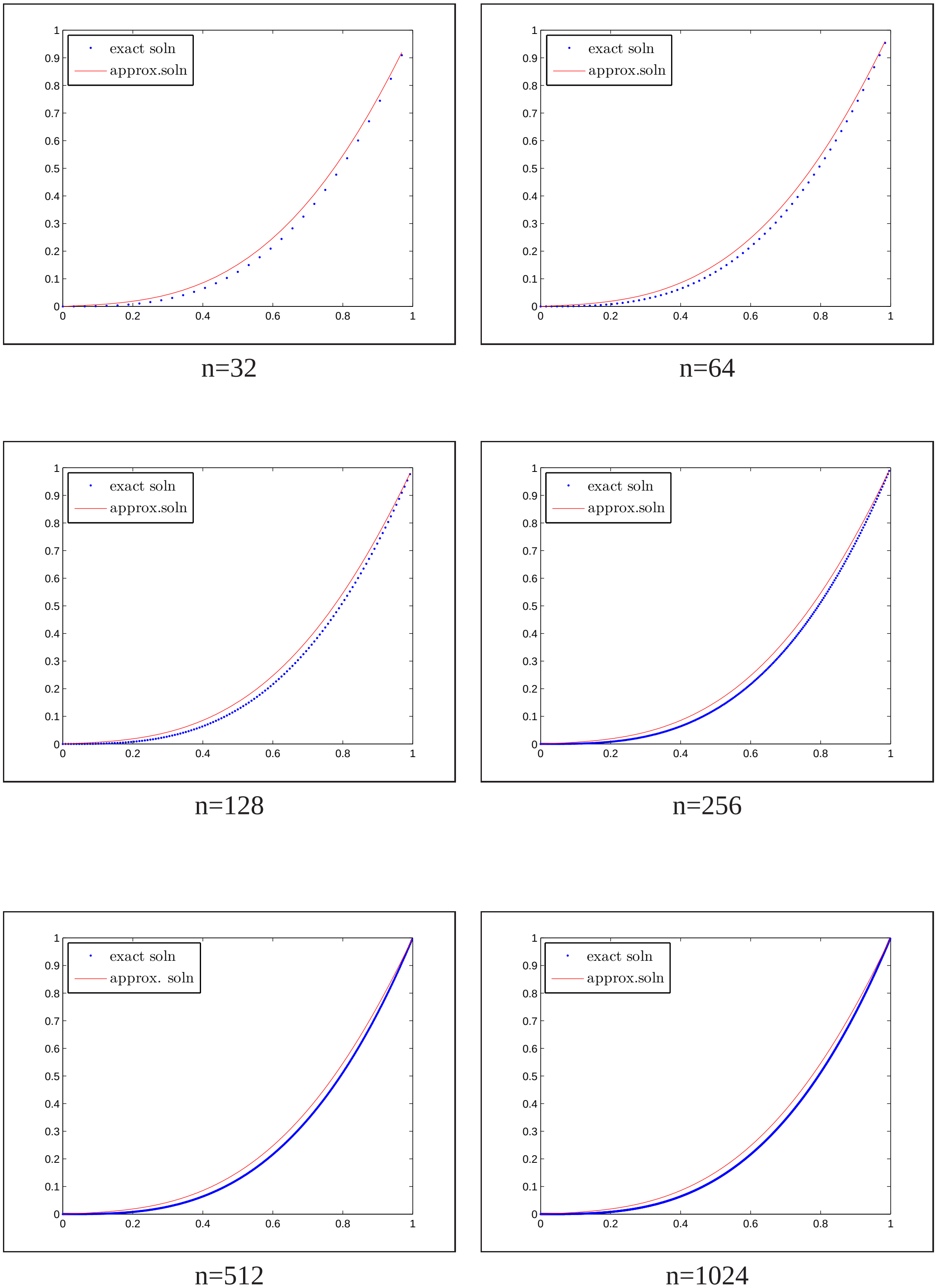


Figure 3.4: Curve of the exact and approximate solutions of Example 3.5.2

Chapter 4

TSNTM WITH CUBIC CONVERGENCE

A locally cubic convergence yielding Two Step Newton-Tikhonov method and its finite dimensional realization is proposed. Two implementations are discussed and applied to nonlinear ill-posed Hammerstein type operator equations (2.1.1). For both cases, local cubic convergence is established and order optimal error bounds are obtained by choosing the regularization parameter according to the the balancing principle of Pereverzev and Schock (2005). Also numerical examples are given to confirm the efficiency of the method.

4.1 INTRODUCTION

In this chapter, we consider a cubic convergence yielding Two Step Newton-Tikhonov Method for approximately solving (2.1.1). As in Chapter 2, we consider this method for two cases of operator F.

The IFD Class $F'(u)^{-1}$ exists and is a bounded operator for all $u \in B_r(x_0)$; i.e., $\|F'(u)^{-1}\| \leq \beta,\ \forall u \in B_r(x_0)$.

MFD Class F is a monotone operator and $F'(u)^{-1}$ does not exists.

This chapter is organized as follows. In Section 4.2 we present TSNTM method yielding cubic convergence and in Section 4.3 we give the finite dimensional realization of method considered in Section 4.2. Section 4.4 deals with the algorithm for implementing the proposed method and in Section 4.5 we provide a numerical example to prove the efficiency of the proposed method.

4.2 CONVERGENCE ANALYSIS OF TSNTM

4.2.1 Analysis of IFD Class

For an initial guess $x_0 \in X$ the TSNTM for IFD Class is defined as;

$$v^\delta_{n,\alpha_k} = u^\delta_{n,\alpha_k} - F'(u^\delta_{n,\alpha_k})^{-1}(F(u^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k}), \tag{4.2.1}$$

$$u^\delta_{n+1,\alpha_k} = v^\delta_{n,\alpha_k} - F'(u^\delta_{n,\alpha_k})^{-1}(F(v^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k}), \tag{4.2.2}$$

where $u^\delta_{0,\alpha_k} = x_0$. Let

$$\sigma^\delta_{n,\alpha_k} := \|v^\delta_{n,\alpha_k} - u^\delta_{n,\alpha_k}\|, \qquad \forall n \geq 0 \tag{4.2.3}$$

and for $0 < k_0 \leq 1$, let $g : (0,1) \to (0,1)$ be the function defined by

$$g(t) = \frac{k_0^2}{8}(4 + 3k_0 t)t^2, \qquad \forall t \in (0,1). \tag{4.2.4}$$

For convenience we will use the notation u_n, v_n and σ_n for u^δ_{n,α_k}, v^δ_{n,α_k} and $\sigma^\delta_{n,\alpha_k}$ respectively.

Further we assume that $\delta \in (0, \delta_0]$ where $\delta_0 < \frac{\sqrt{\alpha_0}}{\beta}$. Let $\|\hat{x} - x_0\| \leq \rho$,

$$\rho < \frac{1}{M}(\frac{1}{\beta} - \frac{\delta_0}{\sqrt{\alpha_0}}) \tag{4.2.5}$$

and

$$\gamma_\rho := \beta[M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}].$$

THEOREM 4.2.1 *Let σ_n and $g(\sigma_n)$ be as in equation (4.2.3) and (4.2.4) respectively, u_n and v_n be as in (4.2.1) and (4.2.2) respectively with $\delta \in (0, \delta_0]$. Then under the assumptions of Theorem 2.2.3 and Assumption 2.3.1, the following hold:*

(a) $\|u_n - v_{n-1}\| \leq \frac{k_0 \sigma_{n-1}}{2}\|v_{n-1} - u_{n-1}\|$;

(b) $\|u_n - u_{n-1}\| \leq (1 + \frac{k_0 \sigma_{n-1}}{2})\|v_{n-1} - u_{n-1}\|$;

(c) $\|v_n - u_n\| \leq g(\sigma_{n-1})\|v_{n-1} - u_{n-1}\|$;

(d) $g(\sigma_n) \leq g(\gamma_\rho)^{3^n}, \qquad \forall n \geq 0$;

(e) $\sigma_n \leq g(\gamma_\rho)^{(3^n-1)/2}\gamma_\rho, \qquad \forall n \geq 0.$

Proof. Observe that

$$\begin{aligned} u_n - v_{n-1} &= v_{n-1} - u_{n-1} - F'(u_{n-1})^{-1}(F(v_{n-1}) - F(u_{n-1})) \\ &= F'(u_{n-1})^{-1}[F'(u_{n-1})(v_{n-1} - u_{n-1}) - (F(v_{n-1}) - F(u_{n-1}))] \\ &= F'(u_{n-1})^{-1}\int_0^1 [F'(u_{n-1}) - F'(u_{n-1} + t(v_{n-1} - u_{n-1}))](v_{n-1} - u_{n-1})dt \end{aligned}$$

and hence by Assumption 2.3.1, we have

$$\begin{aligned} \|u_n - v_{n-1}\| &\leq \|\int_0^1 \Phi(u_{n-1}, u_{n-1} + t(v_{n-1} - u_{n-1}), v_{n-1} - u_{n-1})dt\| \\ &\leq \frac{k_0}{2}\|v_{n-1} - u_{n-1}\|^2. \end{aligned}$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|u_n - u_{n-1}\| \leq \|u_n - v_{n-1}\| + \|v_{n-1} - u_{n-1}\|.$$

To prove (c) we observe that

$$
\begin{aligned}
e_n = \|v_n - u_n\| &\leq \|u_n - v_{n-1} - F'(u_n)^{-1}(F(u_n) - z_\alpha^\delta)\| \\
&\quad + \|F'(u_{n-1})^{-1}(F(v_{n-1}) - z_\alpha^\delta)\| \\
&\leq \|u_n - v_{n-1} - F'(u_n)^{-1}(F(u_n) - F(v_{n-1}))\| \\
&\quad + \|[F'(u_{n-1})^{-1} - F'(u_n)^{-1}](F(v_{n-1}) - z_\alpha^\delta)\| \\
&\leq \|F'(u_n)^{-1}[F'(u_n)(u_n - v_{n-1}) - (F(u_n) - F(v_{n-1}))]\| \\
&\quad + \|[F'(u_{n-1})^{-1} - F'(u_n)^{-1}](F(v_{n-1}) - z_\alpha^\delta)\| \\
&\leq \|F'(u_n)^{-1}\int_0^1 [F'(u_n) - F'(v_{n-1} + t(u_n - v_{n-1})]dt(u_n - v_{n-1})\| \\
&\quad + \|F'(u_n)^{-1}(F'(u_n) - F'(u_{n-1}))F'(u_{n-1})^{-1}(F(v_{n-1}) - z_\alpha^\delta)\| \\
&\leq \|F'(u_n)^{-1}\int_0^1 [F'(u_n) - F'(v_{n-1} + t(u_n - v_{n-1})]dt(u_n - v_{n-1})\| \\
&\quad + \|F'(u_n)^{-1}(F'(u_n) - F'(u_{n-1}))(v_{n-1} - u_n)\| \\
&\leq \|\int_0^1 \Phi(u_n, v_{n-1} + t(u_n - v_{n-1}), u_n - v_{n-1})dt\| \\
&\quad + \|\Phi(u_n, u_{n-1}, v_{n-1} - u_n)\| \\
&\leq \frac{k_0}{2}\|u_n - v_{n-1}\|^2 + k_0\|u_n - u_{n-1}\|\|u_n - v_{n-1}\|.
\end{aligned}
$$

Therefore by (a) and (b) we have

$$
\begin{aligned}
\sigma_n &\leq (\frac{k_0^2}{2} + \frac{3k_0^3}{8}\|v_{n-1} - u_{n-1}\|)\|v_{n-1} - u_{n-1}\|^3 \\
&\leq g(\sigma_{n-1})\sigma_{n-1}. \qquad (4.2.6)
\end{aligned}
$$

This completes the proof of (c).

Since for $\mu \in (0,1)$, $g(\mu t) \leq \mu^2 g(t)$, for all $t \in (0,1)$, by (4.2.6) we have,

$$g(\sigma_n) \leq g(\sigma_0)^{3^n}$$

and

$$
\begin{aligned}
\sigma_n \leq g^3(\sigma_{n-2})\sigma_{n-1} &\leq g^3(\sigma_{n-2})g^3(\sigma_{n-3})\sigma_{n-2}\cdots g(\sigma_0)\sigma_0 \\
&\leq g(\sigma_0)^{3^{n-1}+3^{n-2}+\cdots+1}\sigma_0 \\
&\leq g(\sigma_0)^{(3^n-1)/2}\sigma_0, \qquad (4.2.7)
\end{aligned}
$$

provided $\sigma_n < 1, \forall n \geq 0$. From (4.2.7) it is clear that, $\sigma_n \leq 1$ if $\sigma_0 \leq 1$, but

$$\begin{aligned}
\sigma_0 = \|v_0 - x_0\| &= \|F'(x_0)^{-1}(F(x_0) - z^{\delta}_{\alpha_k})\| \\
&\leq \|F'(x_0)^{-1}\|\|F(x_0) - z^{\delta}_{\alpha_k}\| \\
&\leq \beta\|F(x_0) - z_{\alpha_k} + z_{\alpha_k} - z^{\delta}_{\alpha_k}\| \\
&\leq \beta[\|F(x_0) - F(\hat{x})\| + \|z_{\alpha_k} - z^{\delta}_{\alpha_k}\|] \\
&\leq \beta[\|\int_0^1 F'(\hat{x} + t(x_0 - \hat{x}))(x_0 - \hat{x})dt\| + \frac{\delta}{\sqrt{\alpha_k}}] \\
&\leq \beta[M\rho + \frac{\delta}{\sqrt{\alpha_k}}] \\
&\leq \beta[M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}] \\
&= \gamma_\rho < 1. \qquad (4.2.8)
\end{aligned}$$

As g is monotonically increasing and $\sigma_0 \leq \gamma_\rho$, we have $g(\sigma_0) \leq g(\gamma_\rho)$. This completes the proof of the Theorem.

THEOREM 4.2.2 *Let* $r = (\frac{1}{1-g(\gamma_\rho)} + \frac{k_0}{2}\frac{\gamma_\rho}{1-g(\gamma_\rho)^2})\gamma_\rho$ *with* $g(\gamma_\rho) < 1$ *and let the hypothesis of Theorem 4.2.1 holds. Then* $u_n, v_n \in B_r(x_0)$*, for all* $n \geq 0$*.*

Proof. Note that by (b) of Theorem 4.2.1 we have

$$\begin{aligned}
\|u_1 - x_0\| &\leq [1 + \frac{k_0}{2}\sigma_0]\sigma_0 \qquad (4.2.9) \\
&\leq [1 + \frac{k_0}{2}\gamma_\rho]\gamma_\rho \\
&\leq r,
\end{aligned}$$

i.e., $u_1 \in B_r(x_0)$. Again note that by (4.2.9) and (c) of Theorem 4.2.1 we have

$$\begin{aligned}
\|v_1 - x_0\| &\leq \|v_1 - u_1\| + \|u_1 - x_0\| \\
&\leq (1 + g(\sigma_0) + \frac{k_0}{2}\sigma_0)\sigma_0 \\
&\leq (1 + g(\gamma_\rho) + \frac{k_0}{2}\gamma_\rho)\gamma_\rho \\
&\leq r,
\end{aligned}$$

i.e., $v_1 \in B_r(x_0)$. Further by (4.2.9) and (b) of Theorem 4.2.1 we have

$$\begin{aligned}
\|u_2 - x_0\| &\leq \|u_2 - u_1\| + \|u_1 - x_0\| \\
&\leq (1 + \frac{k_0}{2}\sigma_1)\sigma_1 + (1 + \frac{k_0}{2}\sigma_0)\sigma_0 \\
&\leq (1 + \frac{k_0}{2}g(\sigma_0)\sigma_0)g(\sigma_0)\sigma_0 + (1 + \frac{k_0}{2}\sigma_0)\sigma_0 \\
&\leq (1 + g(\sigma_0) + \frac{k_0}{2}\sigma_0(1 + g(\sigma_0)^2))\sigma_0 \qquad (4.2.10)\\
&\leq (1 + g(\gamma_\rho) + \frac{k_0}{2}\gamma_\rho(1 + g(\gamma_\rho)^2))\gamma_\rho \\
&\leq r
\end{aligned}$$

and by (4.2.10) and (c) of Theorem 4.2.1 we have

$$\begin{aligned}
\|v_2 - x_0\| &\leq \|v_2 - u_2\| + \|u_2 - x_0\| \\
&\leq g(\sigma_1)\sigma_1 + (1 + g(\sigma_0) + \frac{k_0}{2}\sigma_0(1 + g(\sigma_0)^2))\sigma_0 \\
&\leq g(\sigma_0)^4\sigma_0 + (1 + g(\sigma_0) + \frac{k_0}{2}\sigma_0(1 + g(\sigma_0)^2))\sigma_0 \\
&\leq (1 + g(\sigma_0) + g(\sigma_0)^4 + \frac{k_0}{2}\sigma_0(1 + g(\sigma_0)^2))\sigma_0 \\
&\leq (1 + g(\sigma_0) + g(\sigma_0)^2 + \frac{k_0}{2}\sigma_0(1 + g(\sigma_0)^2))\sigma_0 \\
&\leq (1 + g(\gamma_\rho) + g(\gamma_\rho)^2 + \frac{k_0}{2}\gamma_\rho(1 + g(\gamma_\rho)^2))\gamma_\rho \\
&\leq r,
\end{aligned}$$

i.e., $u_2, v_2 \in B_r(x_0)$. Continuing this way one can prove that $u_n, v_n \in B_r(x_0), \forall n \geq 0$. This completes the proof.

The main result of this section is the following Theorem.

THEOREM 4.2.3 *Let v_n and u_n be as in (4.2.1) and (4.2.2) respectively, assumptions of Theorem 4.2.2 hold and let $0 < g(\gamma_\rho) < 1$. Then (u_n) is a Cauchy sequence in $B_r(x_0)$ and converges to $x^\delta_{\alpha_k} \in \overline{B_r(x_0)}$. Further $F(x^\delta_{\alpha_k}) = z^\delta_{\alpha_k}$ and*

$$\|u_n - x^\delta_{\alpha_k}\| \leq C_3 e^{-\gamma 3^n}$$

where $C_3 = (\frac{1}{1-g(\gamma_\rho)^3} + \frac{k_0\gamma_\rho}{2}\frac{1}{1-(g(\gamma_\rho)^2)^3}g(\gamma_\rho)^{3^n})\gamma_\rho$ and $\gamma = -\log g(\gamma_\rho)$.

Proof. Using the relation (b) and (e) of Theorem 4.2.1, we obtain

$$\begin{aligned}
\|u_{n+m}-u_n\| &\leq \sum_{i=0}^{i=m-1}\|u_{n+i+1}-u_{n+i}\| \\
&\leq \sum_{i=0}^{i=m-1}(1+\frac{k_0\sigma_{n+i}}{2})\sigma_{n+i} \\
&\leq \sum_{i=0}^{i=m-1}(1+\frac{k_0\sigma_0}{2}g(\sigma_0)^{3^{n+i}})g(\sigma_0)^{3^{n+i}}\sigma_0 \\
&= (1+\frac{k_0\sigma_0}{2}g(\sigma_0)^{3^n})g(\sigma_0)^{3^n}\sigma_0 \\
&\quad +(1+\frac{k_0\sigma_0}{2}g(\sigma_0)^{3^{n+1}})g(\sigma_0)^{3^{n+1}}\sigma_0+\cdots \\
&\quad +(1+\frac{k_0\sigma_0}{2}g(\sigma_0)^{3^{n+m}})g(\sigma_0)^{3^{n+m}}\sigma_0 \\
&\leq [(1+g(\sigma_0)^3+g(\sigma_0)^{3^2}+\cdots+g(e_0)^{3^m})+ \\
&\quad \frac{k_0\sigma_0}{2}(1+(g(\sigma_0)^2)^3+(g(\sigma_0)^2)^{3^2}+\cdots+(g(\sigma_0)^2)^{3^m})g(\sigma_0)^{3^n}]g(\sigma_0)^{3^n}\sigma_0 \\
&\leq [(1+g(\gamma_\rho)^3+g(\gamma_\rho)^{3^2}+\cdots+g(\gamma_\rho)^{3^m})+ \\
&\quad \frac{k_0\gamma_\rho}{2}(1+(g(\gamma_\rho)^2)^3+(g(\gamma_\rho)^2)^{3^2}+\cdots+(g(\gamma_\rho)^2)^{3^m})g(\gamma_\rho)^{3^n}]g(\gamma_\rho)^{3^n}\gamma_\rho \\
&\leq C_3 g(\gamma_\rho)^{3^n} \\
&\leq C_3 e^{-\gamma 3^n}.
\end{aligned}$$

Thus (u_n) is a Cauchy sequence in $B_r(x_0)$ and hence it converges, say to $x_{\alpha_k}^\delta \in \overline{B_r(x_0)}$. Observe that

$$\begin{aligned}
\|F(u_n)-z_{\alpha_k}^\delta\| &= \|F'(u_n)(u_n-v_n)\| \\
&\leq \|F'(u_n)\|\|(u_n-v_n)\| \\
&\leq M\sigma_n \leq Mg(\gamma_\rho)^{3^n}\gamma_\rho. \qquad (4.2.11)
\end{aligned}$$

Now by letting $n\to\infty$ in (4.2.11) we obtain $F(x_{\alpha_k}^\delta)=z_{\alpha_k}^\delta$. This completes the proof.

REMARK 4.2.4 *Note that* $0<g(\gamma_\rho)<1$ *and hence* $\gamma>0$. *So by (2.1.8), sequence* (u_n) *converges cubically to* $x_{\alpha_k}^\delta$.

Hereafter we assume that

$$\rho \leq r < \frac{1}{k_0}.$$

REMARK 4.2.5 *Note that the above assumption is satisfied if*

$$k_0 \leq \min\left\{1, \frac{1-g(\gamma_\rho)^2}{\gamma_\rho}[\frac{-1}{1-g(\gamma_\rho)} + \sqrt{\frac{1}{(1-g(\gamma_\rho))^2} + \frac{2}{1-g(\gamma_\rho)^2}}]\right\}.$$

The following theorem is a consequence of Theorem 4.2.3 and Theorem 2.3.4.

THEOREM 4.2.6 *Let u_n be as in (4.2.2), assumptions in Theorem 4.2.3 and Theorem 2.3.4 hold. Then*

$$\|\hat{x} - u_n\| \leq C_3 e^{-\gamma 3^n} + \frac{\beta}{1-k_0 r}\|F(\hat{x}) - z^\delta_{\alpha_k}\|$$

where C_3 and γ are as in Theorem 4.2.3.

Now since $l \leq k$ and $\alpha_\delta \leq \alpha_{l+1} \leq \mu\alpha_l$ we have

$$\frac{\delta}{\sqrt{\alpha_k}} \leq \frac{\delta}{\sqrt{\alpha_l}} \leq \mu\frac{\delta}{\sqrt{\alpha_\delta}} = \mu\varphi(\alpha_\delta) = \mu\psi^{-1}(\delta).$$

This leads to the following theorem,

THEOREM 4.2.7 *Let u_n be as in (4.2.2), assumptions in Theorem 2.2.3 and Theorem 4.2.6 hold. Let*

$$n_k := \min\{n : e^{-\gamma 3^n} \leq \frac{\delta}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - u_{n_k}\| = O(\psi^{-1}(\delta)).$$

4.2.2 Analysis of MFD Class

Let X be a real Hilbert space and let Assumption 2.3.1 holds with $\tilde{r}$ in place of r, $\rho \leq \tilde{r} < \frac{1}{k_0}$ and let $c \leq \alpha_k$.

First we consider a TSNTM for approximating the zero x^δ_{c,α_k} of

$$F(u) + \frac{\alpha_k}{c}(u - x_0) = z^\delta_{\alpha_k} \qquad (4.2.12)$$

and then we show that x^δ_{c,α_k} is an approximation to the solution $\hat{x}$ of (2.1.1). For an initial guess $x_0 \in X$ and for $R(x) := F'(u) + \frac{\alpha_k}{c}I$, the TSNTM for MFD Class is defined as:

$$\tilde{v}^\delta_{n,\alpha_k} = \tilde{u}^\delta_{n,\alpha_k} - R(\tilde{u}^\delta_{n,\alpha_k})^{-1}[F(\tilde{u}^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{u}^\delta_{n,\alpha_k} - x_0)] \qquad (4.2.13)$$

and

$$\tilde{u}^\delta_{n+1,\alpha_k} = \tilde{v}^\delta_{n,\alpha_k} - R(\tilde{u}^\delta_{n,\alpha_k})^{-1}[F(\tilde{v}^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{v}^\delta_{n,\alpha_k} - x_0)] \tag{4.2.14}$$

where $\tilde{u}_{0,\alpha_k} := x_0$. Note that with the above notation

$$\|R(u)^{-1}F'(u)\| \leq 1.$$

Let

$$\tilde{\sigma}^\delta_{n,\alpha_k} := \|\tilde{v}^\delta_{n,\alpha_k} - \tilde{v}^\delta_{n,\alpha_k}\|, \qquad \forall n \geq 0. \tag{4.2.15}$$

Here also for convenience we use the notation $\tilde{u}_n$, $\tilde{v}_n$ and $\tilde{\sigma}_n$ for $\tilde{u}^\delta_{n,\alpha_k}$, $\tilde{v}^\delta_{n,\alpha_k}$ and $\tilde{\sigma}^\delta_{n,\alpha_k}$ respectively.

Let

$$\rho \leq \frac{1}{M}(1 - \frac{\delta_0}{\sqrt{\alpha_0}}) \tag{4.2.16}$$

with $\delta_0 < \sqrt{\alpha_0}$ and

$$\tilde{\gamma}_\rho := M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}. \tag{4.2.17}$$

THEOREM 4.2.8 *Let $\tilde{\sigma}_n$ and g be as in equation (4.2.15) and (4.2.4) respectively, $\tilde{u}_n$ and $\tilde{v}_n$ be as in (4.2.14) and (4.2.13) respectively with $\delta \in (0, \delta_0]$. Then the following hold:*

(a) $\|\tilde{u}_n - \tilde{v}_{n-1}\| \leq \frac{k_0\tilde{\sigma}_{n-1}}{2}\|\tilde{v}_{n-1} - \tilde{u}_{n-1}\|$;

(b) $\|\tilde{u}_n - \tilde{u}_{n-1}\| \leq (1 + \frac{k_0\tilde{\sigma}_{n-1}}{2})\|\tilde{v}_{n-1} - \tilde{u}_{n-1}\|$;

(c) $\|\tilde{v}_n - \tilde{u}_n\| \leq g(\tilde{\sigma}_{n-1})\|\tilde{v}_{n-1} - \tilde{u}_{n-1}\|$;

(d) $g(\tilde{\sigma}_n) \leq g(\tilde{\gamma}_\rho)^{3^n}, \qquad \forall n \geq 0$;

(e) $\tilde{\sigma}_n \leq g(\tilde{\gamma}_\rho)^{(3^n-1)/2}\tilde{\gamma}_\rho, \qquad \forall n \geq 0.$

Proof. Observe that

$$\begin{aligned} \tilde{u}_n - \tilde{v}_{n-1} &= \tilde{v}_{n-1} - \tilde{u}_{n-1} - R(\tilde{u}_{n-1})^{-1}(F(\tilde{v}_{n-1}) - F(\tilde{u}_{n-1}) \\ &\quad + \frac{\alpha_k}{c}(\tilde{v}_{n-1} - \tilde{u}_{n-1})) \\ &= R(\tilde{u}_{n-1})^{-1}[R(\tilde{u}_{n-1})(\tilde{v}_{n-1} - \tilde{u}_{n-1}) \\ &\quad -(F(\tilde{v}_{n-1}) - F(\tilde{u}_{n-1})) - \frac{\alpha_k}{c}(\tilde{v}_{n-1} - \tilde{u}_{n-1})] \\ &= R(\tilde{u}_{n-1})^{-1}\int_0^1[F'(\tilde{u}_{n-1}) - F'(\tilde{u}_{n-1} + t(\tilde{v}_{n-1} - \tilde{u}_{n-1}))] \\ &\quad \times(\tilde{v}_{n-1} - \tilde{u}_{n-1})dt. \end{aligned}$$

Now since $\|R(\tilde{u}_{n-1})^{-1}F'(\tilde{u}_{n-1})\| \leq 1$, the proof of (a) and (b) follows as in Theorem 4.2.1. To prove (c) we observe that

$$\begin{aligned}
\tilde{\sigma}_n &\leq \|\tilde{u}_n - \tilde{v}_{n-1} - R(\tilde{u}_n)^{-1}(F(\tilde{u}_n) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{u}_n - x_0))\| \\
&\quad + \|R(\tilde{u}_{n-1})^{-1}(F(\tilde{v}_{n-1}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{v}_{n-1} - x_0))\| \\
&\leq \|\tilde{u}_n - \tilde{v}_{n-1} - R(\tilde{u}_n)^{-1}(F(\tilde{u}_n) - F(\tilde{v}_{n-1}) + \frac{\alpha_k}{c}(\tilde{u}_n - \tilde{v}_{n-1}))\| \\
&\quad + \|[R(\tilde{u}_{n-1})^{-1} - R(\tilde{u}_n)^{-1}](F(\tilde{v}_{n-1}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{v}_{n-1} - x_0))\| \\
&\leq \|R(\tilde{u}_n)^{-1}[R(\tilde{u}_n)(\tilde{x}_n - \tilde{v}_{n-1}) - (F(\tilde{u}_n) - F(\tilde{v}_{n-1})) \\
&\quad - \frac{\alpha_k}{c}(\tilde{u}_n - \tilde{v}_{n-1})]\| \\
&\quad + \|[R(\tilde{u}_{n-1})^{-1} - R(\tilde{u}_n)^{-1}](F(\tilde{v}_{n-1}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{v}_{n-1} - x_0))\| \\
&\leq \|R(\tilde{u}_n)^{-1}\int_0^1 [F'(\tilde{u}_n) - F'(\tilde{v}_{n-1} + t(\tilde{u}_n - \tilde{v}_{n-1})]dt(\tilde{u}_n - \tilde{v}_{n-1})\| \\
&\quad + \|R(\tilde{u}_n)^{-1}(F'(\tilde{u}_n) - F'(\tilde{u}_{n-1}))R(\tilde{u}_{n-1})^{-1}(F(\tilde{v}_{n-1}) - z^\delta_{\alpha_k} \\
&\quad + \frac{\alpha_k}{c}(\tilde{v}_{n-1} - x_0))\|.
\end{aligned}$$

The remaining part of the proof is analogous to the proof of Theorem 4.2.1.

THEOREM 4.2.9 *Let* $\tilde{r} = (\frac{1}{1-g(\tilde{\gamma}_\rho)} + \frac{k_0}{2}\frac{\tilde{\gamma}_\rho}{1-g(\tilde{\gamma}_\rho)^2})\tilde{\gamma}_\rho$ *with* $g(\tilde{\gamma}_\rho) < 1$ *and the assumptions of Theorem 4.2.8 hold. Then* $\tilde{u}_n, \tilde{v}_n \in B_{\tilde{r}}(x_0)$*, for all* $n \geq 0$*.*

Proof. Analogous to the proof of Theorem 4.2.2.

THEOREM 4.2.10 *Let* $\tilde{v}_n$ *and* $\tilde{u}_n$ *be as in (4.2.13) and (4.2.14) respectively and assumptions of Theorem 4.2.9 hold. Then* $(\tilde{u}_n)$ *is a Cauchy sequence in* $B_{\tilde{r}}(x_0)$ *and converges to* $x^\delta_{c,\alpha_k} \in \overline{B_{\tilde{r}}(x_0)}$*. Further* $F(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^\delta_{c,\alpha_k} - x_0) = z^\delta_{\alpha_k}$ *and*

$$\|\tilde{u}_n - x^\delta_{c,\alpha_k}\| \leq \tilde{C}_3 e^{-\tilde{\gamma}3^n}$$

where $\tilde{C}_3 = (\frac{1}{1-g(\tilde{\gamma}_\rho)^3} + \frac{k_0\tilde{\gamma}_\rho}{2}\frac{1}{1-(g(\tilde{\gamma}_\rho)^2)^3}g(\tilde{\gamma}_\rho)^{3^n})\tilde{\gamma}_\rho$ *and* $\tilde{\gamma} = -\log g(\tilde{\gamma}_\rho)$.

Proof. Analogous to the proof of Theorem 4.2.3 one can prove that $(\tilde{u}_n)$ is a Cauchy sequence in $B_{\tilde{r}}(x_0)$ and hence it converges, say to $x^\delta_{c,\alpha_k} \in \overline{B_{\tilde{r}}(x_0)}$. Observe that

$$\begin{aligned}
\|F(\tilde{u}_n) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{u}_n - x_0)\| &= \|R(\tilde{u}_n)(\tilde{u}_n - \tilde{v}_n)\| \\
&\leq \|R(\tilde{u}_n)\|\|\tilde{u}_n - \tilde{v}_n\| \\
&\leq (\|F'(u_n)\| + \frac{\alpha_k}{c})\tilde{\sigma}_n \\
&\leq (\|F'(u_n)\| + \frac{\alpha_k}{c})g(\tilde{\sigma}_0)^{3^n}\tilde{\sigma}_0 \\
&\leq (\|F'(u_n)\| + \frac{\alpha_k}{c})g(\tilde{\gamma}_\rho)^{3^n}\tilde{\gamma}_\rho. \qquad (4.2.18)
\end{aligned}$$

Now by letting $n \to \infty$ in (4.2.18) we obtain $F(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^\delta_{c,\alpha_k} - x_0) = z^\delta_{\alpha_k}$. This completes the proof.

Assume that $k_2 < \frac{1-k_0\tilde{r}}{1-c}$ with $k_0\tilde{r} < 1,\ \ c < 1$ and $\varphi_1(\alpha) \leq \varphi(\alpha)$ for $\alpha > 0$.

The following Theorem is a consequence of Theorem 4.2.10 and Theorem 2.3.11.

THEOREM 4.2.11 *Let $\tilde{u}_n$ be as in (4.2.14), assumptions in Theorem 4.2.10 and Theorem 2.3.11 hold. Then*

$$\|\hat{x} - \tilde{u}_n\| \leq \tilde{C}_3 e^{-\tilde{\gamma}3^n} + O(\psi^{-1}(\delta))$$

where $\tilde{C}_3$ and $\tilde{\gamma}$ are as in Theorem 4.2.10.

THEOREM 4.2.12 *Let $\tilde{u}_n$ be as in (4.2.14), assumptions in Theorem 2.2.3 and Theorem 4.2.11 hold. Let*

$$n_k := \min\{n : e^{-\tilde{\gamma}3^n} \leq \frac{\delta}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - \tilde{u}_{n_k}\| = O(\psi^{-1}(\delta)).$$

4.3 DISCRETIZED TWO STEP NEWTON-TIKHONOV METHOD (DTSNTM)

In this Section we consider the finite dimensional realization of the iterative method consider in Section 4.2. As in Section 4.2, we considered two cases of F : in the first case $F'(.)^{-1}$ exists in a neighbourhood of the initial guess x_0 and in the second case F is monotone and $F(.)^{-1}$ does not exist.

4.3.1 Convergence Analysis of IFD Class

For an initial guess $x_0 \in X$ the Discretized Newton Tikhonov Method is defined as;

$$v^{h,\delta}_{n,\alpha_k} = u^{h,\delta}_{n,\alpha_k} - P_h F'(u^{h,\delta}_{n,\alpha_k})^{-1} P_h(F(u^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k}), \tag{4.3.19}$$

$$u^{h,\delta}_{n+1,\alpha_k} = v^{h,\delta}_{n,\alpha_k} - P_h F'(u^{h,\delta}_{n,\alpha_k})^{-1} P_h(F(v^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k}), \tag{4.3.20}$$

where $u^{h,\delta}_{0,\alpha_k} := P_h x_0$.

Note: Observe that if $b_0 < \frac{1}{k_0}$ and if $u \in B_r(P_h x_0)$ where $r < \frac{1}{k_0} - b_0$, then $F'(u)^{-1}$ exists and is bounded. This can be seen as follows:

$$\begin{aligned} \|F'(u)^{-1}\| &= \sup_{\|v\|\leq 1} \|[I + F'(x_0)^{-1}(F'(u) - F'(x_0))]^{-1} F'(x_0)^{-1} v\| \\ &\leq \sup_{\|v\|\leq 1} \frac{\|F'(x_0)^{-1}\|}{1 - \|F'(x_0)^{-1}(F'(u) - F'(x_0))v\|} \end{aligned} \tag{4.3.21}$$

Now by Assumption 2.3.1 and the triangle inequality;

$$\|u - x_0\| \leq \|u - P_h x_0\| + \|P_h x_0 - x_0\|,$$

we have

$$\|F'(x_0)^{-1}(F'(u) - F'(x_0))v\| \leq k_0(r + b_0).$$

And hence by (3.3.1) and (4.3.21) we have

$$\|F'(u)^{-1}\| \leq \frac{\beta_1}{1 - k_0(r + b_0)}.$$

Thus without loss of generality we can assume that

$$\|F'(u)^{-1}\| \leq \beta, \ \ \forall u \in B_r(P_h x_0) \tag{4.3.22}$$

and for some $\beta > 0$.

LEMMA 4.3.1 *Let* $u \in B_r(P_h x_0)$, $b_0 < \frac{1}{k_0}$ *and* $r < \frac{1}{k_0} - b_0$. *Then* $\|P_h F'(u)^{-1} P_h F'(u)\| \leq 1 + \beta\tau_0$.

Proof. Proof is analogous to the proof of Lemma 3.3.1.

Let

$$\sigma_{n,\alpha_k}^{h,\delta} := \|v_{n,\alpha_k}^{h,\delta} - u_{n,\alpha_k}^{h,\delta}\|, \qquad \forall n \geq 0 \tag{4.3.23}$$

and let $g_h : (0,1) \to (0,1)$ be defined by

$$g_h(t) = \frac{k_0^2}{8}(4 + 3k_0(1+\beta\tau_0)t)(1+\beta\tau_0)^2 t^2 \qquad \forall t \in (0,1), \tag{4.3.24}$$

where $k_0 < \min\{1, \frac{1}{1+\beta\tau_0}\sqrt{\frac{8}{4+3(1+\beta\tau_0)}}\}$. Hereafter we assume that $\delta_0 + \varepsilon_0 < \frac{2}{\beta(2M+3)}\sqrt{\alpha_0}$. Let $\|\hat{x} - x_0\| \leq \rho$ where

$$\rho < \frac{1}{M}[\frac{1}{\beta} - (M + 1 + C_{M\rho})\frac{\delta_0 + \varepsilon_0}{\sqrt{\alpha_0}}]$$

and let

$$\gamma_\rho := \beta[M\rho + (M + 1 + C_{M\rho})(\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}})].$$

REMARK 4.3.2 *Note that $\gamma_\rho < 1$ and hence $g_h(\gamma_\rho) < 1$.*

THEOREM 4.3.3 *Let $\sigma_{n,\alpha_k}^{h,\delta}$ and $g_h(\sigma_{n,\alpha_k}^{h,\delta})$ be as in equation (4.3.23) and (4.3.24) respectively, $v_{n,\alpha_k}^{h,\delta}$ and $u_{n,\alpha_k}^{h,\delta}$ be as in (4.3.19) and (4.3.20) respectively with $\delta \in (0, \delta_0]$, $\alpha = \alpha_k$ and $\varepsilon_h \in (0, \varepsilon_0]$. If $u_{n,\alpha_k}^{h,\delta},\ v_{n,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$, then by Assumption 2.3.1, Lemma 4.3.1 and Theorem 3.2.2, the following hold:*

(a) $\|u_{n,\alpha_k}^{h,\delta} - v_{n-1,\alpha_k}^{h,\delta}\| \leq (1+\beta\tau_0)\frac{k_0\sigma_{n-1,\alpha_k}^{h,\delta}}{2}\|v_{n-1,\alpha_k}^{h,\delta} - u_{n-1,\alpha_k}^{h,\delta}\|;$

(b) $\|u_{n,\alpha_k}^{h,\delta} - u_{n-1,\alpha_k}^{h,\delta}\| \leq (1 + (1+\beta\tau_0)\frac{k_0\sigma_{n-1,\alpha_k}^{h,\delta}}{2})\|v_{n-1,\alpha_k}^{h,\delta} - u_{n-1,\alpha_k}^{h,\delta}\|;$

(c) $\|v_{n,\alpha_k}^{h,\delta} - u_{n,\alpha_k}^{h,\delta}\| \leq g_h(\sigma_{n-1,\alpha_k}^{h,\delta})\|v_{n-1,\alpha_k}^{h,\delta} - u_{n-1,\alpha_k}^{h,\delta}\|;$

(d) $g_h(\sigma_{n,\alpha_k}^{h,\delta}) \leq g_h(\gamma_\rho)^{3^n}, \qquad \forall n \geq 0;$

(e) $\sigma_{n,\alpha_k}^{h,\delta} \leq g_h(\gamma_\rho)^{(3^n-1)/2}\gamma_\rho, \qquad \forall n \geq 0.$

Proof. Observe that

$$\begin{aligned} u^{h,\delta}_{n,\alpha_k} - v^{h,\delta}_{n-1,\alpha_k} &= v^{h,\delta}_{n-1,\alpha_k} - u^{h,\delta}_{n-1,\alpha_k} - P_h F'(u^{h,\delta}_{n-1,\alpha_k})^{-1} P_h \\ &\quad \times (F(v^{h,\delta}_{n-1,\alpha_k}) - F(u^{h,\delta}_{n-1,\alpha_k})) \\ &= P_h F'(u^{h,\delta}_{n-1,\alpha_k})^{-1} [P_h F'(u^{h,\delta}_{n-1,\alpha_k})(v^{h,\delta}_{n-1,\alpha_k} - u^{h,\delta}_{n-1,\alpha_k}) \\ &\quad - P_h(F(v^{h,\delta}_{n-1,\alpha_k}) - F(u^{h,\delta}_{n-1,\alpha_k}))] \\ &= P_h F'(u^{h,\delta}_{n-1,\alpha_k})^{-1} P_h \int_0^1 [F'(u^{h,\delta}_{n-1,\alpha_k}) - F'(u^{h,\delta}_{n-1,\alpha_k} \\ &\quad + t(v^{h,\delta}_{n-1,\alpha_k} - u^{h,\delta}_{n-1,\alpha_k}))](v^{h,\delta}_{n-1,\alpha_k} - u^{h,\delta}_{n-1,\alpha_k}) dt \end{aligned}$$

and hence by Assumption 2.3.1 and Lemma 4.3.1 we have

$$\begin{aligned} \|u^{h,\delta}_{n,\alpha_k} - v^{h,\delta}_{n-1,\alpha_k}\| &\leq (1+\beta\tau_0) \| \int_0^1 \Phi(u^{h,\delta}_{n-1,\alpha_k}, u^{h,\delta}_{n-1,\alpha_k} + t(v^{h,\delta}_{n-1,\alpha_k} \\ &\quad - u^{h,\delta}_{n-1,\alpha_k}), v^{h,\delta}_{n-1,\alpha_k} - u^{h,\delta}_{n-1,\alpha_k}) dt \| \\ &\leq (1+\beta\tau_0) \frac{k_0}{2} \|v^{h,\delta}_{n-1,\alpha_k} - u^{h,\delta}_{n-1,\alpha_k}\|^2. \end{aligned}$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|u^{h,\delta}_{n,\alpha_k} - u^{h,\delta}_{n-1,\alpha_k}\| \leq \|u^{h,\delta}_{n,,\alpha_k} - v^{h,\delta}_{n-1,\alpha_k}\| + \|v^{h,\delta}_{n-1,\alpha_k} - u^{h,\delta}_{n-1,\alpha_k}\|.$$

To prove (c) we observe that

$$\begin{aligned} \sigma^{h,\delta}_{n,\alpha_k} &= \|u^{h,\delta}_{n,\alpha_k} - v^{h,\delta}_{n-1,\alpha_k} - (P_h F'(u^{h,\delta}_{n,\alpha_k}))^{-1} P_h (F(u^{h,\delta}_{n,\alpha_k}) \\ &\quad - z^{h,\delta}_{\alpha_k}) + P_h F'(u^{h,\delta}_{n-1,\alpha_k})^{-1} P_h (F(v^{h,\delta}_{n-1,\alpha_k}) - z^{h,\delta}_{\alpha_k})\| \\ &= \|u^{h,\delta}_{n,\alpha_k} - v^{h,\delta}_{n-1,\alpha_k} - P_h F'(u^{h,\delta}_{n,\alpha_k})^{-1} P_h (F(u^{h,\delta}_{n,\alpha_k}) \\ &\quad - F(v^{h,\delta}_{n-1,\alpha_k})) + P_h [F'(u^{h,\delta}_{n-1,\alpha_k})^{-1} - F'(u^{h,\delta}_{n,\alpha_k})^{-1}] \\ &\quad \times P_h (F(v^{h,\delta}_{n-1,\alpha_k}) - z^{h,\delta}_{\alpha_k})\| \\ &\leq \Lambda_1 + \Lambda_2 \end{aligned} \tag{4.3.25}$$

where

$$\Lambda_1 := \|u^{h,\delta}_{n,\alpha_k} - v^{h,\delta}_{n-1,\alpha_k} - P_h F'(u^{h,\delta}_{n,\alpha_k})^{-1} P_h (F(u^{h,\delta}_{n,\alpha_k}) - F(v^{h,\delta}_{n-1,\alpha_k}))\|$$

and

$$\Lambda_2 := \|P_h[F'(u_{n-1,\alpha_k}^{h,\delta})^{-1} - F'(u_{n,\alpha_k}^{h,\delta})^{-1}]P_h(F(v_{n-1,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta})\|.$$

Note that

$$\begin{aligned}
\Lambda_1 &\leq \|P_hF'(u_{n,\alpha_k}^{h,\delta})^{-1}P_h[F'(u_{n,\alpha_k}^{h,\delta})(u_{n,\alpha_k}^{h,\delta} - v_{n-1,\alpha_k}^{h,\delta}) \\
&\quad -(F(u_{n,\alpha_k}^{h,\delta}) - F(v_{n-1,\alpha_k}^{h,\delta}))]\| \\
&\leq \|P_hF'(u_{n,\alpha_k}^{h,\delta})^{-1}P_h\int_0^1 [F'(u_{n,\alpha_k}^{h,\delta}) - F'(v_{n-1,\alpha_k}^{h,\delta} \\
&\quad +t(u_{n,\alpha_k}^{h,\delta} - v_{n-1,\alpha_k}^{h,\delta})]dt(u_{n,\alpha_k}^{h,\delta} - v_{n-1,\alpha_k}^{h,\delta})\| \\
&\leq (1+\beta\tau_0)\|\int_0^1 \Phi(u_{n,\alpha_k}^{h,\delta}, v_{n-1,\alpha_k}^{h,\delta} + t(u_{n,\alpha_k}^{h,\delta} - v_{n-1,\alpha_k}^{h,\delta}), \\
&\quad u_{n,\alpha_k}^{h,\delta} - v_{n-1,\alpha_k}^{h,\delta})dt\| \\
&\leq (1+\beta\tau_0)\frac{k_0}{2}\|u_{n,\alpha_k}^{h,\delta} - v_{n-1,\alpha_k}^{h,\delta}\|^2. \qquad (4.3.26)
\end{aligned}$$

The last but one step follows from Assumption 2.3.1 and Lemma 4.3.1. Similarly

$$\begin{aligned}
\Lambda_2 &\leq \|P_h[F'(u_{n-1,\alpha_k}^{h,\delta})^{-1} - F'(u_{n,\alpha_k}^{h,\delta})^{-1}]P_h(F(v_{n-1,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta})\| \\
&\leq \|P_hF'(u_{n,\alpha_k}^{h,\delta})^{-1}P_h(F'(u_{n,\alpha_k}^{h,\delta}) - F'(u_{n-1,\alpha_k}^{h,\delta}))P_h \\
&\quad \times F'(u_{n-1,\alpha_k}^{h,\delta})^{-1}P_h(F(v_{n-1,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta})\| \\
&\leq \|P_hF'(u_{n,\alpha_k}^{h,\delta})^{-1}P_h(F'(u_{n,\alpha_k}^{h,\delta}) - F'(u_{n-1,\alpha_k}^{h,\delta})) \\
&\quad \times P_h(v_{n-1,\alpha_k}^{h,\delta} - u_{n,\alpha_k}^{h,\delta})\| \\
&\leq (1+\beta\tau_0)\|\Phi(u_{n,\alpha_k}^{h,\delta}, u_{n-1,\alpha_k}^{h,\delta}, v_{n-1,\alpha_k}^{h,\delta} - u_{n,\alpha_k}^{h,\delta})\| \\
&\leq k_0(1+\beta\tau_0)\|u_{n,\alpha_k}^{h,\delta} - u_{n-1,\alpha_k}^{h,\delta}\|\|u_{n,\alpha_k}^{h,\delta} - v_{n-1,\alpha_k}^{h,\delta}\|. \qquad (4.3.27)
\end{aligned}$$

Hence from (4.3.25), (4.3.26), (4.3.27), (a) and (b) we have

$$\begin{aligned}
\sigma_{n,\alpha_k}^{h,\delta} &\leq (1+\beta\tau_0)^2(\frac{k_0^2}{2} + \frac{3k_0^3(1+\beta\tau_0)}{8}\|v_{n-1,\alpha_k}^{h,\delta} - u_{n-1,\alpha_k}^{h,\delta}\|) \\
&\quad \times\|v_{n-1,\alpha_k}^{h,\delta} - u_{n-1,\alpha_k}^{h,\delta}\|^3 \\
&\leq g_h(\sigma_{n-1,\alpha_k}^{h,\delta})\sigma_{n-1,\alpha_k}^{h,\delta}. \qquad (4.3.28)
\end{aligned}$$

This completes the proof of (c).

Note that for $\mu \in (0,1)$, $g_h(\mu t) \leq \mu^2 g_h(t)$, for all $t \in (0,1)$, so by (4.3.28) we have, $g_h(\sigma^{h,\delta}_{n,\alpha_k}) \leq g_h(\sigma^{h,\delta}_{0,\alpha_k})^{3^n}$ and

$$\sigma^{h,\delta}_{n,\alpha_k} \leq g_h(\sigma^{h,\delta}_{0,\alpha_k})^{(3^n-1)/2}\sigma^{h,\delta}_{0,\alpha_k} \tag{4.3.29}$$

provided $\sigma^{h,\delta}_{n,\alpha_k} < 1, \forall n \geq 0$. Further from (4.3.29) observe that, $\sigma^{h,\delta}_{n,\alpha_k} \leq 1$ if $\sigma^{h,\delta}_{0,\alpha_k} \leq 1$, but

$$\begin{aligned} \sigma^{h,\delta}_{0,\alpha_k} &\leq \beta[(M+1)b_h + (1+\frac{\varepsilon_h}{2\sqrt{\alpha_k}})M\rho + \frac{\delta}{2\sqrt{\alpha_k}}] \\ &\leq \beta[M\rho + (M+1+C_{M\rho})(\frac{\varepsilon_0+\delta_0}{\sqrt{\alpha_0}})]. \end{aligned}$$

As g_h is monotonic increasing and $\sigma^{h,\delta}_{0,\alpha_k} \leq \gamma_\rho$, we have $g_h(\sigma^{h,\delta}_{0,\alpha_k}) \leq g_h(\gamma_\rho)$. This completes the proof of the Theorem.

THEOREM 4.3.4 *Let* $r = (\frac{1}{1-g_h(\gamma_\rho)} + \frac{(1+\beta\tau_0)k_0}{2}\frac{\gamma_\rho}{1-g_h(\gamma_\rho)^2})\gamma_\rho$ *with* $g_h(\gamma_\rho) < 1$ *and let the hypothesis of Theorem 4.3.3 holds. Then* $u^{h,\delta}_{n,\alpha_k}, v^{h,\delta}_{n,\alpha_k} \in B_r(P_h x_0)$*, for all* $n \geq 0$.

Proof. The proof is analogous to the proof of Theorem 4.2.2.

The next theorem is the main result of this section.

THEOREM 4.3.5 *Let* $v^{h,\delta}_{n,\alpha_k}$ *and* $u^{h,\delta}_{n,\alpha_k}$ *be as in (4.3.19) and (4.3.20) respectively, assumptions of Theorem 4.3.4 hold and let* $0 < g_h(\gamma_\rho) < 1$*. Then* $(u^{h,\delta}_{n,\alpha_k})$ *is a Cauchy sequence in* $B_r(P_h x_0)$ *and converges to* $x^{h,\delta}_{\alpha_k} \in \overline{B_r(P_h x_0)}$*. Further* $P_h F(x^{h,\delta}_{\alpha_k}) = z^{h,\delta}_{\alpha_k}$ *and*

$$\|u^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{\alpha_k}\| \leq C_4 e^{-\gamma_1 3^n}$$

where $C_4 = (\frac{1}{1-g_h(\gamma_\rho)^3} + (1+\beta\tau_0)\frac{k_0\gamma_\rho}{2}\frac{1}{1-(g_h(\gamma_\rho)^2)^3} g_h(\gamma_\rho)^{3^n})\gamma_\rho$ *and* $\gamma_1 = -\log g_h(\gamma_\rho)$.

Proof. Analogous to the proof of Theorem 2.3.3 in Chapter 1, one can show that $(u^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_r(P_h x_0)$ and hence it converges, say to $x^{h,\delta}_{\alpha_k} \in \overline{B_r(P_h x_0)}$. Observe that

$$\begin{aligned} \|P_h(F(u^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k})\| &= \|P_h F'(u^{h,\delta}_{n,\alpha_k})(u^{h,\delta}_{n,\alpha_k} - v^{h,\delta}_{n,\alpha_k})\| \\ &\leq \|F'(u^{h,\delta}_{n,\alpha_k})\|\|u^{h,\delta}_{n,\alpha_k} - v^{h,\delta}_{n,\alpha_k}\| \\ &\leq M\sigma^{h,\delta}_{n,\alpha_k} \leq M g_h(\gamma_\rho)^{3^n}\gamma_\rho. \end{aligned} \tag{4.3.30}$$

Now by letting $n \to \infty$ in (4.3.30) we obtain $P_h F(x^{h,\delta}_{\alpha_k}) = z^{h,\delta}_{\alpha_k}$. This completes the proof.

REMARK 4.3.6 *Note that $0 < g_h(\gamma_\rho) < 1$ and hence $\gamma > 0$. So sequence $(u_{n,\alpha_k}^{h,\delta})$ converges cubically to $x_{\alpha_k}^{h,\delta}$.*

Hereafter we assume that $\rho \leq r < \frac{1}{(1+\beta\tau_0)k_0}$.

REMARK 4.3.7 *The above assumption is satisfied if $\rho \leq r$ and*

$$k_0 < \frac{1-g_h(\gamma_\rho)^2}{\gamma_\rho}[\frac{-1}{1-g_h(\gamma_\rho)} + \sqrt{\frac{1}{(1-g_h(\gamma_\rho))^2} + \frac{2}{(1-g_h(\gamma_\rho)^2)(1+\beta\tau_0)}}].$$

The following Theorem is a consequence of Theorem 4.3.5 and Theorem 3.3.5.

THEOREM 4.3.8 *Let $u_{n,\alpha_k}^{h,\delta}$ be as in (4.3.20), assumptions in Theorem 4.3.5 and Theorem 3.3.5 hold. Then*

$$\|\hat{x} - u_{n,\alpha_k}^{h,\delta}\| \leq C_4 e^{-\gamma_1 3^n} + \frac{\beta}{(1-(1+\beta\tau_0)k_0 r)}\|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\|$$

where C_4 and γ_1 are as in Theorem 4.3.5.

Now since $l \leq k$ and $\alpha_\delta \leq \alpha_{l+1} \leq \mu\alpha_l$ we have

$$\frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_l}} \leq \mu\frac{\delta+\varepsilon_h}{\sqrt{\alpha_\delta}} = \mu\varphi(\alpha(\delta,h)) = \mu\psi^{-1}(\delta+\varepsilon_h).$$

This leads to the following theorem,

THEOREM 4.3.9 *Let $u_{n,\alpha_k}^{h,\delta}$ be as in (4.3.20) and assumptions in Theorem 4.3.8 hold. Let*

$$n_k := \min\{n : e^{-\gamma_1 3^n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - u_{n_k,\alpha_k}^{h,\delta}\| = O(\psi^{-1}(\delta+\varepsilon_h)).$$

4.3.2 Convergence Analysis of MFD Class

Let X be a real Hilbert space. We need the Assumptions 2.3.1, 2.3.9 and 2.3.10 for the convergence of DTSNTM and to obtain the error estimate.

First we consider a DTSNTM for approximating the zero $x_{c,\alpha_k}^{h,\delta}$ of

$$P_h(F(u) + \frac{\alpha_k}{c}(u-x_0)) = P_h z_{\alpha_k}^{h,\delta} \tag{4.3.31}$$

and then we show that $x_{c,\alpha_k}^{h,\delta}$ is an approximation to the solution $\hat{x}$ of $KF(x) = f$ where $c \le \alpha_k$. For an initial guess $x_0 \in X$ and for $R(u) := P_h F'(u) P_h + \frac{\alpha_k}{c} P_h$, the DTSNTM is defined as:

$$\tilde{v}_{n,\alpha_k}^{h,\delta} = \tilde{u}_{n,\alpha_k}^{h,\delta} - R(\tilde{u}_{n,\alpha_k}^{h,\delta})^{-1} P_h [F(\tilde{u}_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} + \frac{\alpha_k}{c}(\tilde{u}_{n,\alpha_k}^{h,\delta} - \tilde{u}_{0,\alpha_k}^{h,\delta})], \tag{4.3.32}$$

$$\tilde{u}_{n+1,\alpha_k}^{h,\delta} = \tilde{v}_{n,\alpha_k}^{h,\delta} - R(\tilde{u}_{n,\alpha_k}^{h,\delta})^{-1} P_h [F(\tilde{v}_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} + \frac{\alpha_k}{c}(\tilde{v}_{n,\alpha_k}^{h,\delta} - \tilde{u}_{0,\alpha_k}^{h,\delta})], \tag{4.3.33}$$

where $\tilde{u}_{0,\alpha_k}^{h,\delta} := P_h x_0$. Note that with the above notation, as in Equation (3.3.18) of Chapter 3, we have

$$\|R(\tilde{u}_{n,\alpha_k}^{h,\delta})^{-1} P_h F'(\tilde{u}_{n,\alpha_k}^{h,\delta})\| \le 1 + \tau_0. \tag{4.3.34}$$

Let

$$\tilde{\sigma}_{n,\alpha_k}^{h,\delta} := \|\tilde{v}_{n,\alpha_k}^{h,\delta} - \tilde{u}_{n,\alpha_k}^{h,\delta}\|, \qquad \forall n \ge 0. \tag{4.3.35}$$

and let k_0 be such that

$$\frac{k_0^2}{8}(4 + 3k_0(1+\tau_0))(1+\tau_0)^2 < 1.$$

REMARK 4.3.10 *Note that the above assumption is satisfied if we choose*

$$k_0 < \min\{1, \frac{1}{1+\tau_0}\sqrt{\frac{8}{4+3(1+\tau_0)}}\}.$$

Let $\tilde{g_h} : (0,1) \to (0,1)$ be the function defined by

$$\tilde{g_h}(t) = \frac{k_0^2}{8}(4 + 3k_0(1+\tau_0)t)(1+\tau_0)^2 t^2 \qquad \forall t \in (0,1). \tag{4.3.36}$$

Let $\|\hat{x} - x_0\| \le \rho$, with

$$\rho < \frac{1}{M}(1 - (\frac{3}{2} + M)\frac{\delta_0 + \varepsilon_0}{\sqrt{\alpha_0}}),$$

$$\tilde{\gamma}_\rho := M\rho + (\frac{3}{2} + M)(\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}}).$$

THEOREM 4.3.11 *Let $\tilde{\sigma}_{n,\alpha_k}^{h,\delta}$ and $\tilde{g_h}$ be as in equation (4.3.35) and (4.3.36) respectively, and let $\tilde{u}_{n,\alpha_k}^{h,\delta}$ and $\tilde{v}_{n,\alpha_k}^{h,\delta}$ be as in (4.3.33) and (4.3.32) respectively, with $\delta \in (0, \delta_0]$, $\alpha = \alpha_k$ and $\varepsilon_h \in (0, \varepsilon_0]$. Then the following hold:*

(a) $\|\tilde{u}^{h,\delta}_{n,\alpha_k} - \tilde{v}^{h,\delta}_{n-1,\alpha_k}\| \leq (1+\tau_0)\frac{k_0\tilde{\sigma}^{h,\delta}_{n-1,\alpha_k}}{2}\|\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}\|$;

(b) $\|\tilde{u}^{h,\delta}_{n,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}\| \leq (1+(1+\tau_0)\frac{k_0\tilde{\sigma}^{h,\delta}_{n-1,\alpha_k}}{2})\|\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}\|$;

(c) $\|\tilde{v}^{h,\delta}_{n,\alpha_k} - \tilde{u}^{h,\delta}_{n,\alpha_k}\| \leq \tilde{g_h}(\tilde{\sigma}^{h,\delta}_{n-1,\alpha_k})\|\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}\|$;

(d) $\tilde{g_h}(\tilde{\sigma}^{h,\delta}_{n,\alpha_k}) \leq \tilde{g_h}(\tilde{\gamma}_\rho)^{3^n}, \qquad \forall n \geq 0;$

(e) $\tilde{\sigma}^{h,\delta}_{n,\alpha_k} \leq \tilde{g_h}(\tilde{\gamma}_\rho)^{(3^n-1)/2}\tilde{\gamma}_\rho, \qquad \forall n \geq 0.$

Proof. Observe that

$$\begin{aligned}
\tilde{u}^{h,\delta}_{n,\alpha_k} - \tilde{v}^{h,\delta}_{n-1,\alpha_k} &= \tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k} - R(\tilde{u}^{h,\delta}_{n-1,\alpha_k})^{-1}P_h(F(\tilde{v}^{h,\delta}_{n-1,\alpha_k}) \\
&\quad -F(\tilde{u}^{h,\delta}_{n-1,\alpha_k}) + \frac{\alpha_k}{c}(\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k})) \\
&= R(\tilde{u}^{h,\delta}_{n-1,\alpha_k})^{-1}[R(\tilde{u}^{h,\delta}_{n-1,\alpha_k})(\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}) \\
&\quad -P_h(F(\tilde{v}^{h,\delta}_{n-1,\alpha_k}) - F(\tilde{u}^{h,\delta}_{n-1,\alpha_k})) - \frac{\alpha_k}{c}(\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k})] \\
&= R(\tilde{u}^{h,\delta}_{n-1,\alpha_k})^{-1}[(P_hF'(\tilde{u}^{h,\delta}_{n-1,\alpha_k})P_h + \frac{\alpha_k}{c}P_h)(\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}) \\
&\quad -P_h(F(\tilde{v}^{h,\delta}_{n-1,\alpha_k}) - F(\tilde{u}^{h,\delta}_{n-1,\alpha_k})) - \frac{\alpha_k}{c}(\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k})] \\
&= R(\tilde{u}^{h,\delta}_{n-1,\alpha_k})^{-1}P_h\int_0^1[F'(\tilde{u}^{h,\delta}_{n-1,\alpha_k}) - F'(\tilde{u}^{h,\delta}_{n-1,\alpha_k} \\
&\quad +t(\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}))]P_h(\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k})dt.
\end{aligned}$$

Now by Assumption 2.3.1 and (3.3.18) we have

$$\begin{aligned}
\|\tilde{u}^{h,\delta}_{n,\alpha_k} - \tilde{v}^{h,\delta}_{n-1,\alpha_k}\| &\leq (1+\tau_0)\|\int_0^1\Phi(\tilde{u}^{h,\delta}_{n-1,\alpha_k}, \tilde{u}^{h,\delta}_{n-1,\alpha_k} + t(\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}), \\
&\quad \tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k})dt\| \\
&\leq (1+\tau_0)\frac{k_0}{2}\|\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}\|^2.
\end{aligned}$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|\tilde{u}^{h,\delta}_{n,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}\| \leq \|\tilde{u}^{h,\delta}_{n,\alpha_k} - \tilde{v}^{h,\delta}_{n-1,\alpha_k}\| + \|\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{n-1,\alpha_k}\|.$$

To prove (c) we observe that

$$\begin{aligned}\tilde{\sigma}^{h,\delta}_{n,\alpha_k} &= \|\tilde{u}^{h,\delta}_{n,\alpha_k} - \tilde{v}^{h,\delta}_{n-1,\alpha_k} - R(\tilde{u}^{h,\delta}_{n,\alpha_k})^{-1}P_h(F(\tilde{u}^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k} \\ &\quad + \frac{\alpha_k}{c}(\tilde{u}^{h,\delta}_{n,\alpha_k} - \tilde{u}^{h,\delta}_{0,\alpha_k})) + R(\tilde{u}^{h,\delta}_{n-1,\alpha_k})^{-1}P_h(F(\tilde{v}^{h,\delta}_{n-1,\alpha_k}) \\ &\quad - z^{h,\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{x}^{h,\delta}_{0,\alpha_k}))\| \\ &= \|\tilde{u}^{h,\delta}_{n,\alpha_k} - \tilde{v}^{h,\delta}_{n-1,\alpha_k} - R(\tilde{u}^{h,\delta}_{n,\alpha_k})^{-1}P_h(F(\tilde{u}^{h,\delta}_{n,\alpha_k}) - F(\tilde{v}^{h,\delta}_{n-1,\alpha_k}) \\ &\quad + \frac{\alpha_k}{c}(\tilde{u}^{h,\delta}_{n,\alpha} - \tilde{v}^{h,\delta}_{n-1,\alpha_k})) + [R(\tilde{u}^{h,\delta}_{n-1,\alpha_k})^{-1} - R(\tilde{u}^{h,\delta}_{n,\alpha_k})^{-1}] \\ &\quad \times P_h(F(\tilde{u}^{h,\delta}_{n-1,\alpha_k}) - z^{h,\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{v}^{h,\delta}_{n-1,\alpha_k} - \tilde{u}^{h,\delta}_{0,\alpha_k}))\|.\end{aligned}$$

The remaining part of the proof is analogous to the proof of Theorem 4.3.3.

THEOREM 4.3.12 *Let* $\tilde{r} = (\frac{1}{1-\tilde{g_h}(\tilde{\gamma}_\rho)} + (1+\tau_0)\frac{k_0}{2}\frac{\tilde{\gamma}_\rho}{1-\tilde{g_h}(\tilde{\gamma}_\rho)^2})\tilde{\gamma}_\rho$ *with* $\tilde{g_h}(\tilde{\gamma}_\rho) < 1$ *and the assumptions of Theorem 4.3.11 hold. Then* $\tilde{u}^{h,\delta}_{n,\alpha_k}, \tilde{v}^{h,\delta}_{n,\alpha_k} \in B_{\tilde{r}}(P_h x_0)$*, for all* $n \geq 0$.

Proof. Proof is analogous to the proof of Theorem 4.2.2. The main result of this section is the following Theorem.

THEOREM 4.3.13 *Let* $\tilde{v}^{h,\delta}_{n,\alpha_k}$ *and* $\tilde{u}^{h,\delta}_{n,\alpha_k}$ *be as in (4.3.32) and (4.3.33) respectively, and let assumptions of Theorem 4.3.11 and 4.3.12 hold. Then* $(\tilde{u}^{h,\delta}_{n,\alpha_k})$ *is a Cauchy sequence in* $B_{\tilde{r}}(P_h x_0)$ *and converges to* $x^{h,\delta}_{c,\alpha_k} \in \overline{B_{\tilde{r}}(P_h x_0)}$*. Further* $P_h[F(x^{h,\delta}_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^{h,\delta}_{c,\alpha_k} - x_0)] = P_h z^{h,\delta}_{\alpha_k}$ *and*

$$\|\tilde{u}^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{c,\alpha_k}\| \leq \tilde{C}_4 e^{-\tilde{\gamma}_1 3^n},$$

where $\tilde{C}_4 = (\frac{1}{1-\tilde{g_h}(\tilde{\gamma}_\rho)^3} + (1+\tau_0)\frac{k_0\tilde{\gamma}_\rho}{2}\frac{1}{1-(\tilde{g_h}(\tilde{\gamma}_\rho)^2)^3}\tilde{g_h}(\tilde{\gamma}_\rho)^{3^n})\tilde{\gamma}_\rho$ *and* $\tilde{\gamma_1} = -\log \tilde{g_h}(\tilde{\gamma}_\rho)$.

Proof. Analogous to the proof of Theorem 4.2.3 one can show that $(\tilde{u}^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_{\tilde{r}}(P_h x_0)$ and hence it converges, say to $x^{h,\delta}_{c,\alpha_k} \in \overline{B_{\tilde{r}}(P_h x_0)}$. Observe that from (4.3.32)

$$\begin{aligned}\|P_h(F(\tilde{u}^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k}) + \frac{\alpha_k}{c}(\tilde{u}^{h,\delta}_{n,\alpha_k} - P_h x_0)\| &= \|R(\tilde{u}^{h,\delta}_{n,\alpha_k})(\tilde{u}^{h,\delta}_{n,\alpha_k} - \tilde{v}^{h,\delta}_{n,\alpha_k})\| \\ &\leq \|R(\tilde{u}^{h,\delta}_{n,\alpha_k})\|\|\tilde{v}^{h,\delta}_{n,\alpha_k} - \tilde{u}^{h,\delta}_{n,\alpha_k}\| \\ &\leq (\|P_h F'(\tilde{u}^{h,\delta}_{n,\alpha_k})P_h\| + \frac{\alpha_k}{c})\tilde{\sigma}^{h,\delta}_{n,\alpha_k} \\ &\leq (\|P_h F'(\tilde{u}^{h,\delta}_{n,\alpha_k})P_h\| + \frac{\alpha_k}{c}) \\ &\quad \times \tilde{g_h}(\tilde{\sigma}^{h,\delta}_{0,\alpha_k})^{3^n}\tilde{\sigma}^{h,\delta}_{0,\alpha_k} \\ &\leq (M + \frac{\alpha_k}{c})\tilde{g_h}(\tilde{\gamma}_\rho)^{3^n}\tilde{\gamma}_\rho. \qquad (4.3.37)\end{aligned}$$

Now by letting $n \to \infty$ in (4.3.37) we obtain $P_h F(x_{c,\alpha_k}^{h,\delta}) + \frac{\alpha_k}{c}(x_{c,\alpha_k}^{h,\delta} - P_h x_0) = P_h z_{\alpha_k}^{h,\delta}$. This completes the proof.

Hereafter we assume that $\tilde{r} < \frac{1}{k_0}$ and $k_2 < \frac{1-k_0\tilde{r}}{1-c}$ with $c < 1$.

The following theorem is a consequence of Theorems 2.3.11, 4.3.13 and (3.3.23).

THEOREM 4.3.14 *Let $\tilde{u}_{n,\alpha_k}^{h,\delta}$ be as in (4.3.33), assumptions in Theorem 4.3.13, Theorem 2.3.11 and (3.3.23) hold. Then*

$$\|\hat{x} - \tilde{u}_{n,\alpha_k}^{h,\delta}\| \leq \tilde{C}_4 e^{-\tilde{\gamma}_1 3^n} + \frac{\varphi_1(\alpha_k) + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta + \varepsilon_h)}{1 - (1-c)k_2 - k_0\tilde{r}} + \frac{2}{1-\tau_0}(\frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}})$$

where $\tilde{C}_4$ and $\tilde{\gamma}_1$ are as in Theorem 4.3.13.

THEOREM 4.3.15 *Let $\tilde{u}_{n,\alpha_k}^{h,\delta}$ be as in (4.3.33) and assumptions in Theorem 4.3.14 hold. Further let $\varphi_1(\alpha_k) \leq \varphi(\alpha_k)$ and*

$$n_k := \min\{n : e^{-\tilde{\gamma}_1 3^n} \leq \frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - \tilde{u}_{n_k,\alpha_k}^{h,\delta}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

4.4 ALGORITHM

Note that for $i, j \in \{0, 1, 2, \cdots, N\}$

$$z_{\alpha_i}^{h,\delta} - z_{\alpha_j}^{h,\delta} = (\alpha_j - \alpha_i)(P_h K^* K P_h + \alpha_j I)^{-1}(P_h K^* K P_h + \alpha_i I)^{-1} P_h K^*(f^\delta - KF(x_0)).$$

Hence, the adaptive algorithm associated with the choice of the parameter specified in Theorems 3.2.2, 4.3.9 and 5.3.10 involve the following steps.

Part I:

- $\alpha_0 = \mu^2(\delta + \varepsilon_h)^2$,
- $\alpha_i = \mu^{2i}\alpha_0$, $\mu > 1$
- solve for w_i:

$$(P_h K^* K P_h + \alpha_i I)w_i = P_h K^*(f^\delta - KF(x_0)); \qquad (4.4.1)$$

- solve for $j < i$, z_{ij}^h : $\quad (P_h K^* K P_h + \alpha_j I) z_{ij} = (\alpha_j - \alpha_i) w_i$;
- if $\|z_{ij}^h\| > \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$, then take $k = i - 1$;
- otherwise, repeat with $i + 1$ in place of i.

<u>Part II:</u>

- choose $n_k = \min\{n : e^{-\gamma_1 3^n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$ for IFD Class and $n_k = \min\{n : e^{-\tilde{\gamma}_1 3^n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$ for MFD Class;

<u>Part III:</u>

- solve $u_{n_k,\alpha_k}^{h,\delta}$ using the iteration (4.3.20) and $\tilde{u}_{n_k,\alpha_k}^{h,\delta}$ using the iteration (4.3.33).

In the next sections we consider two examples to illustrate the above algorithm. The computational results provided endorse the reliability and effectiveness of our method.

4.5 IMPLEMENTATION OF THE METHODS

We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with $dimV_n = n + 1$ and let $P_h = P_{\frac{1}{n}}$ denote the orthogonal projection on X with range $R(P_h) = V_n$. We assume that $\|P_h x - x\| \to 0 \; as \; h \to 0$ for all $x \in X$. Precisely we choose V_n as the space of linear splines $\{v_1, v_2, \cdots, v_{n+1}\}$ in a uniform grid of $n + 1$ points in $[0, 1]$ as a basis of V_n.

Since $w_i \in V_n, \quad w_i$ is of the form $\sum_{i=1}^{n+1} \lambda_i v_i$ for some scalars $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$. It can be seen that w_i is a solution of (4.4.1) if and only if $\bar{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{n+1})^T$ is the unique solution of

$$(M_n + \alpha_i B_n)\bar{\lambda} = \bar{a}$$

where

$$M_n = \langle K v_i, K v_j \rangle, i, j = 1, 2, \cdots, n + 1$$

$$B_n = \langle v_i, v_j \rangle, i, j = 1, 2, \cdots, n + 1$$

and

$$\bar{a} = (\langle P_h K^*(f^\delta - KF(x_0)), v_i\rangle)^T, i = 1, 2, \cdots, n+1.$$

Observe that $z_{ij}^{h,\delta}$ is in V_n and hence $z_{ij}^{h,\delta} = \sum_{k=1}^{n+1} \mu_k^{ij} v_k$ for some $\mu_k^{ij}, k = 1, 2, \cdots, n+1$. One can see that for $j < i, \ \ z_{ij}^{h,\delta}$ is a solution of

$$(P_h K^* K P_h + \alpha_j I) z_{ij}^{h,\delta} = (\alpha_j - \alpha_i) w_i$$

if and only if $\overline{\mu^{ij}} = (\mu_1^{ij}, \mu_2^{ij}, \cdots, \mu_{n+1}^{ij})^T$ is the unique solution of

$$(M_n + \alpha_j B_n)\overline{\mu^{ij}} = \bar{b}$$

where $\bar{b} = (\alpha_j - \alpha_i) B_n \bar{\lambda}$. Compute $z_{ij}^{h,\delta}$ till $\|z_{ij}^{h,\delta}\| > \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$ and fix $k = i-1$. Now we choose $n_k = \min\{n : e^{-\gamma_1 3^n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$.

Case I:IFD Class. Since $v_{n_k,\alpha_k}^{h,\delta}, \ u_{n_k,\alpha_k}^{h,\delta} \in V_n$, let $v_{n_k,\alpha_k}^{h,\delta} = \sum_{i=1}^{n+1} \xi_i^n v_i$ and $u_{n_k,\alpha_k}^{h,\delta} = \sum_{i=1}^{n+1} \eta_i^n v_i$, where ξ_i^n and η_i^n are some scalars. Then from (4.3.19) we have

$$P_h F'(u_{n_k,\alpha_k}^{h,\delta})(v_{n_k,\alpha_k}^{h,\delta} - u_{n_k,\alpha_k}^{h,\delta}) = P_h[z_{\alpha_k}^{h,\delta} - F(u_{n_k,\alpha_k}^{h,\delta})]. \tag{4.5.1}$$

Observe that $(v_{n_k,\alpha_k}^{h,\delta} - u_{n_k,\alpha_k}^{h,\delta})$ is a solution of (4.3.19) if and only if $(\overline{\xi^n - \eta^n}) = (\xi_1^n - \eta_1^n, \xi_2^n - \eta_2^n, \cdots, \xi_{n+1}^n - \eta_{n+1}^n)^T$ is the unique solution of

$$Q_n(\overline{\xi^n - \eta^n}) = B_n[\overline{\lambda^n} - F_{h1}]$$

where $Q_n = \langle F'(u_{n_k,\alpha_k}^{h,\delta}) v_i, v_j\rangle, \ \ i, j = 1, 2, \cdots, n+1,$

$$F_{h1} = [F(u_{n_k,\alpha_k}^{h,\delta})(t_1), F(u_{n_k,\alpha_k}^{h,\delta})(t_2), \cdots, F(u_{n_k,\alpha_k}^{h,\delta})(t_{n+1})]^T,$$

where $t_1, t_2, \cdots, t_{n+1}$ are the grid points.

Further from (4.3.20) it follows that

$$P_h F'(u_{n_k,\alpha_k}^{h,\delta})(u_{n_k+1,\alpha_k}^{h,\delta} - v_{n_k,\alpha_k}^{h,\delta}) = P_h[z_{\alpha_k}^{h,\delta} - F(v_{n_k,\alpha_k}^{h,\delta})]. \tag{4.5.2}$$

Thus $(u_{n_k+1,\alpha_k}^{h,\delta} - v_{n_k,\alpha_k}^{h,\delta})$ is a solution of (4.5.2) if and only if $(\overline{\eta^{n+1} - \xi^n}) = (\eta_1^{n+1} - \xi_1^n, \eta_2^{n+1} - \xi_2^n, \cdots, \eta_{n+1}^{n+1} - \xi_{n+1}^n)^T$ is the unique solution of

$$Q_n(\overline{\eta^{n+1} - \xi^n}) = B_n[\overline{\lambda^n} - F_{h2}]$$

where $F_{h2} = [F(v^{h,\delta}_{n_k,\alpha_k})(t_1), F(v^{h,\delta}_{n_k,\alpha_k})(t_2), \cdots, F(v^{h,\delta}_{n_k,\alpha_k})(t_{n+1})]^T$.

Case II: MFD Class. Let $\xi^n = (\xi_1^n, \xi_2^n, \cdots, \xi_{n+1}^n)$, $\eta^n = (\eta_1^n, \eta_2^n, \cdots, \eta_{n+1}^n)$, $\tilde{v}^{h,\delta}_{n,\alpha_k} = \sum_{i=1}^{n+1} \xi_i^n v_i$ and $\tilde{u}^{h,\delta}_{n,\alpha_k} = \sum_{i=1}^{n+1} \eta_i^n v_i$. Then from (4.3.32) we have

$$\begin{aligned}(P_h F'(\tilde{u}^{h,\delta}_{n,\alpha_k}) + \frac{\alpha_k}{c}) \sum_{i=1}^{n+1} (\xi_i^n - \eta_i^n) v_i &= \sum_{i=1}^{n+1} \lambda_i v_i - \sum_{i=1}^{n+1} P_h F(\tilde{u}^{h,\delta}_{n,\alpha_k}) v_i \\ &\quad + \frac{\alpha_k}{c} \sum_{i=1}^{n+1} (x_0(t_i) - \eta_i^n) v_i,\end{aligned}$$

where $t_1, t_2, \cdots, t_{n+1}$ are the grid points.

Observe that $(\tilde{v}^{h,\delta}_{n,\alpha_k} - \tilde{u}^{h,\delta}_{n,\alpha_k})$ is a solution of (4.3.32) if and only if $(\overline{\xi^n - \eta^n}) = (\xi_1^n - \eta_1^n, \xi_2^n - \eta_2^n, \cdots, \xi_{n+1}^n - \eta_{n+1}^n)^T$ is the unique solution of

$$(Q_n + \frac{\alpha_k}{c} B_n)(\overline{\xi^n - \eta^n}) = B_n[\overline{\lambda} - F_{h1} + \frac{\alpha_k}{c}(X_0 - \overline{\eta^n})],$$

where $Q_n = \langle F'(\tilde{u}^{h,\delta}_{n,\alpha_k}) v_i, v_j \rangle, \quad i, j = 1, 2, \cdots, n+1,$

$$F_{h1} = [F(\tilde{u}^{h,\delta}_{n,\alpha_k})(t_1), F(\tilde{u}^{h,\delta}_{n,\alpha_k})(t_2), \cdots, F(\tilde{u}^{h,\delta}_{n,\alpha_k})(t_{n+1})]^T$$

and $X_0 = [x_0(t_1), x_0(t_2), \cdots, x_0(t_{n+1})]^T$.

Further from (4.3.33) it follows that

$$(P_h F'(\tilde{u}^{h,\delta}_{n,\alpha_k}) + \frac{\alpha_k}{c})(\tilde{u}^{h,\delta}_{n+1,\alpha_k} - \tilde{v}^{h,\delta}_{n,\alpha_k}) = P_h[z^{h,\delta}_{\alpha_k} - F(\tilde{v}^{h,\delta}_{n,\alpha_k}) + \frac{\alpha_k}{c}(\tilde{u}^{h,\delta}_{0,\alpha_k} - \tilde{v}^{h,\delta}_{n,\alpha_k})]. \quad (4.5.3)$$

Thus $(u^{h,\delta}_{n+1,\alpha_k} - v^{h,\delta}_{n,\alpha_k})$ is a solution of (4.5.3) if and only if $(\overline{\eta^{n+1} - \xi^n}) = (\eta_1^{n+1} - \xi_1^n, \eta_2^{n+1} - \xi_2^n, \cdots, \eta_{n+1}^{n+1} - \xi_{n+1}^n)^T$ is the unique solution of

$$(Q_n + \frac{\alpha_k}{c} B_n)(\overline{\eta^{n+1} - \xi^n}) = B_n[\overline{\lambda} - F_{h2} + \frac{\alpha_k}{c}(X_0 - \overline{\xi^n})],$$

where $F_{h2} = [F(\tilde{v}^{h,\delta}_{n,\alpha_k})(t_1), F(\tilde{v}^{h,\delta}_{n,\alpha_k})(t_2), \cdots, F(\tilde{v}^{h,\delta}_{n,\alpha_k})(t_{n+1})]^T$.

4.6 NUMERICAL EXAMPLES

EXAMPLE 4.6.1 *We consider the operator* $KF : L^2(0,1) \longrightarrow L^2(0,1)$ *where* $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$F(u) := u^3$$

and $K : L^2(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds$$

where

$$k(t,s) = \begin{cases} (1-t)s, 0 \leq s \leq t \leq 1 \\ (1-s)t, 0 \leq t \leq s \leq 1 \end{cases} .$$

The Fréchet derivative of F *is given by*

$$F'(u)w = 3(u^2)w.$$

Observe that

$$\begin{aligned} [F'(v) - F'(u)]w &= 3(v^2 - u^2)w \\ &= 3u^2(\frac{v^2}{u^2} - 1)w \\ &= F'(u)\Phi(u,v,w), \end{aligned}$$

where $\Phi(u,v,w) = (\frac{v^2}{u^2} - 1)w = \frac{(v+u)(v-u)}{u^2}w$. *Thus* Φ *satisfies the Assumption 2.3.1 (cf. Scherzer, Engl and Kunisch (1993), Example 2.7).*

We take $f(t) = \frac{6\sin\pi t + \sin^3(\pi t)}{9\pi^2}$ *and* $f^\delta = f + \delta$. *Then the exact solution*

$$\hat{x}(t) = \sin\pi t.$$

We use

$$x_0(t) = \sin\pi t + 1/10$$

as our initial guess, so that the function $F(x_0) - F(\hat{x})$ *satisfies the source condition*

$$F(x_0) - F(\hat{x}) = \varphi(F'(\hat{x}))(\frac{3\sin^2(\pi t) + 3.3\sin(\pi t) + 0.91}{30(1/2 + \sin\pi t)^2})$$

where $\varphi(\lambda) = \lambda$. *Thus we expect to have an accuracy of order at least* $O(\delta^{\frac{1}{2}})$.

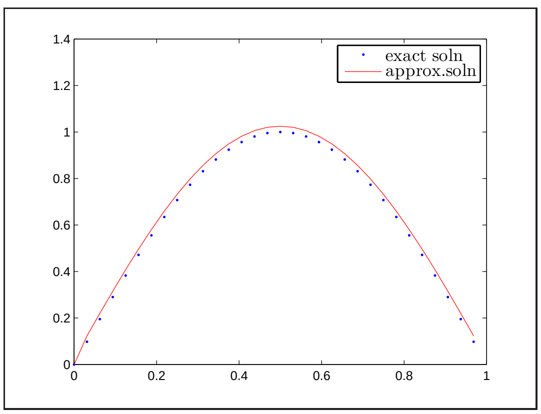
We choose $\alpha_0 = (1.5))\delta + \varepsilon_h)^2, \mu = 1.5, \delta = 0.0667, \beta = 0.925, \rho = 0.1, \gamma_\rho = 0.8212$ *and* $g_h(\gamma_\rho) = 0.54$ *approximately. In this example, for all* n, *the number of iteration* $n_k = 2$. *The results of the computation are presented in Table 4.1. The plots of the exact and the approximate solution obtained are given in Figures 4.1 and 4.2.*

EXAMPLE 4.6.2 *(cf. Semenova (2010), section 4.3) To illustrate the method for MFD class, we consider the space* $X = Y = L^2[0,1]$ *and the Fredholm integral operator* $K : L^2(0,1) \to L^2(0,1)$. *Then for all* $x(t), y(t) : x(t) > y(t)$:

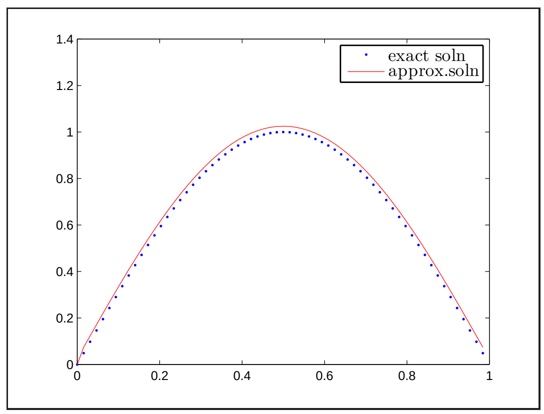
$$\langle F(x) - F(y), x - y \rangle = \int_0^1 \left[\int_0^1 k(t,s)(x^3 - y^3)(s)ds \right] (x-y)(t)dt \geq 0.$$

n	k	α_k	$\|u_k^h - \hat{x}\|$	$\frac{\|u_k^h - \hat{x}\|}{(\delta+\varepsilon_h)^{1/2}}$
32	4	0.1714	0.0246	0.0953
64	4	0.1710	0.0248	0.0960
128	4	0.1709	0.0249	0.0964
256	4	0.1709	0.0250	0.0966
512	4	0.1709	0.0250	0.0967
1024	4	0.1709	0.0250	0.0968

Table 4.1: Iterations and corresponding Error Estimates of Example 4.6.1



n=32



n=64

Figure 4.1: Curve of the exact and approximate solutions of Example 4.6.1

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)(u(s))^2 w(s)ds.$$

So for any $u \in B_r(x_0), {x_0}^2(s) \geq k_3 > 0, \forall s \in (0,1)$, we have

$$F'(u)w = F'(x_0)G(u,x_0)w,$$

where $G(u,x_0) = (\frac{u}{x_0})^2$.

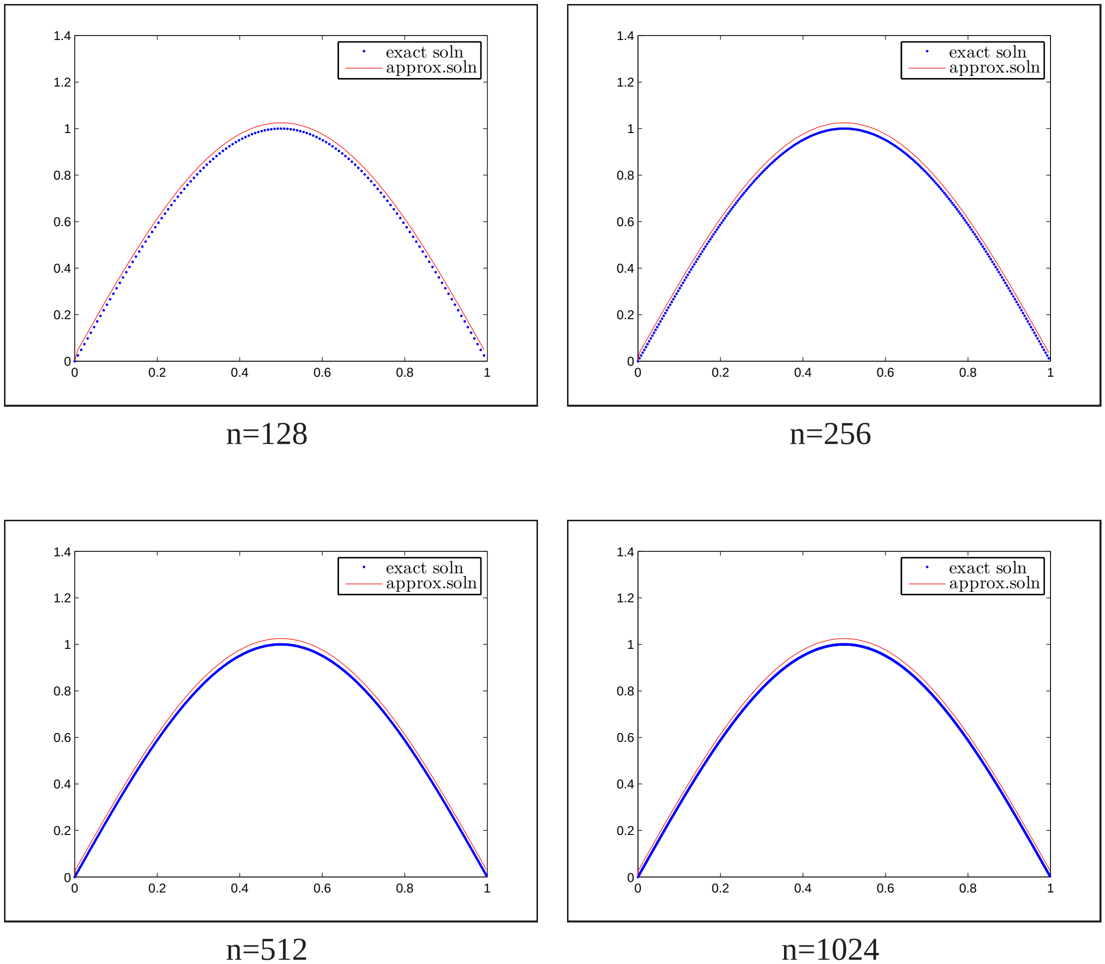


Figure 4.2: Curve of the exact and approximate solutions of Example 4.6.1

Further observe that

$$\begin{aligned}[F'(v) - F'(u)]w(s) &= 3\int_0^1 k(t,s)(v^2(s) - u^2(s))w(s)ds \\ &:= F'(u)\Phi(u,v,w),\end{aligned}$$

where $\Phi(u,v,w) = [\frac{v^2}{u^2} - 1]w$.

In our computation, we take

$$\begin{aligned}f(t) &= \frac{1}{36\pi^2}(27\sin\pi t - \sin 3\pi t) + \frac{1}{36\pi}(27t^2\cos\pi t - 3t^2\cos 3\pi t \\ &\quad +6t\cos 3\pi t - 3\cos 3\pi t - 27t\cos\pi t)\end{aligned}$$

and $f^\delta = f + \delta$*. Then the exact solution is*

$$\hat{x}(t) = \sin \pi t.$$

We use

$$x_0(t) = \sin \pi t + \frac{3}{4\pi^2}(1 + t\pi^2 - t^2\pi^2 - \cos^2(\pi t))$$

as our initial guess, so that the function $x_0 - \hat{x}$ *satisfies the source condition*

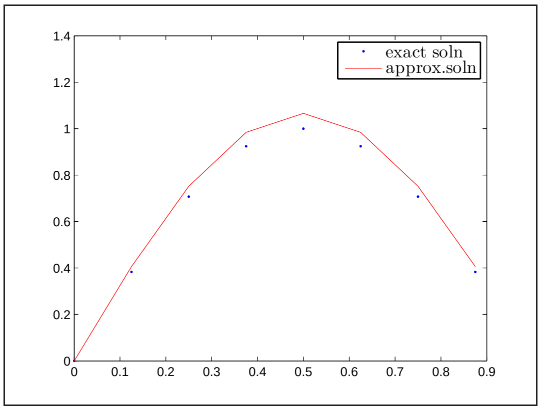
$$x_0 - \hat{x} = F'(\hat{x})1 = \varphi_1(F'(x_0))G(x_0, \hat{x})$$

where $\varphi_1(\lambda) = \lambda$*. Thus we expect to have an accuracy of order at least* $O((\delta + \varepsilon_h)^{\frac{1}{2}})$.

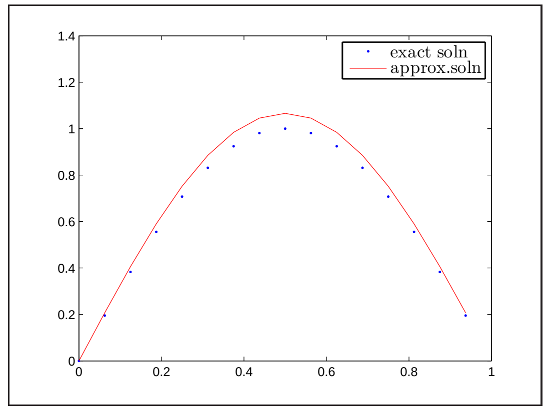
We choose $\alpha_0 = (1.5)\delta^2$, $\mu = 1.5$, $\delta = 0.0667 = c$, $\varepsilon_h = \frac{1}{10n^2}$, $\rho = 0.19$, $\tilde{\gamma}_\rho = 0.8173$ *and* $\tilde{g}_h(\tilde{\gamma}_\rho) = 0.54$ *approximately. For all* n, *the number of iteration* $n_k = 3$ *in this example. The results of the computation are presented in Table 4.2. The plots of the exact and the approximate solution obtained are given in Figures 4.3 and 4.4.*

n	k	α_k	$\|\tilde{u}_k^h - \hat{x}\|$	$\frac{\|\tilde{u}_k^h - \hat{x}\|}{(\delta+\varepsilon_h)^{1/2}}$
8	4	0.1790	0.0363	0.1388
16	4	0.1729	0.0432	0.1669
32	4	0.1714	0.0450	0.1742
64	4	0.1710	0.0455	0.1761
128	4	0.1709	0.0456	0.1765
256	4	0.1709	0.0456	0.1767
512	4	0.1709	0.0456	0.1767
1024	4	0.1709	0.0456	0.1767

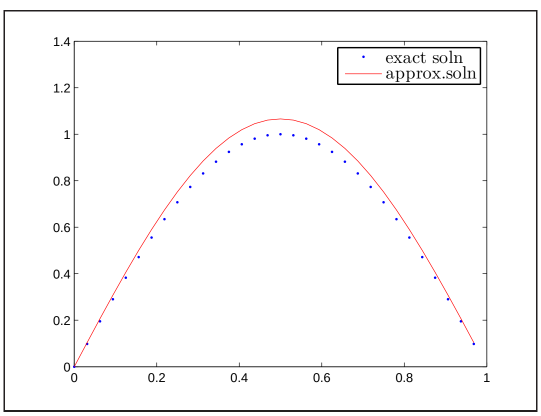
Table 4.2: Iterations and corresponding Error Estimates of Example 4.6.2



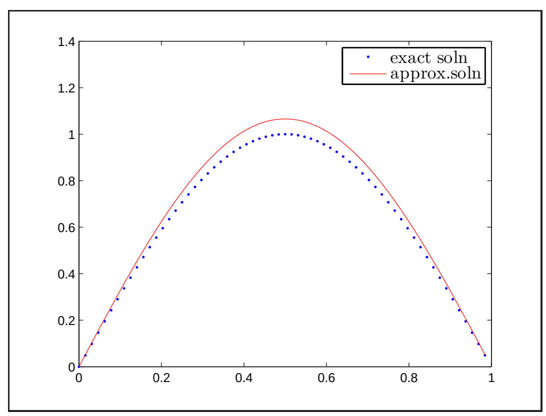
n=8



n=16

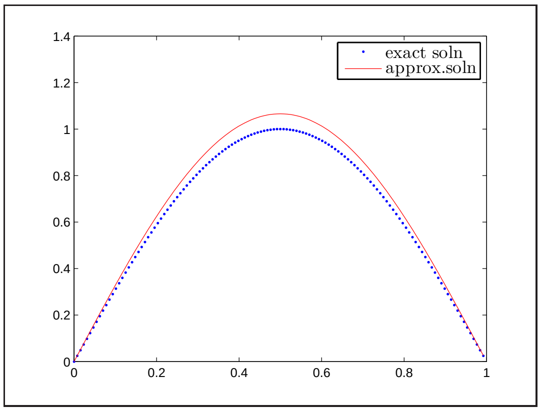


n=32

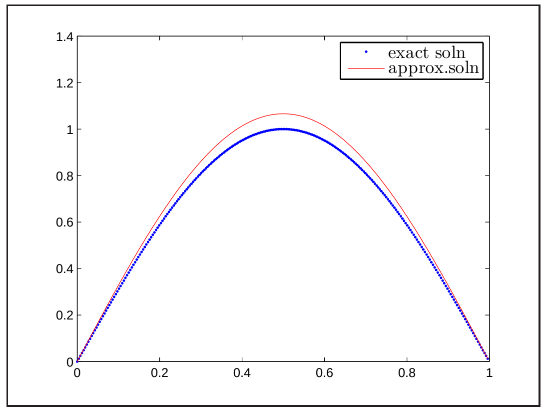


n=64

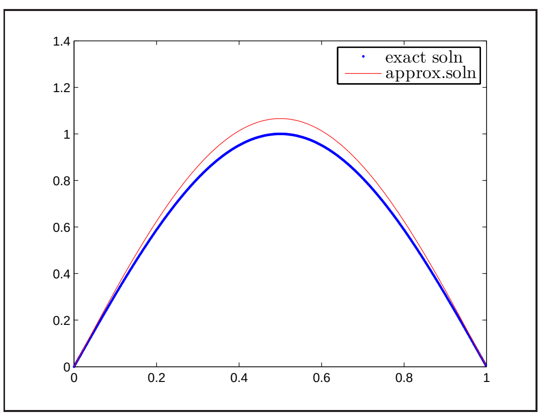
Figure 4.3: Curve of the exact and approximate solutions of Example 4.6.2



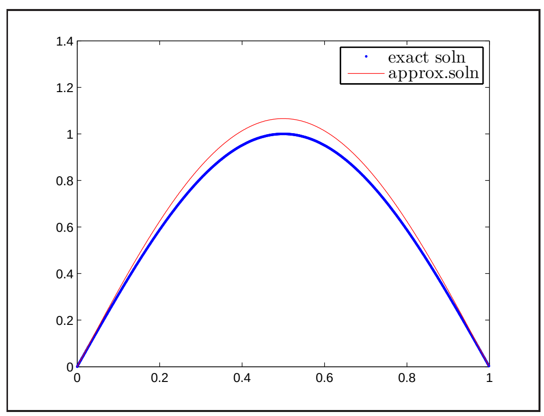
n=128



n=256



n=512



n=1024

Figure 4.4: Curve of the exact and approximate solutions of Example 4.6.2

Chapter 5

MODIFIED TSNTM WITH QUARTIC CONVERGENCE

The main aim of this chapter is to improve the rate of convergence of the methods considered in Chapter 2, 3 and 4 for obtaining an approximate solution for Ill-posed Hammerstein Operator equation (2.1.1). As in earlier Chapters we consider two regularity classes of the operator F, i.e., IFD Class and MFD Class. Regularization parameter is chosen according to the adaptive scheme suggested by Perverzev and Schock(2005). The error bounds obtained are of optimal order with respect to the general source conditions and we have obtained a quartic convergance rate.

5.1 INTRODUCTION

The preliminaries and adaptive scheme for choosing the regularization parameter α for Tikhonov regularization of (2.1.5) follows as in Chapter 2, 3 and 4. The proposed Modified Two Step Newton Tikhonov Method (MTSNTM) for both IFD and MFD class are given in Section 5.2. The finite dimensional approximation of the proposed method is given in Section 5.3 along with a numerical example in Section 5.5 to test the efficiency of the approach.

5.2 MODIFIED TWO STEP NEWTON-TYPE ITERATIVE METHOD

5.2.1 MTSNTM for IFD Class

For an initial guess $x_0 \in X$ the MTSNTM is defined as;

$$r^{\delta}_{n,\alpha_k} = s^{\delta}_{n,\alpha_k} - F'(s^{\delta}_{n,\alpha_k})^{-1}(F(s^{\delta}_{n,\alpha_k}) - z^{\delta}_{\alpha_k}), \tag{5.2.1}$$

$$s^{\delta}_{n+1,\alpha_k} = r^{\delta}_{n,\alpha_k} - F'(r^{\delta}_{n,\alpha_k})^{-1}(F(r^{\delta}_{n,\alpha_k}) - z^{\delta}_{\alpha_k}). \tag{5.2.2}$$

Throughout this section $s^{\delta}_{0,\alpha_k} = x_0$. Let

$$\varrho^{\delta}_{n,\alpha_k} := \|r^{\delta}_{n,\alpha_k} - s^{\delta}_{n,\alpha_k}\|, \qquad \forall n \geq 0 \tag{5.2.3}$$

and for $0 < k_0 \leq 1$, let $g_q : (0,1) \to (0,1)$ be the function defined by

$$g_q(t) = \frac{27k_0^3}{8}t^3, \qquad \forall t \in (0,1). \tag{5.2.4}$$

For convenience will use the notation s_n, r_n and ϱ_n for s^{δ}_{n,α_k}, r^{δ}_{n,α_k} and $\varrho^{\delta}_{n,\alpha_k}$ respectively. Assume that $\delta \in (0, \delta_0]$ where $\delta_0 < \frac{\sqrt{\alpha_0}}{\beta}$. Let $\|\hat{x} - x_0\| \leq \rho$,

$$\rho < \frac{1}{M}(\frac{1}{\beta} - \frac{\delta_0}{\sqrt{\alpha_0}})$$

and

$$\gamma_\rho := \beta[M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}].$$

THEOREM 5.2.1 *Let ϱ_n and $g_q(\varrho_n)$ be as in equation (5.2.3)and (5.2.4) respectively, s_n and r_n be as in (5.2.2) and (5.2.1) respectively with $\delta \in (0, \delta_0]$. Then under the assumptions of Theorem 2.2.3, the following hold:*

(a) $\|s_n - r_{n-1}\| \leq \frac{3k_0\varrho_{n-1}}{2}\|r_{n-1} - s_{n-1}\|$;

(b) $\|s_n - s_{n-1}\| \leq (1 + \frac{3k_0\varrho_{n-1}}{2})\|r_{n-1} - s_{n-1}\|$;

(c) $\|r_n - s_n\| \leq g_q(\varrho_{n-1})\|r_{n-1} - s_{n-1}\|$;

(d) $g_q(\varrho_n) \leq g_q(\gamma_\rho)^{4^n}, \qquad \forall n \geq 0$;

(e) $\varrho_n \leq g_q(\gamma_\rho)^{(4^n-1)/2}\gamma_\rho \qquad \forall n \geq 0.$

Proof. Observe that

$$\begin{aligned} s_{n+1} - r_n &= r_n - s_n - F'(r_n)^{-1}(F(r_n) - z^\delta_{\alpha_k}) + F'(s_n)^{-1}(F(s_n) - z^\delta_{\alpha_k})) \\ &= r_n - s_n - F'(r_n)^{-1}(F(r_n) - F(s_n)) - (F'(r_n)^{-1} \\ &\quad -F'(s_n)^{-1})(F(s_n) - z^\delta_{\alpha_k})) \\ &= F'(r_n)^{-1}\int_0^1 [F'(r_n) - F'(s_n + t(r_n - s_n))](r_n - s_n)dt \\ &\quad -F'(r_n)^{-1}(F'(r_n) - F'(s_n))(r_n - s_n) \end{aligned}$$

and hence by Assumption 2.3.1, we have

$$\begin{aligned} \|s_{n+1} - r_n\| &\leq \|\int_0^1 \Phi(r_n, s_n + t(r_n - s_n), r_n - s_n)dt\| \\ &\quad +\|\Phi(r_n, s_n, r_n - s_n)\| \\ &\leq \frac{3k_0}{2}\|r_n - s_n\|^2. \end{aligned}$$

This proves (a). The proof of (b) and (c) are analogous to the proof of corresponding results in Theorem 4.2.1.

Further, since for $\mu \in (0,1)$, $g_q(\mu t) \leq \mu^3 g_q(t)$, for all $t \in (0,1)$, by (c) we have,

$$g_q(\varrho_n) \leq g_q(\varrho_0)^{4^n}$$

and

$$\begin{aligned} \varrho_n \leq {g_q}^4(\varrho_{n-2})\varrho_{n-1} &\leq {g_q}^4(\varrho_{n-2}){g_q}^4(\varrho_{n-3})\varrho_{n-2}\cdots g_q(\varrho_0)\varrho_0 \\ &\leq g_q(\varrho_0)^{4^{n-1}+4^{n-2}+\cdots+1}\varrho_0 \\ &\leq g_q(\varrho_0)^{(4^n-1)/2}\varrho_0 \end{aligned} \tag{5.2.5}$$

provided $\varrho_n < 1, \forall n \geq 0$.

From (5.2.5) it is clear that, $\varrho_n \leq 1$ if $\varrho_0 \leq 1$, but by (4.2.8), $\varrho_0 \leq \gamma_\rho < 1$.

As g_q is monotonic increasing and $\varrho_0 \leq \gamma_\rho$, we have $g_q(\varrho_0) \leq g_q(\gamma_\rho)$. This completes the proof of the Theorem.

THEOREM 5.2.2 *Let $r = (\frac{1}{1-g_q(\gamma_\rho)} + \frac{3k_0}{2}\frac{\gamma_\rho}{1-g_q(\gamma_\rho)^2})\gamma_\rho$ with $g_q(\gamma_\rho) < 1$ and let the hypothesis of Theorem 5.2.1 holds. Then $s_n, r_n \in B_r(x_0)$, for all $n \geq 0$.*

Proof. Proof is analogous to the proof of Theorem 4.2.2 in Chapter 4.

Next we have the main theorem of this section.

THEOREM 5.2.3 *Let r_n and s_n be as in (5.2.1) and (5.2.2) respectively, assumptions of Theorem 5.2.2 hold and let $0 < g_q(\gamma_\rho) < 1$. Then (s_n) is a Cauchy sequence in $B_r(x_0)$ and converges to $x^\delta_{\alpha_k} \in \overline{B_r(x_0)}$. Further $F(x^\delta_{\alpha_k}) = z^\delta_{\alpha_k}$ and*

$$\|s_n - x^\delta_{\alpha_k}\| \leq C_q e^{-\gamma_2 4^n}$$

where $C_q = (\frac{1}{1-g_q(\gamma_\rho)^4} + \frac{3k_0\gamma_\rho}{2}\frac{1}{1-(g_q(\gamma_\rho)^2)^4} g_q(\gamma_\rho)^{4^n})\gamma_\rho$ and $\gamma_2 = -\log g_q(\gamma_\rho)$.

Proof. Proof is analogous to the proof of Theorem 4.2.3 in Chapter 4.

REMARK 5.2.4 *Note that $0 < g_q(\gamma_\rho) < 1$ and $\gamma > 0$. Hence the sequence (s_n) converges quartically to $x^\delta_{\alpha_k}$.*

Next we assume that $\rho \leq r < \frac{1}{k_0}$ and note that this assumption is satisfied if

$$k_0 \leq \min\left\{1, \frac{1-g_q(\gamma_\rho)^2}{3\gamma_\rho}[\frac{-1}{1-g_q(\gamma_\rho)} + \sqrt{\frac{1}{(1-g_q(\gamma_\rho))^2} + \frac{6}{1-g_q(\gamma_\rho)^2}}]\right\}.$$

The next Theorem is a consequence of Theorem 5.2.3 and Theorem 2.3.4.

THEOREM 5.2.5 *Let s_n be as in (5.2.2), assumptions in Theorem 5.2.3 and Theorem 2.3.4 hold. Then*

$$\|\hat{x} - s_n\| \leq C_q e^{-\gamma 4^n} + \frac{\beta}{1-k_0 r}\|F(\hat{x}) - z^\delta_{\alpha_k}\|$$

where C_q and γ are as in Theorem 5.2.3.

THEOREM 5.2.6 *Let s_n be as in (5.2.2), assumptions in Theorem 2.2.3 and Theorem 5.2.5 hold. Let*

$$n_k := \min\{n : e^{-\gamma 4^n} \leq \frac{\delta}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - s_{n_k}\| = O(\psi^{-1}(\delta)).$$

5.2.2 MTSNTM for MFD Class

Let X be a real Hilbert space. For an initial guess $x_0 \in X$ and for $R(x) := F'(x) + \frac{\alpha_k}{c} I$, the MTSNTM in this case is defined as:

$$\tilde{r}^{\delta}_{n,\alpha_k} = \tilde{s}^{\delta}_{n,\alpha_k} - R(\tilde{s}^{\delta}_{n,\alpha_k})^{-1}[F(\tilde{s}^{\delta}_{n,\alpha_k}) - z^{\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{s}^{\delta}_{n,\alpha_k} - x_0)] \tag{5.2.6}$$

and

$$\tilde{s}^{\delta}_{n+1,\alpha_k} = \tilde{r}^{\delta}_{n,\alpha_k} - R(\tilde{r}^{\delta}_{n,\alpha_k})^{-1}[F(\tilde{r}^{\delta}_{n,\alpha_k}) - z^{\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{r}^{\delta}_{n,\alpha_k} - x_0)]. \tag{5.2.7}$$

where $\tilde{s}_{0,\alpha_k} := x_0$. Note that with the above notation

$$\|R(x)^{-1}F'(x)\| \leq 1.$$

Let

$$\tilde{\varrho}^{\delta}_{n,\alpha_k} := \|\tilde{r}^{\delta}_{n,\alpha_k} - \tilde{s}^{\delta}_{n,\alpha_k}\|, \qquad \forall n \geq 0. \tag{5.2.8}$$

Here also for convenience we use the notation $\tilde{s}_n$, $\tilde{r}_n$ and $\tilde{\varrho}_n$ for $\tilde{s}^{\delta}_{n,\alpha_k}$, $\tilde{r}^{\delta}_{n,\alpha_k}$ and $\tilde{\varrho}^{\delta}_{n,\alpha_k}$ respectively. Let Assumption 2.3.1 holds with $\tilde{r}$ in place of r, $\rho \leq \tilde{r} < \frac{1}{k_0}$ and let $c \leq \alpha_k$. Let ρ and $\tilde{\gamma}_\rho$ be as defined in (4.2.16) and (4.2.17) respectively. Then we have the following Theorem.

THEOREM 5.2.7 *Let $\tilde{\varrho}_n$ and g_q be as in equation (5.2.8)and (5.2.4) respectively, $\tilde{s}_n$ and $\tilde{r}_n$ be as in (5.2.7) and (5.2.6) respectively with $\delta \in (0, \delta_0]$. Then the following hold:*

(a) $\|\tilde{s}_n - \tilde{r}_{n-1}\| \leq \frac{3k_0\tilde{\varrho}_{n-1}}{2}\|\tilde{r}_{n-1} - \tilde{s}_{n-1}\|$;

(b) $\|\tilde{s}_n - \tilde{s}_{n-1}\| \leq (1 + \frac{3k_0\tilde{\varrho}_{n-1}}{2})\|\tilde{r}_{n-1} - \tilde{s}_{n-1}\|$;

(c) $\|\tilde{r}_n - \tilde{s}_n\| \leq g_q(\tilde{\varrho}_{n-1})\|\tilde{r}_{n-1} - \tilde{s}_{n-1}\|$;

(d) $g_q(\tilde{\varrho}_n) \leq g_q(\tilde{\gamma}_\rho)^{4^n}, \qquad \forall n \geq 0$;

(e) $\tilde{\varrho}_n \leq g_q(\tilde{\gamma}_\rho)^{(4^n-1)/2}\tilde{\gamma}_\rho, \qquad \forall n \geq 0.$

Proof. Observe that

$$\begin{aligned}
\tilde{s}_n - \tilde{r}_{n-1} &= \tilde{r}_{n-1} - \tilde{s}_{n-1} - R(\tilde{r}_{n-1})^{-1}(F(\tilde{r}_{n-1}) - z^{\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{r}_{n-1} - x_0)) \\
&\quad + R(\tilde{s}_{n-1})^{-1}(F(\tilde{s}_{n-1}) - z^{\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{s}_{n-1} - x_0)) \\
&= \tilde{r}_{n-1} - \tilde{s}_{n-1} - R(\tilde{r}_{n-1})^{-1}(F(\tilde{r}_{n-1}) - F(\tilde{s}_{n-1}) + \frac{\alpha_k}{c}(\tilde{r}_{n-1} - \tilde{s}_{n-1})) \\
&\quad + (R(\tilde{s}_{n-1})^{-1} - R(\tilde{r}_{n-1})^{-1})(F(\tilde{s}_{n-1}) - z^{\delta}_{\alpha} + \frac{\alpha_k}{c}(\tilde{s}_{n-1} - x_0)) \\
&= R(\tilde{r}_{n-1})^{-1}[R(\tilde{r}_{n-1})(\tilde{r}_{n-1} - \tilde{s}_{n-1}) - (F(\tilde{r}_{n-1}) - F(\tilde{s}_{n-1})) - \frac{\alpha_k}{c}(\tilde{r}_{n-1} \\
&\quad - \tilde{s}_{n-1})] - R(\tilde{r}_{n-1})^{-1}[F'(\tilde{r}_{n-1}) - F'(\tilde{s}_{n-1})](\tilde{r}_{n-1} - \tilde{s}_{n-1}) \\
&= R(\tilde{r}_{n-1})^{-1}\int_0^1 [F'(\tilde{r}_{n-1}) - F'(\tilde{s}_{n-1} + t(\tilde{r}_{n-1} - \tilde{s}_{n-1}))] \\
&\quad \times(\tilde{r}_{n-1} - \tilde{s}_{n-1})dt - R(\tilde{r}_{n-1})^{-1}[F'(\tilde{r}_{n-1}) - F'(\tilde{s}_{n-1})](\tilde{r}_{n-1} - \tilde{s}_{n-1}).
\end{aligned}$$

Now since $\|R(\tilde{s}_{n-1})^{-1}F'(\tilde{s}_{n-1})\| \leq 1$, the proof of (a) and (b) follows as in Theorem 5.2.1. To prove (c) we observe that

$$\begin{aligned}
\tilde{\varrho}_n &\leq \|\tilde{s}_n - \tilde{r}_{n-1} - R(\tilde{s}_n)^{-1}(F(\tilde{s}_n) - F(\tilde{r}_{n-1}) + \frac{\alpha_k}{c}(\tilde{s}_n - \tilde{r}_{n-1}))\| \\
&\quad + \|R(\tilde{x}_n)^{-1}(R(\tilde{x}_n) - R(\tilde{r}_{n-1}))(\tilde{s}_n - \tilde{r}_{n-1})\| \\
&\leq \|R(\tilde{s}_n)^{-1}[R(\tilde{s}_n)(\tilde{s}_n - \tilde{r}_{n-1}) - (F(\tilde{s}_n) - F(\tilde{r}_{n-1})) \\
&\quad - \frac{\alpha_k}{c}(\tilde{s}_n - \tilde{r}_{n-1})]\| \\
&\quad + \|R(\tilde{x}_n)^{-1}(F'(\tilde{x}_n) - F'(\tilde{r}_{n-1}))(\tilde{s}_n - \tilde{r}_{n-1})\|.
\end{aligned}$$

The remaining part of the proof is analogous to the proof of Theorem 5.2.1.

We state the following Theorems whose proofs are analogous to the proof of Theorems 4.2.9, 4.2.10, 4.2.11 and 4.2.12 respectively.

THEOREM 5.2.8 *Let* $\tilde{r} = (\frac{1}{1-g_q(\tilde{\gamma}_\rho)} + \frac{3k_0}{2}\frac{\tilde{\gamma}_\rho}{1-g_q(\tilde{\gamma}_\rho)^2})\tilde{\gamma}_\rho$ *with* $g_q(\tilde{\gamma}_\rho) < 1$ *and the assumptions of Theorem 5.2.7 hold. Then* $\tilde{s}_n, \tilde{r}_n \in B_{\tilde{r}}(x_0)$*, for all* $n \geq 0$*.*

THEOREM 5.2.9 *Let* $\tilde{r}_n$ *and* $\tilde{s}_n$ *be as in (5.2.6) and (5.2.7) respectively and assumptions of Theorem 5.2.8 hold. Then* $(\tilde{s}_n)$ *is a Cauchy sequence in* $B_{\tilde{r}}(x_0)$ *and converges to* $x^{\delta}_{c,\alpha_k} \in \overline{B_{\tilde{r}}(x_0)}$*. Further* $F(x^{\delta}_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^{\delta}_{c,\alpha_k} - x_0) = z^{\delta}_{\alpha_k}$ *and*

$$\|\tilde{s}_n - x^{\delta}_{c,\alpha_k}\| \leq \tilde{C}_q e^{-\tilde{\gamma}_2 4^n}$$

where $\tilde{C}_q = (\frac{1}{1-g(\tilde{\gamma}_\rho)^4} + \frac{3k_0\tilde{\gamma}_\rho}{2}\frac{1}{1-(g_q(\tilde{\gamma}_\rho)^2)^4}g(\tilde{\gamma}_\rho)^{4^n})\tilde{\gamma}_\rho$ *and* $\tilde{\gamma}_2 = -\log g_q(\tilde{\gamma}_\rho)$*.*

THEOREM 5.2.10 *Let $\tilde{s}_n$ be as in (5.2.7), assumptions in Theorem 5.2.9 and Theorem 2.3.11 hold. Then*

$$\|\hat{x} - \tilde{s}_n\| \leq \tilde{C}_q e^{-\tilde{\gamma}_2 4^n} + O(\psi^{-1}(\delta))$$

where $\tilde{C}_q$ and $\tilde{\gamma}_2$ are as in Theorem 5.2.9.

THEOREM 5.2.11 *Let $\tilde{s}_n$ be as in (5.2.7), assumptions in Theorem 2.2.3 and Theorem 5.2.10 hold. Let*

$$n_k := \min\{n : e^{-\gamma_1 4^n} \leq \frac{\delta}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - \tilde{s}_{n_k}\| = O(\psi^{-1}(\delta)).$$

5.3 PROJECTION SCHEME OF MTSNTM

In this section we consider the convergence analysis of MTSNTM in the finite dimensional setting. The method is analyzed for both the cases of operator F i.e., IFD and MFD Class. The finite dimensional realization of the method and the associated algorithm are proposed. Local-quartic convergence is established for the method and is validated numerically. The proofs of the results are analogous to the corresponding results in section DTSNTM of Chapter 3.

5.3.1 Discretization of MTSNTM for IFD Class

For an initial guess $x_0 \in X$ the method is defined as;

$$r^{h,\delta}_{n,\alpha_k} = s^{h,\delta}_{n,\alpha_k} - P_h F'(s^{h,\delta}_{n,\alpha_k})^{-1} P_h(F(s^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k}), \tag{5.3.9}$$

$$s^{h,\delta}_{n+1,\alpha_k} = r^{h,\delta}_{n,\alpha_k} - P_h F'(r^{h,\delta}_{n,\alpha_k})^{-1} P_h(F(r^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k}), \tag{5.3.10}$$

where $s^{h,\delta}_{0,\alpha_k} := P_h x_0$.

Note that if $b_0 < \frac{1}{k_0}$ and if $x \in B_r(P_h x_0)$ where $r < \frac{1}{k_0} - b_0$, then $F'(x)^{-1}$ exists and is bounded i.e.,

$$\|F'(x)^{-1}\| \leq \beta, \ \ \forall x \in B_r(P_h x_0), \ \ \beta > 0. \tag{5.3.11}$$

Let

$$\varrho_{n,\alpha_k}^{h,\delta} := \|r_{n,\alpha_k}^{h,\delta} - s_{n,\alpha_k}^{h,\delta}\|, \qquad \forall n \geq 0 \tag{5.3.12}$$

and let $g_p : (0,1) \to (0,1)$ be defined by

$$g_p(t) = \frac{27k_0^3}{8}(1+\beta\tau_0)^3 t^3 \qquad \forall t \in (0,1), \tag{5.3.13}$$

where

$$k_0 < \min\{1, \frac{2}{3(1+\beta\tau_0)}\}.$$

Hereafter we assume that $\delta_0 + \varepsilon_0 < \frac{2}{\beta(2M+3)}\sqrt{\alpha_0}$. Let $\|\hat{x} - x_0\| \leq \rho$ where

$$\rho < \frac{1}{M}[\frac{1}{\beta} - (M + \frac{3}{2})\frac{\delta_0 + \varepsilon_0}{\sqrt{\alpha_0}}]$$

and let

$$\gamma_\rho := \beta[M\rho + (M + \frac{3}{2})(\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}})].$$

One can see that $\gamma_\rho < 1$ and hence $g_p(\gamma_\rho) < 1$.

In the next theorem we obtain an estimate for $\varrho_{n,\alpha_k}^{h,\delta}$ in terms of $g_p(\gamma_\rho)$ under the assumption that $s_{n,\alpha_k}^{h,\delta}$ and $r_{n,\alpha_k}^{h,\delta}$ are in $B_r(P_h x_0)$. Later in Theorem 5.3.2 we prove that $s_{n,\alpha_k}^{h,\delta},\ r_{n,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$, by induction.

THEOREM 5.3.1 *Let $\varrho_{n,\alpha_k}^{h,\delta}$ and $g_p(\varrho_{n,\alpha_k}^{h,\delta})$ be as in equation (5.3.12) and (5.3.17) respectively, $r_{n,\alpha_k}^{h,\delta}$ and $s_{n,\alpha_k}^{h,\delta}$ be as in (5.3.9) and (5.3.10) respectively with $\delta \in (0, \delta_0]$, $\alpha = \alpha_k$ and $\varepsilon_h \in (0, \varepsilon_0]$. If $s_{n,\alpha_k}^{h,\delta},\ r_{n,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$, then by Assumption 2.3.1 and Lemma 4.3.1, the following hold:*

(a) $\|s_{n,\alpha_k}^{h,\delta} - r_{n-1,\alpha_k}^{h,\delta}\| \leq (1+\beta\tau_0)\frac{3k_0\varrho_{n-1,\alpha_k}^{h,\delta}}{2}\|r_{n-1,\alpha_k}^{h,\delta} - s_{n-1,\alpha_k}^{h,\delta}\|$;

(b) $\|s_{n,\alpha_k}^{h,\delta} - s_{n-1,\alpha_k}^{h,\delta}\| \leq (1 + (1+\beta\tau_0)\frac{3k_0\varrho_{n-1,\alpha_k}^{h,\delta}}{2})\|r_{n-1,\alpha_k}^{h,\delta} - s_{n-1,\alpha_k}^{h,\delta}\|$;

(c) $\|r_{n,\alpha_k}^{h,\delta} - s_{n,\alpha_k}^{h,\delta}\| \leq g(\varrho_{n-1,\alpha_k}^{h,\delta})\|r_{n-1,\alpha_k}^{h,\delta} - s_{n-1,\alpha_k}^{h,\delta}\|$;

(d) $g_p(\varrho_{n,\alpha_k}^{h,\delta}) \leq g_p(\gamma_\rho)^{4^n}, \qquad \forall n \geq 0$;

(e) $\varrho_{n,\alpha_k}^{h,\delta} \leq g_p(\gamma_\rho)^{(4^n-1)/2}\gamma_\rho, \qquad \forall n \geq 0.$

THEOREM 5.3.2 *Let $r = (\frac{1}{1-g_p(\gamma_\rho)} + \frac{(1+\beta\tau_0)3k_0}{2}\frac{\gamma_\rho}{1-g_p(\gamma_\rho)^2})\gamma_\rho$ with $g_p(\gamma_\rho) < 1$ and let the hypothesis of Theorem 5.3.1 holds. Then $s_{n,\alpha_k}^{h,\delta}, r_{n,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$, for all $n \geq 0$.*

THEOREM 5.3.3 *Let $r^{h,\delta}_{n,\alpha_k}$ and $s^{h,\delta}_{n,\alpha_k}$ be as in (5.3.9) and (5.3.10) respectively, assumptions of Theorem 5.3.2 hold and let $0 < g_p(\gamma_\rho) < 1$. Then $(s^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_r(P_h x_0)$ and converges to $x^{h,\delta}_{\alpha_k} \in \overline{B_r(P_h x_0)}$. Further $P_h F(x^{h,\delta}_{\alpha_k}) = z^{h,\delta}_{\alpha_k}$ and*

$$\|s^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{\alpha_k}\| \leq C_p e^{-\gamma_3 4^n}$$

where $C_p = (\frac{1}{1-g_p(\gamma_\rho)^4} + (1+\beta\tau_0)\frac{3k_0\gamma_\rho}{2}\frac{1}{1-(g_p(\gamma_\rho)^2)^4} g_p(\gamma_\rho)^{4^n})\gamma_\rho$ and $\gamma_3 = -\log g_p(\gamma_\rho)$.

Note that $\rho \leq r$ and $k_0 < \frac{1-g_p(\gamma_\rho)^2}{3(1+\beta\tau_0)\gamma_\rho}[\frac{-1}{1-g_p(\gamma_\rho)} + \sqrt{\frac{1}{(1-g_p(\gamma_\rho))^2} + \frac{6}{(1-g_p(\gamma_\rho)^2)}}]$. Hereafter we assume that $\rho \leq r < \frac{1}{(1+\beta\tau_0)k_0}$.

THEOREM 5.3.4 *Let $s^{h,\delta}_{n,\alpha_k}$ be as in (5.3.10), assumptions in Theorem 5.3.3 and Theorem 3.3.5 hold. Then*

$$\|\hat{x} - s^{h,\delta}_{n,\alpha_k}\| \leq C_p e^{-\gamma_3 4^n} + \frac{\beta}{(1-(1+\beta\tau_0)k_0 r)}\|F(\hat{x}) - z^{h,\delta}_{\alpha_k}\|$$

where C_p and γ_3 are as in Theorem 5.3.3.

THEOREM 5.3.5 *Let $s^{h,\delta}_{n,\alpha_k}$ be as in (5.3.10), assumptions in Theorem 5.3.4 hold. Let*

$$n_k := \min\{n : e^{-\gamma_3 4^n} \leq \frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - s^{h,\delta}_{n_k,\alpha_k}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

5.3.2 Discretization of MTSNTM for MFD Class

For an initial guess $x_0 \in X$ and for $R(x) := P_h F'(x) P_h + \frac{\alpha_k}{c} P_h$, the discretization of MTSNTM is defined as:

$$\tilde{r}^{h,\delta}_{n,\alpha_k} = \tilde{s}^{h,\delta}_{n,\alpha_k} - R(\tilde{s}^{h,\delta}_{n,\alpha_k})^{-1} P_h[F(\tilde{s}^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{s}^{h,\delta}_{n,\alpha_k} - \tilde{s}^{h,\delta}_{0,\alpha_k})] \tag{5.3.14}$$

and

$$x^{h,\delta}_{n+1,\alpha_k} = \tilde{r}^{h,\delta}_{n,\alpha_k} - R(\tilde{r}^{h,\delta}_{n,\alpha_k})^{-1} P_h[F(\tilde{r}^{h,\delta}_{n,\alpha_k}) - z^{h,\delta}_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{r}^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{0,\alpha_k})] \tag{5.3.15}$$

where $\tilde{s}^{h,\delta}_{0,\alpha_k} := P_h x_0$.

Let

$$\tilde{\varrho}^{h,\delta}_{n,\alpha_k} := \|\tilde{r}^{h,\delta}_{n,\alpha_k} - \tilde{s}^{h,\delta}_{n,\alpha_k}\|, \qquad \forall n \geq 0 \tag{5.3.16}$$

and let k_0 be such that $k_0 < \min\{1, \frac{2}{3(1+\tau_0)}\}$. Let $\tilde{g_p} : (0,1) \to (0,1)$ be the function defined by

$$\tilde{g_p}(t) = \frac{27k_0^3}{8}(1+\tau_0)^3 t^3, \qquad \forall t \in (0,1). \tag{5.3.17}$$

Let $\|\hat{x} - x_0\| \leq \rho$, with

$$\rho < \frac{1}{M}(1 - (\frac{3}{2} + M)\frac{\delta_0 + \varepsilon_0}{\sqrt{\alpha_0}})$$

and

$$\tilde{\gamma}_\rho := M\rho + (\frac{3}{2} + M)(\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}}).$$

THEOREM 5.3.6 *Let $\tilde{\varrho}^{h,\delta}_{n,\alpha_k}$ and $\tilde{g_p}$ be as in equation (5.3.16)and (5.3.17) respectively, $\tilde{s}^{h,\delta}_{n,\alpha_k}$ and $\tilde{r}^{h,\delta}_{n,\alpha_k}$ be as in (5.3.15) and (5.3.14) respectively with $\delta \in (0, \delta_0]$, $\alpha = \alpha_k$ and $\varepsilon_h \in (0, \varepsilon_0]$. Then by Assumption 2.3.1 and (4.3.34) the following hold:*

(a) $\|\tilde{s}^{h,\delta}_{n,\alpha_k} - \tilde{r}^{h,\delta}_{n-1,\alpha_k}\| \leq (1+\tau_0)\frac{3k_0\tilde{\varrho}^{h,\delta}_{n-1,\alpha_k}}{2}\|\tilde{r}^{h,\delta}_{n-1,\alpha_k} - \tilde{s}^{h,\delta}_{n-1,\alpha_k}\|$;

(b) $\|\tilde{s}^{h,\delta}_{n,\alpha_k} - \tilde{s}^{h,\delta}_{n-1,\alpha_k}\| \leq (1 + (1+\tau_0)\frac{3k_0\tilde{\varrho}^{h,\delta}_{n-1,\alpha_k}}{2})\|\tilde{r}^{h,\delta}_{n-1,\alpha_k} - \tilde{s}^{h,\delta}_{n-1,\alpha_k}\|$;

(c) $\|\tilde{r}^{h,\delta}_{n,\alpha_k} - \tilde{s}^{h,\delta}_{n,\alpha_k}\| \leq \tilde{g_p}(\tilde{\varrho}^{h,\delta}_{n-1,\alpha_k})\|\tilde{r}^{h,\delta}_{n-1,\alpha_k} - \tilde{s}^{h,\delta}_{n-1,\alpha_k}\|$;

(d) $\tilde{g_p}(\tilde{\varrho}^{h,\delta}_{n,\alpha_k}) \leq \tilde{g_p}(\tilde{\gamma}_\rho)^{4^n}, \qquad \forall n \geq 0$;

(e) $\tilde{\varrho}^{h,\delta}_{n,\alpha_k} \leq \tilde{g_p}(\tilde{\gamma}_\rho)^{(4^n-1)/2}\tilde{\gamma}_\rho, \qquad \forall n \geq 0.$

THEOREM 5.3.7 *Let $\tilde{r} = (\frac{1}{1-\tilde{g_p}(\tilde{\gamma}_\rho)} + (1+\tau_0)\frac{3k_0}{2}\frac{\tilde{\gamma}_\rho}{1-\tilde{g_p}(\tilde{\gamma}_\rho)^2})\tilde{\gamma}_\rho$ with $\tilde{g_p}(\tilde{\gamma}_\rho) < 1$ and the assumptions of Theorem 5.3.6 hold. Then $\tilde{s}^{h,\delta}_{n,\alpha_k}, \tilde{r}^{h,\delta}_{n,\alpha_k} \in B_{\tilde{r}}(P_h x_0)$, for all $n \geq 0$.*

The main result of this section is the following Theorem.

THEOREM 5.3.8 *Let $\tilde{r}^{h,\delta}_{n,\alpha_k}$ and $\tilde{s}^{h,\delta}_{n,\alpha_k}$ be as in (5.3.14) and (5.3.15) respectively and assumptions of Theorem 5.3.7 hold. Then $(\tilde{s}^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_{\tilde{r}}(P_h x_0)$ and converges to $x^{h,\delta}_{c,\alpha_k} \in \overline{B_{\tilde{r}}(P_h x_0)}$. Further $P_h[F(x^{h,\delta}_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^{h,\delta}_{c,\alpha_k} - x_0)] = P_h z^{h,\delta}_{\alpha_k}$ and*

$$\|\tilde{s}^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{c,\alpha_k}\| \leq \overline{C_p} e^{-\tilde{\gamma}_3 4^n}$$

where $\overline{C_p} = (\frac{1}{1-\tilde{g_p}(\tilde{\gamma}_\rho)^4} + (1+\tau_0)\frac{3k_0\tilde{\gamma}_\rho}{2}\frac{1}{1-(\tilde{g_p}(\tilde{\gamma}_\rho)^2)^4}\tilde{g_p}(\tilde{\gamma}_\rho)^{4^n})\tilde{\gamma}_\rho$ and $\tilde{\gamma}_3 = -\log \tilde{g_p}(\tilde{\gamma}_\rho)$.

THEOREM 5.3.9 *Let $\tilde{s}^{h,\delta}_{n,\alpha_k}$ be as in (5.3.15), assumptions in Theorem 5.3.8, (2.3.17) and (3.3.23) hold. Then*

$$\|\hat{x} - \tilde{s}^{h,\delta}_{n,\alpha_k}\| \leq \overline{C_p} e^{-\tilde{\gamma}_3 4^n} + \frac{\varphi_1(\alpha_k) + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta + \varepsilon_h)}{1 - (1-c)k_2 - k_0\tilde{r}} + \frac{2}{1-\tau_0}(\frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}})$$

where $\overline{C_p}$ and $\tilde{\gamma}_3$ are as in Theorem 5.3.8.

THEOREM 5.3.10 *Let $\tilde{s}^{h,\delta}_{n,\alpha_k}$ be as in (5.3.15) and assumptions in Theorem 5.3.9 hold. Further let $\varphi_1(\alpha_k) \leq \varphi(\alpha_k)$ and*

$$n_k := \min\{n : e^{-\tilde{\gamma}_3 4^n} \leq \frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - \tilde{s}^{h,\delta}_{n,\alpha_k}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

5.4 ALGORITHM

Note that for $i, j \in \{0, 1, 2, \cdots, N\}$,

$$z^{h,\delta}_{\alpha_i} - z^{h,\delta}_{\alpha_j} = (\alpha_j - \alpha_i)(P_h K^* K P_h + \alpha_j I)^{-1}(P_h K^* K P_h + \alpha_i I)^{-1} P_h K^*(f^\delta - KF(x_0)).$$

Therefore the balancing principle algorithm associated with the choice of the parameter involves the following steps.

Step 1: (a) Choose α_0 such that $\delta_0 + \varepsilon_0 < \frac{2\sqrt{\alpha_0}}{\beta(2M+3)}$, $\mu > \{1, \frac{\beta(2M+3)}{2}\}$ for IFD Class and $\delta_0 + \varepsilon_0 < \frac{2\sqrt{\alpha_0}}{2M+3}$ and $\mu > 1$ for MFD Class;

Step 2: $\alpha_i = \mu^{2i}\alpha_0$;

Step 3: solve for w_i:

$$(P_h K^* K P_h + \alpha_i I)w_i = P_h K^*(f^\delta - KF(x_0)); \tag{5.4.18}$$

Step 4: solve for $j < i$, $z^{h,\delta}_{ij}$: $(P_h K^* K P_h + \alpha_j I)z^{h,\delta}_{ij} = (\alpha_j - \alpha_i)w_i$;

Step 5: if $\|z^{h,\delta}_{ij}\| > \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$, then take $k = i - 1$;

Step 6: otherwise, repeat with $i + 1$ in place of i.

Step 7: choose $n_k = \min\{n : e^{-\gamma_3 4^n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$ for IFD Class and $n_k = \min\{n : e^{-\tilde{\gamma}_3 4^n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$ for MFD Class.

Step 8: solve $s^{h,\delta}_{n_k,\alpha_k}$ using the iteration (5.3.10) or $\tilde{s}^{h,\delta}_{n_k,\alpha_k}$ using the iteration (5.3.15) .

5.5 NUMERICAL EXAMPLES

In this section we give an example for IFD Class and MFD Class for illustrating the algorithm considered in the above section. We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with dim $V_n = n+1$. Precisely we choose V_n as the space of linear splines in a uniform grid of $n+1$ points in $[0,1]$. The implementation of the method is analogous to that given in Chapter 4.

EXAMPLE 5.5.1 *To illustrate the method for IFD Class, we consider the operator* $KF : L^2(0,1) \longrightarrow L^2(0,1)$ *where* $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$F(u) := u^3,$$

and $K : L^2(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds$$

where $k(t,s) = \begin{cases} (1-t)s, 0 \leq s \leq t \leq 1 \\ (1-s)t, 0 \leq t \leq s \leq 1 \end{cases}$.

The Fréchet derivative of F *is given by* $F'(u)w = 3(u^2)w$. *So*

$$[F'(v) - F'(u)]w = 3(v^2-u^2)w = 3u^2(\frac{v^2}{u^2}-1)w = F'(u)\Phi(u,v,w),$$

where $\Phi(u,v,w) = (\frac{v^2}{u^2}-1)w = \frac{(v+u)(v-u)}{u^2}w$. *Thus* F *satisfies the Assumption 2.3.1.*

We take

$$f(t) = \frac{-1}{144\pi^2}[-54 + 63\pi^2t^2 - 220\sin(\pi t) + 16\sin(\pi t)\cos^2(\pi t) + 54\cos^2(\pi t) - 63\pi^2 t]$$

and $f^\delta = f + \delta$. *Then the exact solution*

$$\hat{x}(t) = 1/2 + \sin \pi t.$$

We use

$$x_0(t) = \sin \pi t + 3/5$$

as our initial guess, then

$$F(x_0) - F(\hat{x}) = x_0^3 - \hat{x}^3.$$

Even though we are unable to write

$$F(x_0) - F(\hat{x}) = \varphi(K^*K)w$$

for some function φ, we use the function $\varphi(\lambda) = \lambda$ and obtain the results as given in the last column of the Table 1. Thus we expect to have an accuracy of order at least $O(\delta^{\frac{1}{2}})$.

We choose $\alpha_0 = (1.5)(\delta + \varepsilon_h)^2$, $\mu = 1.3$, $(\delta + \varepsilon_h) = 0.1$, $g_p(\gamma_\rho) = 0.54$ approximately. In this example, for all n, the number of iteration $n_k = 1$. The results of the computation are presented in Table 5.1. The plots of the exact and the approximate solution obtained for n=8 to 1024 are given in Figures 5.1 and 5.2.

n	k	α	$\|s_k^h - \hat{x}\|$	$\frac{\|s_k^h - \hat{x}\|}{(\delta+\varepsilon_h)^{1/2}}$
8	4	0.1094	0.2010	0.6307
16	4	0.1069	0.1361	0.4296
32	4	0.1063	0.0959	0.3031
64	4	0.1061	0.0701	0.2218
128	4	0.1061	0.0536	0.1696
256	4	0.1060	0.0434	0.1371
512	4	0.1060	0.0373	0.1178
1024	4	0.1060	0.0338	0.1069

Table 5.1: Iterations and corresponding Error Estimates of Example 5.5.1

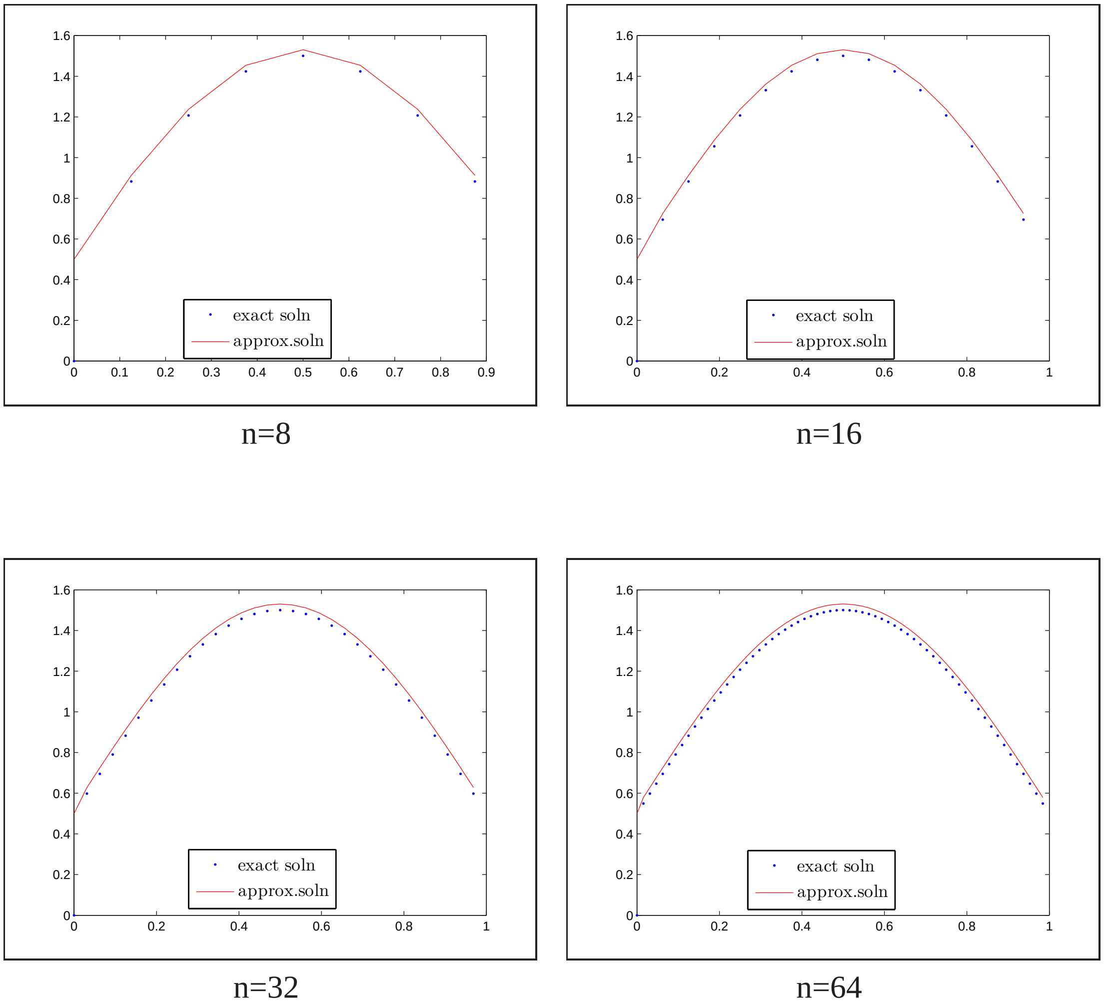


Figure 5.1: Curve of the exact and approximate solutions of Example 5.5.1

EXAMPLE 5.5.2 *To illustrate the method for Case 2, we consider the operator*

$$KF : L^2(0,1) \longrightarrow L^2(0,1)$$

where $K : L^2(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds$$

and $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$F(u) := \int_0^1 k(t,s)u^3(s)ds,$$

where $k(t,s) = \begin{cases} (1-t)s, 0 \le s \le t \le 1 \\ (1-s)t, 0 \le t \le s \le 1 \end{cases}.$

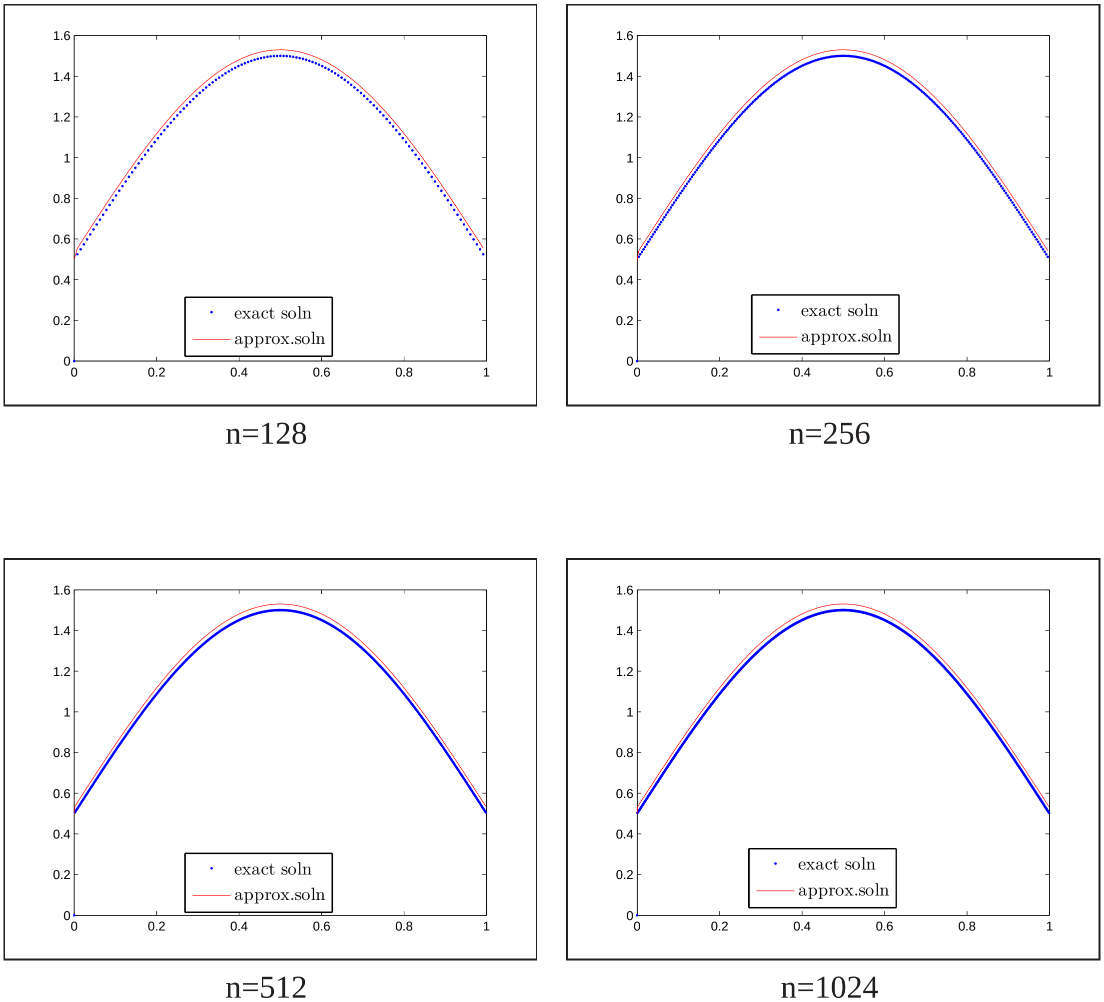


Figure 5.2: Curve of the exact and approximate solutions of Example 5.5.1

Then for all $x(t), y(t) : x(t) > y(t)$:

$$\begin{aligned}\langle F(x) - F(y), x - y\rangle &= \int_0^1 \left(\int_0^1 k(t,s)(x^3 - y^3)(s)ds\right) \\ &\quad \times (x-y)(t)dt \geq 0.\end{aligned}$$

Thus the operator F *is monotone. The Fréchet derivative of* F *is given by*

$$F'(u)w = 3\int_0^1 k(t,s)(u(s))^2 w(s)ds.$$

So for any $u \in B_r(x_0), {x_0}^2(s) \geq k_3 > 0, \forall s \in (0,1)$, *we have*

$$F'(u)w = F'(x_0)G(u, x_0)w,$$

where $G(u, x_0) = (\frac{u}{x_0})^2$.

Further observe that

$$\begin{aligned} [F'(v) - F'(u)]w(s) &= 3\int_0^1 k(t,s)(v^2(s) - u^2(s))w(s)ds \\ &:= F'(u)\Phi(u, v, w), \end{aligned}$$

where $\Phi(u, v, w) = [\frac{v^2}{u^2} - 1]w$.

Thus Φ *satisfies the Assumption 2.3.1 (cf. Scherzer, Engl and Kunisch (1993), Example 2.7).*

In our computation, we take

$$\begin{aligned} f(t) &= (\frac{1}{18\pi^2})(1-t)(14t - 7 + cos^3(\pi t) + 6cos(\pi t))t^2 - (\frac{1}{18\pi^2})t(14t - 7 \\ &\quad + cos^3(\pi t) + 6cos(\pi t))(1 - t^2) + (\frac{1}{9\pi^2})t(1-t)(14t - 7 \\ &\quad + cos^3(\pi t) + 6cos(\pi t)) \end{aligned}$$

and $f^\delta = f + \delta$. *Then the exact solution is*

$$\hat{x}(t) = cos\pi t.$$

We use

$$\begin{aligned} x_0(t) &= cos(\pi t) + 3[\frac{-1}{4\pi^2}(1 - t + 2\pi t^2 cos(\pi t)sin(\pi t) + \pi^2 t^3 \\ &\quad + tcos^2(\pi t) - 2\pi tcos(\pi t)\ sin(\pi t) - \pi^2 t^2 - cos^2(\pi t)) \\ &\quad + \frac{1}{4\pi^2}t(-2cos(\pi t)sin(\pi t)\pi - 2\pi^2 t + 2\pi tcos(\pi t)sin(\pi t) \\ &\quad + \pi^2 t^2 + cos^2(\pi t) + \pi^2 - cos^2(\pi t))] \end{aligned}$$

as our initial guess, so that the function $x_0 - \hat{x}$ *satisfies the source condition*

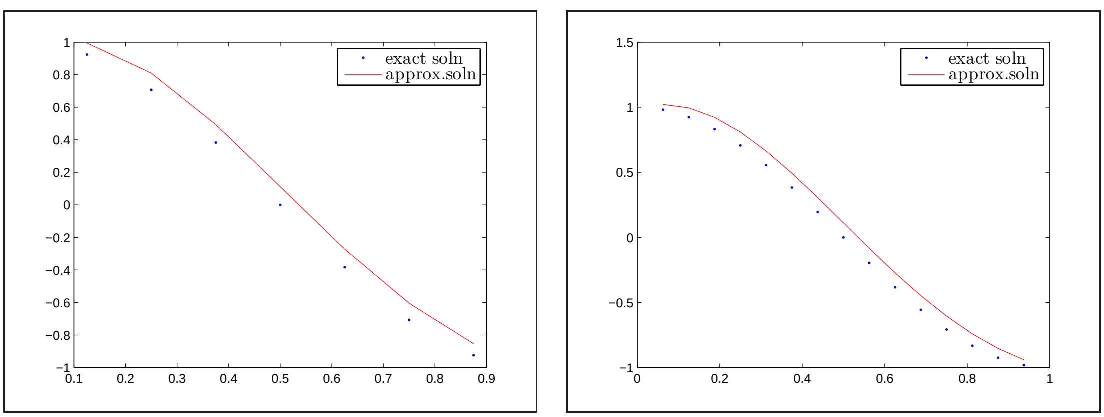
$$x_0 - \hat{x} = \varphi_1(F'(x_0))1$$

where $\varphi_1(\lambda) = \lambda$. *Thus we expect to have an accuracy of order at least* $O(\delta^{\frac{1}{2}})$.

We choose $\alpha_0 = (1.3)(\delta + \varepsilon_h)^2$, $\mu = 1.3$, $\delta + \varepsilon_h = 0.1 = c$, $\rho = 0.19$, $\tilde{\gamma}_\rho = 0.8173$ *and* $\tilde{g}_p(\tilde{\gamma}_\rho) = 0.54$ *approximately. For all* n *the number of iteration* $n_k = 1$. *The results of the computation are presented in Table 5.2. The plots of the exact and the approximate solution obtained are given in Figures 5.3 and 5.4.*

n	k	$\delta + \varepsilon_h$	α	$\|\tilde{s}_k^h - \hat{x}\|$	$\frac{\|\tilde{s}_k^h - \hat{x}\|}{(\delta+\varepsilon_h)^{1/2}}$
8	4	0.1016	0.1094	0.3652	1.1458
16	4	0.1004	0.1069	0.2664	0.8408
32	4	0.1001	0.1063	0.1994	0.6303
64	4	0.1000	0.1061	0.1554	0.4914
128	4	0.1000	0.1061	0.1278	0.4042
256	4	0.1000	0.1060	0.1115	0.3526
512	4	0.1000	0.1060	0.1024	0.3238
1024	4	0.1000	0.1060	0.0975	0.3083

Table 5.2: Iterations and corresponding Error Estimates of Example 5.5.2



n=8 n=16

Figure 5.3: Curve of the exact and approximate solutions of Example 5.5.2

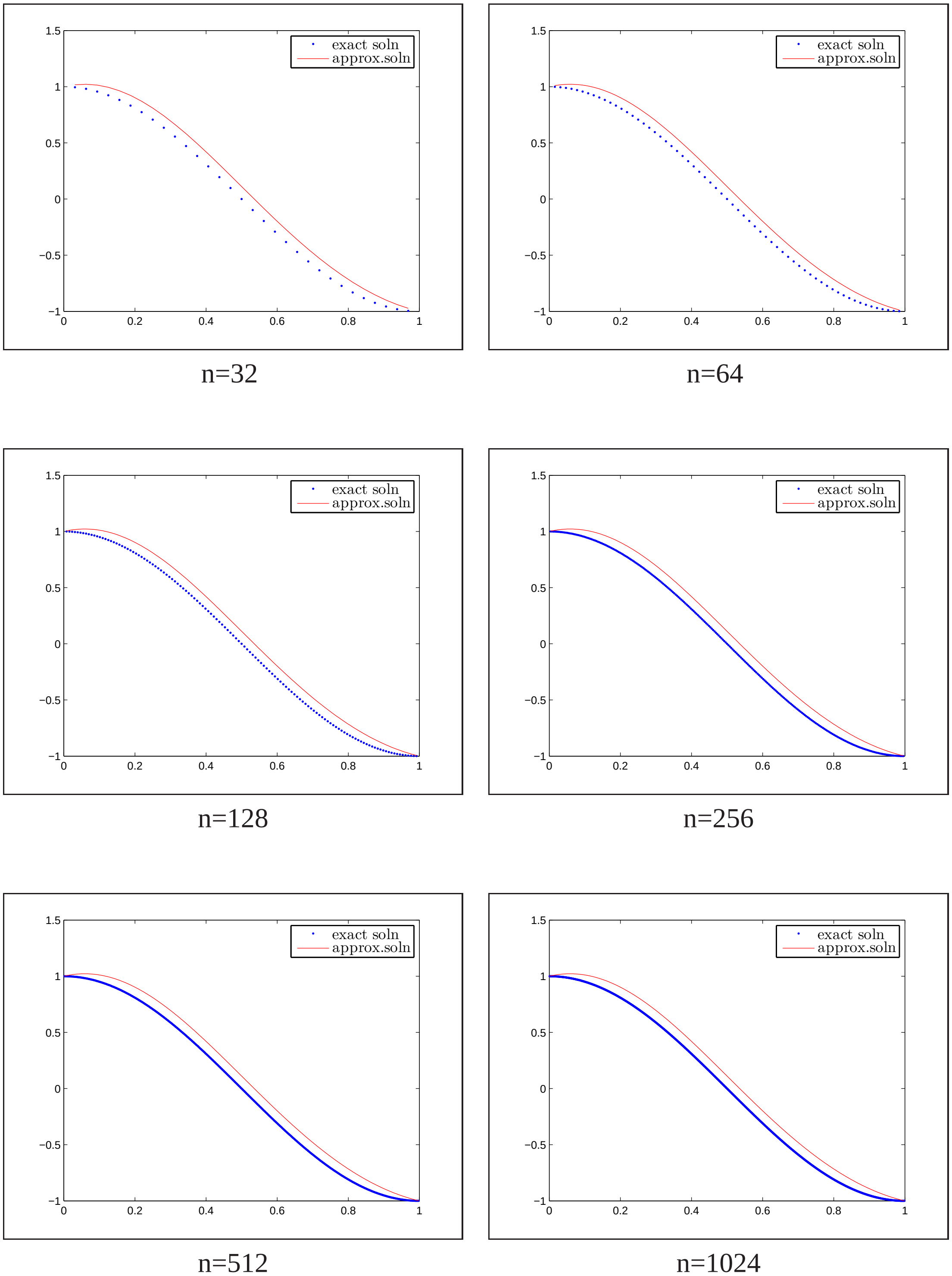


Figure 5.4: Curve of the exact and approximate solutions of Example 5.5.2

Chapter 6

TWO STEP NEWTON-TIKHONOV METHOD IN HILBERT SCALES

A Hilbert scale variant of modified Newton-Tikhonov method is considered for approximately solving ill-posed Hammerstein type operator equations. We derive order optimal error bounds by choosing the regularization parameter according to an adaptive scheme of Pereverzev and Schock(2005).

6.1 INTRODUCTION

In this Chapter we present an iterative method which combines Tikhonov regularization with the Modified Newton's method in Hilbert scales, for approximately solving the operator equation (2.1.1). In order to improve the rate of convergence of Tikhonov regularization of linear ill-posed problems many authors have considered the Hilbert scale variant of the regularization methods for solving ill-posed operator equations, for example see Natterer (1984), Egger and Neubauer (2005), Qi-nian (2000), Lu *et al*. (2010), Mathe and Tautenhahn (2007), Neubauer (2000), Jin and Tautenhahn (2011b) and Jin and Tautenhahn (2011a).

For the regularization of (2.1.1) in the setting of Hilbert scales, we consider a Hilbert scale $\{X_t\}_{t\in R}$ generated by a strictly positive operator $L : D(L) \to X$ with $D(L)$ dense in X satisfying

$$\|Lx\| \geq \|x\|, \quad x \in D(L).$$

Recall Qi-nian (2000), Tautenhahn (1998), that the space X_t is the completion of $D :=$

$\cap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle x_1, x_2 \rangle_t = \langle L^{t/2} x_1, L^{t/2} x_2 \rangle$$

i.e.,

$$\|x\|_t = \|L^{t/2} x\|, \quad t \in R.$$

Moreover, if $\beta \leq \gamma$, then the embedding $X_\gamma \hookrightarrow X_\beta$ is continuous, and therefore the norm $\|.\|_\beta$ is also defined in X_γ and there is a constant $c_{\beta,\gamma}$ such that

$$\|x\|_\beta \leq c_{\beta,\gamma} \|x\|_\gamma, \quad x \in X_\gamma.$$

As in chapter 2, we consider two cases of the operator F in $KF(x) = f$;

IFD Class: $F'(x_0)^{-1}$ exists and is bounded. Thus the ill-posedness of (2.1.1) is essentially due to the non-closedness of the range of the linear operator K. In this case we consider the sequence $(x_{n,\alpha,s}^\delta)$ defined iteratively by

$$y_{n,\alpha,s}^\delta = x_{n,\alpha,s}^\delta - F'(x_0)^{-1}[F(x_{n,\alpha,s}^\delta) - z_{\alpha,s}^\delta] \tag{6.1.1}$$

and

$$x_{n+1,\alpha,s}^\delta = y_{n,\alpha,s}^\delta - F'(x_0)^{-1}[F(y_{n,\alpha,s}^\delta) - z_{\alpha,s}^\delta] \tag{6.1.2}$$

where $x_{0,\alpha,s}^\delta := x_0$, is the initial approximation for the solution $\hat{x}$ of (2.1.1). Here

$$z_{\alpha,s}^\delta := F(x_0) + (L^{-s}K^*K + \alpha I)^{-1} L^{-s} K^* (f^\delta - KF(x_0)) \tag{6.1.3}$$

and α is the regularization parameter to be chosen appropriately from the finite set $D_N := \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\}$ depending on the inexact data f^δ and the error level δ satisfying $\|f - f^\delta\| \leq \delta$. We use the adaptive parameter selection procedure suggested by Pereverzev and Schock (2005) for the selection of regularization parameter.

MFD Class: $F'(x_0)$ is non-invertible and F is a monotone operator: In this case we consider the sequence $(\tilde{x}_{n,\alpha,s}^\delta)$ defined iteratively by

$$\tilde{y}_{n,\alpha,s}^\delta = \tilde{x}_{n,\alpha,s}^\delta - (F'(x_0) + \frac{\alpha}{c} L^{s/2})^{-1} [F(\tilde{x}_{n,\alpha,s}^\delta) - z_{\alpha,s}^\delta + \frac{\alpha}{c} L^{s/2} (\tilde{x}_{n,\alpha,s}^\delta - x_0)] \tag{6.1.4}$$

and

$$\tilde{x}^{\delta}_{n+1,\alpha} = \tilde{y}^{\delta}_{n,\alpha,s} - (F'(x_0) + \frac{\alpha}{c}L^{s/2})^{-1}[F(\tilde{y}^{\delta}_{n,\alpha,s}) - z^{\delta}_{\alpha,s} + \frac{\alpha}{c}L^{s/2}(\tilde{y}^{\delta}_{n,\alpha,s} - x_0)] \quad (6.1.5)$$

where $\tilde{x}^{\delta}_{0,\alpha,s} := x_0$, with x_0 and α are as in IFD Class and $0 < c \leq \alpha$.

The Chapter is organized as follows: In Section 6.2, we give the preliminaries and the adaptive scheme for choosing the regularization parameter α for Tikhonov regularization of (2.1.5) in the setting of Hilbert scales. The proposed method and the error estimates for the IFD Class and MFD Class is given in Section 6.3.

6.2 PRELIMINARIES

We assume that the ill-posed nature of the operator K is related to the Hilbert scale $\{X_t\}_{t\in R}$ according to the relation

$$c_1\|x\|_{-a} \leq \|Kx\| \leq c_2\|x\|_{-a}, \quad x \in X, \quad (6.2.6)$$

for some reals a, c_1, and c_2. Observe that from the relation

$$\langle Kx, f\rangle = \langle x, K^*f\rangle = \langle x, L^{-s}K^*f\rangle_s$$

for all $x \in X$ and $f \in Y$, we conclude that $L^{-s}K^* : Y \to X$ is the adjoint of the operator K in X. Consequently $L^{-s}K^*K : X \to X$ is self-adjoint. Further we note that

$$(A_s^*A_s + \alpha I)^{-1}L^{s/2} = L^{s/2}(L^{-s}K^*K + \alpha I)^{-1}$$

where $A_s = KL^{-s/2}$.

One of the crucial results for proving the results in this Chapter is the following proposition:

Let

$$f(t) = min\{c_1^t, c_2^t\}, \quad g(t) = max\{c_1^t, c_2^t\}, \quad t \in R, |t| \leq 1,$$

where c_1 and c_2 are as in (6.2.6).

PROPOSITION 6.2.1 *(See Tautenhahn (1996), Proposition 2.1) For $s \geq 0$ and $|\nu| \leq 1$,*

$$f(\nu)\|x\|_{-\nu(s+a)} \leq \|(A_s^*A_s)^{\nu/2}x\| \leq g(\nu)\|x\|_{-\nu(s+a)}, \quad x \in H.$$

We make use of the relation

$$\|(A_s + \alpha I)^{-1}A_s^p\| \leq \alpha^{p-1}, \;\; p > 0, \;\; 0 < p \leq 1, \tag{6.2.7}$$

which follows from the spectral properties of the positive self-adjoint operator $A_s, \;\; s > 0$.

The following assumption on source condition is based on a source function φ and a property of the source function φ. We will be using this assumption to obtain an error estimate for $\|z_{\alpha,s}^\delta - F(\hat{x})\|$.

ASSUMPTION 6.2.2 *There exists a continuous, strictly monotonically increasing function $\varphi : (0, \|A_s^*A_s\|] \to (0, \infty)$ such that the following conditions hold:*

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0,$
- $$\sup_{\lambda > 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha) \qquad \forall \lambda \in (0, \|A_s^*A_s\|]$$

  and
- *there exists $v \in X$ with $\|v\| \leq \overline{E}, \;\; \overline{E} > 0$ such that*

  $$(A_s^*A_s)^{\frac{s}{2(s+a)}}L^{s/2}(F(\hat{x}) - F(x_0)) = \varphi(A_s^*A_s)v.$$

REMARK 6.2.3 *Note that if $F(\hat{x}) - F(x_0) \in X_t$ i.e., $\|F(\hat{x}) - F(x_0)\|_t \leq E$, for some $0 < t \leq 2s + a$, then the above assumption is satisfied. This can be seen as follows.*

$$\begin{aligned}(A_s^*A_s)^{\frac{s}{2(s+a)}}L^{s/2}(F(\hat{x}) - F(x_0)) &= (A_s^*A_s)^{\frac{t}{2(s+a)}}(A_s^*A_s)^{\frac{(s-t)}{(2s+2a)}}L^{s/2}(F(\hat{x}) - F(x_0)) \\ &= \varphi(A_s^*A_s)v\end{aligned}$$

where $\varphi(\lambda) = \lambda^{\frac{t}{2(s+a)}}$ and $v = (A_s^*A_s)^{\frac{(s-t)}{(2s+2a)}}L^{s/2}(F(\hat{x}) - F(x_0))$.

Further note that

$$\begin{aligned}\|v\| &\leq g(\frac{s-t}{s+a})\|L^{s/2}(F(\hat{x}) - F(x_0))\|_{t-s} \\ &\leq g(\frac{s-t}{s+a})\|(F(\hat{x}) - F(x_0))\|_t \\ &\leq \overline{E}\end{aligned}$$

where $\overline{E} = g(\frac{s-t}{s+a})E$.

THEOREM 6.2.4 *Suppose that Assumption 6.2.2 and Proposition 6.2.1 hold, and let* $z_{\alpha,s} := z^0_{\alpha,s}$*. Then*

1.
$$\|z^\delta_{\alpha,s} - z_{\alpha,s}\| \leq \psi(s)\alpha^{\frac{-a}{2(s+a)}}\delta, \tag{6.2.8}$$

2.
$$\|F(\hat{x}) - z_{\alpha,s}\| \leq \phi(s)\varphi(\alpha), \tag{6.2.9}$$

3.
$$\|F(x_0) - z_{\alpha,s}\| \leq \psi_1(s)\|F(\hat{x}) - F(x_0)\|, \tag{6.2.10}$$

where $\psi(s) = \frac{1}{f(\frac{s}{s+a})}, \quad \phi(s) = \frac{\overline{E}}{f(\frac{s}{s+a})}$ *and* $\psi_1(s) = \frac{g(\frac{s}{s+a})}{f(\frac{s}{s+a})}$.

Proof. Note that

$$\begin{aligned}\|z^\delta_{\alpha,s} - z_{\alpha,s}\| &= \|(L^{-s}K^*K + \alpha I)^{-1}L^{-s}K^*(f^\delta - f)\| \\ &= \|L^{-s/2}(A_s^*A_s + \alpha I)^{-1}A_s^*(f^\delta - f)\|\end{aligned}$$

now by taking $\nu = \frac{s}{s+a}$ and $x = (A_s^*A_s + \alpha I)^{-1}A_s^*(f^\delta - f)$ in Proposition 6.2.1, we have

$$\begin{aligned}\|z^\delta_{\alpha,s} - z_{\alpha,s}\| &\leq \frac{1}{f(\frac{s}{s+a})}\|(A_s^*A_s)^{\frac{s}{2(s+a)}}(A_s^*A_s + \alpha I)^{-1}A_s^*(f^\delta - f)\| \\ &= \frac{1}{f(\frac{s}{s+a})}\|(A_s^*A_s + \alpha I)^{-1}(A_s^*A_s)^{\frac{2s+a}{2(s+a)}}(f^\delta - f)\| \\ &\leq \frac{1}{f(\frac{s}{s+a})}\|(A_s^*A_s + \alpha I)^{-1}(A_s^*A_s)^{\frac{2s+a}{2(s+a)}}\|\delta. \end{aligned} \tag{6.2.11}$$

We note that the relation (6.2.7) with $p = \frac{2s+a}{2(s+a)}$ gives

$$\|(A_s^*A_s + \alpha I)^{-1}(A_s^*A_s)^{\frac{2s+a}{2(s+a)}}\| \leq \alpha^{\frac{-a}{2(s+a)}}. \tag{6.2.12}$$

Now (6.2.8) follows from (6.2.11) and (6.2.12). Further we observe that

$$\begin{aligned}\|z_{\alpha,s} - F(\hat{x})\| &= \|[(L^{-s}K^*K + \alpha I)^{-1}L^{-s}K^*K - I](F(\hat{x}) - F(x_0))\| \\ &= \|\alpha L^{-s/2}(A_s^*A_s + \alpha I)^{-1}L^{s/2}(F(\hat{x}) - F(x_0))\| \\ &\leq \frac{1}{f(\frac{s}{2(s+a)})}\|\alpha(A_s^*A_s)^{\frac{s}{2(s+a)}}(A_s^*A_s + \alpha I)^{-1} \\ &\quad L^{s/2}(F(\hat{x}) - F(x_0))\|. \end{aligned} \tag{6.2.13}$$

So by Assumption 6.2.2 and (6.2.13), we have

$$\|z_{\alpha,s} - F(\hat{x})\| \leq \frac{1}{f(\frac{s}{s+a})}\varphi(\alpha)\overline{E}.$$

Again,

$$\begin{aligned}
\|z_{\alpha,s} - F(x_0)\| &= \|(L^{-s}K^*K + \alpha I)^{-1}L^{-s}K^*K(F(\hat{x}) - F(x_0))\| \\
&= \|L^{-s/2}(A_s^*A_s + \alpha I)^{-1}A_s^*A_sL^{s/2}(F(\hat{x}) - F(x_0))\| \\
&\leq \frac{1}{f(\frac{s}{s+a})}\|(A_s^*A_s)^{\frac{s}{2(s+a)}}(A_s^*A_s + \alpha I)^{-1} \\
&\quad (A_s^*A_s)L^{s/2}(F(\hat{x}) - F(x_0))\| \\
&= \frac{1}{f(\frac{s}{s+a})}\|(A_s^*A_s + \alpha I)^{-1}(A_s^*A_s)\| \\
&\quad \|(A_s^*A_s)^{\frac{s}{2(s+a)}}L^{s/2}(F(\hat{x}) - F(x_0))\| \\
&\leq \frac{g(\frac{s}{s+a})}{f(\frac{s}{s+a})}\|L^{s/2}(F(\hat{x}) - F(x_0))\|_{-s} \\
&\leq \psi_1(s)\|F(\hat{x}) - F(x_0)\|.
\end{aligned}$$

This completes the proof of the Theorem.

6.2.1 Error Bounds and Parameter Choice in Hilbert Scales

Let $C_s = \max\{\phi(s), \psi(s)\}$, then by (6.2.8), (6.2.9) and triangle inequality, we have

$$\|F(\hat{x}) - z_{\alpha,s}^\delta\| \leq C_s(\varphi(\alpha) + \alpha^{\frac{-a}{2(s+a)}}\delta). \tag{6.2.14}$$

The error estimate $\varphi(\alpha) + \alpha^{\frac{-a}{2(s+a)}}\delta$ in (6.2.14) attains minimum for the choice $\alpha := \alpha(\delta, s, a)$ which satisfies $\varphi(\alpha) = \alpha^{\frac{-a}{2(s+a)}}\delta$. Clearly $\alpha(\delta, s, a) = \varphi^{-1}(\psi_{s,a}^{-1}(\delta))$, where

$$\psi_{s,a}(\lambda) = \lambda[\varphi^{-1}(\lambda)]^{\frac{a}{2(s+a)}}, \quad 0 < \lambda \leq \|A_s\|^2 \tag{6.2.15}$$

and in this case

$$\|F(\hat{x}) - z_{\alpha,s}^\delta\| \leq 2C_s\psi_{s,a}^{-1}(\delta),$$

which has at least optimal order with respect to δ, s and a (cf. Pereverzev and Schock (2005)).

6.2.2 Adaptive Scheme and Stopping Rule

In this Chapter we use a modified version of the adaptive scheme suggested by Pereverzev and Schock (2005) for choosing the parameter α to suit the Hilbert scale set up.

Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu = \eta^{2(1+s/a)}, \eta > 1$ and $\alpha_0 = \delta^{2(1+s/a)}$. Let

$$l := max\{i : \varphi(\alpha_i) \leq \alpha_i^{\frac{-a}{2(s+a)}} \delta\} < N \tag{6.2.16}$$

and

$$k := max\{i : \|z^\delta_{\alpha_{i,s}} - z^\delta_{\alpha_{j,s}}\| \leq 4C_s \alpha_j^{\frac{-a}{2(s+a)}} \delta\}, j = 0, 1, 2, \cdots, i-1\}. \tag{6.2.17}$$

Analogous to Theorem 4.3 in George and Kunhanandan (2009), we have the following Theorem.

THEOREM 6.2.5 *Let l be as in (6.2.16), k be as in (6.2.17), $\psi_{s,a}$ be as in (6.2.15) and $z^\delta_{\alpha_k,s}$ be as in (6.1.3) with $\alpha = \alpha_k$. Then $l \leq k$ and*

$$\|F(\hat{x}) - z^\delta_{\alpha_k,s}\| \leq C_s(2 + \frac{4\eta}{\eta - 1})\eta \psi^{-1}_{s,a}(\delta)$$

where C_s is as in (6.2.14).

Proof. To see that $l \leq k$, it is enough to show that, for $i = 1, 2, \cdots, N$,

$$\varphi(\alpha_i) \leq \alpha_j^{-a/2(s+a)} \delta \Longrightarrow \|z^\delta_{\alpha_i,s} - z^\delta_{\alpha_j,s}\| \leq 4C_s \alpha_j^{-a/2(s+a)} \delta, \qquad \forall j = 0, 1, \cdots, i.$$

For $j \leq i$, by (6.2.17)

$$\begin{aligned}
\|z^\delta_{\alpha_i,s} - z^\delta_{\alpha_j,s}\| &\leq \|z^\delta_{\alpha_i,s} - F(\hat{x})\| + \|F(\hat{x}) - z^\delta_{\alpha_j,s}\| \\
&\leq C_s(\varphi(\alpha_i) + \alpha_i^{-a/2(s+a)} \delta) + C_s(\varphi(\alpha_j) + \alpha_j^{-a/2(s+a)} \delta) \\
&\leq 2C_s \alpha_i^{-a/2(s+a)} \delta + 2C_s \alpha_j^{-a/2(s+a)} \delta \\
&\leq 4C_s \alpha_j^{-a/2(s+a)} \delta.
\end{aligned}$$

This proves the relation $l \leq k$.

Thus by the relation $(\alpha_{l+m})^{a/2(s+a)} = \eta^m(\alpha_l)^{a/2(s+a)}$ and by using triangle inequality successively, we obtain

$$\begin{aligned}
\|F(\hat{x}) - z^\delta_{\alpha_k,s}\| &\leq \|F(\hat{x}) - z^\delta_{\alpha_l,s}\| + \sum_{i=l+1}^{k} \|z^\delta_{\alpha_i,s} - z^\delta_{\alpha_{i-1},s}\| \\
&\leq \|F(\hat{x}) - z^\delta_{\alpha_l,s}\| + \sum_{i=l+1}^{k} 4C_s\alpha_{i-1}^{-a/2(s+a)}\delta \\
&\leq \|F(\hat{x}) - z^\delta_{\alpha_l,s}\| + \sum_{m=0}^{k-l-1} 4C_s\alpha_l^{-a/2(s+a)}\eta^{-m}\delta \\
&\leq \|F(\hat{x}) - z^\delta_{\alpha_l,s}\| + \frac{4\eta}{\eta-1}C_s\alpha_l^{-a/2(s+a)}\delta. \qquad (6.2.18)
\end{aligned}$$

Therefore by (6.2.18) and (6.2.16) we have

$$\begin{aligned}
\|F(\hat{x}) - z^\delta_{\alpha_k,s}\| &\leq C_s(\varphi(\alpha_l) + \alpha_l^{-a/2(s+a)}\delta) + \frac{4\eta}{\eta-1}C_s\alpha_l^{-a/2(s+a)}\delta \\
&\leq C_s(2 + \frac{4\eta}{\eta-1})\alpha_l^{-a/2(s+a)}\delta \\
&\leq C_s(2 + \frac{4\eta}{\eta-1})\eta\psi_{s,a}^{-1}(\delta).
\end{aligned}$$

The last step follows from the inequality $\alpha_\delta \leq \alpha_{l+1} = \eta\alpha_l$.

6.3 THE ITERATIVE METHOD AND CONVERGENCE ANALYSIS

6.3.1 Regularization of IFD Class

Consider the two step iterative method defined as (6.1.1) and (6.1.2) with α_k in place of α. We assume that F possess a uniformly bounded Fr$\acute{e}$chet derivative for all $x \in D(F)$ i.e., $\|F'(x_0)\| \leq M$, for some $M > 0$ and $\|F'(x_0)^{-1}\| := \beta,\ \ \beta > 0$. Let

$$e^\delta_{n,\alpha_k,s} := \|y^\delta_{n,\alpha_k,s} - x^\delta_{n,\alpha_k,s}\|, \qquad \forall n \geq 0 \qquad (6.3.1)$$

and let $\delta_0 < \frac{1}{4k_0\beta\psi(s)}\alpha_0^{\frac{a}{2(s+a)}}$ and $\|\hat{x} - x_0\| \leq \rho$, with

$$\rho < \frac{1}{\psi_1(s)M}[\frac{1}{4k_0\beta} - \psi(s)\alpha_0^{\frac{-a}{2(s+a)}}\delta_0]$$

and

$$\gamma_\rho := \beta[\psi_1(s)M\rho + \psi(s)\alpha^{\frac{-a}{2(s+a)}}\delta].$$

Further let

$$r_1 = \frac{1-\sqrt{1-4k_0\gamma_\rho}}{2k_0}$$

and

$$r_2 = min\{\frac{1}{k_0}, \frac{1+\sqrt{1-4k_0\gamma_\rho}}{2k_0}\}.$$

For $r \in (r_1, r_2)$, let

$$q_s = k_0 r, \tag{6.3.2}$$

then $q_s < 1$.

LEMMA 6.3.1 *Let $e^\delta_{0,\alpha_k,s}$ be as in (6.3.1). Then $e^\delta_{0,\alpha_k,s} \leq \gamma_\rho$.*

Proof. Observe that

$$\begin{aligned} e^\delta_{0,\alpha_k,s} = \|y^\delta_{0,\alpha_k,s} - x^\delta_{0,\alpha_k,s}\| &= \|F'(x_0)^{-1}(F(x_0) - z^\delta_{\alpha_k,s})\| \\ &\leq \beta\|F(x_0) - z^\delta_{\alpha_k,s}\| \\ &\leq \beta[\|F(x_0) - z_{\alpha_k,s}\| \\ &\quad + \|z_{\alpha_k,s} - z^\delta_{\alpha_k,s}\|]. \end{aligned} \tag{6.3.3}$$

Now using (6.2.8) and (6.2.10) in (6.3.3), one can see that

$$\begin{aligned} e^\delta_{0,\alpha_k,s} &\leq \beta[\psi_1(s)\|F(\hat{x}) - F(x_0)\| + \psi(s)\alpha^{\frac{-a}{2(s+a)}}\delta] \\ &\leq \beta[\psi_1(s)M\rho + \psi(s)\alpha^{\frac{-a}{2(s+a)}}\delta] = \gamma_\rho. \end{aligned}$$

This completes the proof.

THEOREM 6.3.2 *Let $e^\delta_{n,\alpha_k,s}$ and q_s be as in equation (6.3.1) and (6.3.2) respectively, $y^\delta_{n,\alpha_k,s}$ and $x^\delta_{n,\alpha_k,s}$ be as defined in (6.1.1) and (6.1.2) respectively with $\alpha = \alpha_k$ and $\delta \in (0, \delta_0]$. Then by Assumption 2.3.1 and Lemma 6.3.1, $x^\delta_{n,\alpha_k,s}, y^\delta_{n,\alpha_k,s} \in B_r(x_0)$ and the following estimates hold for all $n \geq 0$.*

(a) $\|x^\delta_{n,\alpha_k,s} - y^\delta_{n-1,\alpha_k,s}\| \leq q_s\|y^\delta_{n-1,\alpha_k,s} - x^\delta_{n-1,\alpha_k,s}\|$;

(b) $\|x^\delta_{n,\alpha_k,s} - x^\delta_{n-1,\alpha_k,s}\| \leq (1+q_s)\|y^\delta_{n-1,\alpha_k,s} - x^\delta_{n-1,\alpha_k,s}\|$;

(c) $\|y^\delta_{n,\alpha_k,s} - x^\delta_{n,\alpha_k,s}\| \leq q_s^2 \|y^\delta_{n-1,\alpha_k,s} - x^\delta_{n-1,\alpha_k,s}\|$;

(d) $e^\delta_{n,\alpha_k,s} \leq q_s^{2n}\gamma_\rho, \quad \forall n \geq 0.$

Proof. Suppose $x^\delta_{n,\alpha_k,s},\ y^\delta_{n,\alpha_k,s} \in B_r(x_0)$. Then

$$\begin{aligned} x^\delta_{n+1,\alpha_k,s} - y^\delta_{n,\alpha_k,s} &= y^\delta_{n,\alpha_k,s} - x^\delta_{n,\alpha_k,s} + F'(x_0)^{-1}[F(x^\delta_{n,\alpha_k,s}) - (F(y^\delta_{n,\alpha_k,s})] \\ &= F'(x_0)^{-1}[F'(x_0)(y^\delta_{n,\alpha_k,s} - x^\delta_{n,\alpha_k,s}) - (F(y^\delta_{n,\alpha_k,s}) - F(x^\delta_{n,\alpha_k,s}))] \end{aligned}$$

and hence by Assumption 2.3.1, we have

$$\begin{aligned} \|x^\delta_{n+1,\alpha_k,s} - y^\delta_{n,\alpha_k,s}\| &= \|F'(x_0)^{-1}\int_0^1 F'(x_0)\Phi(x_0, x^\delta_{n,\alpha_k,s} \\ &\quad +t(y^\delta_{n,\alpha_k,s} - x^\delta_{n,\alpha_k,s}), y^\delta_{n,\alpha_k,s} - x^\delta_{n,\alpha_k,s})dt\| \\ &\leq k_0 r\|y^\delta_{n,\alpha_k,s} - x^\delta_{n,\alpha_k,s}\|. \end{aligned}$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|x^\delta_{n,\alpha_k,s} - x^\delta_{n-1,\alpha_k,s}\| \leq \|x^\delta_{n,\alpha_k,s} - y^\delta_{n-1,\alpha_k,s}\| + \|y^\delta_{n-1,\alpha_k,s} - x^\delta_{n-1,\alpha_k,s}\|.$$

Again (c) follows from (a), Assumption 2.3.1 and the following expression,

$$e^\delta_{n,\alpha_k,s} = \|F'(x_0)^{-1}\int_0^1 [F'(x_0) - F'(x^\delta_{n,\alpha_k,s} + t(x^\delta_{n,\alpha_k,s} - y^\delta_{n-1,\alpha_k,s}))](x^\delta_{n,\alpha_k,s} - y^\delta_{n-1,\alpha_k,s})dt\|$$

and (d) follows from (c). Now we show that $x^\delta_{n,\alpha_k,s}, y^\delta_{n,\alpha_k,s} \in B_r(x_0)$ by induction.

Note that by (b) and Lemma 6.3.1,

$$\begin{aligned} \|x^\delta_{1,\alpha_k,s} - x_0\| &\leq (1+q_s)e^\delta_{0,\alpha_k,s} \\ &\leq \frac{e^\delta_{0,\alpha_k,s}}{1-q_s} \\ &\leq \frac{\gamma_\rho}{1-q_s} \\ &< r, \end{aligned} \tag{6.3.4}$$

i.e., $x^\delta_{1,\alpha_k,s} \in B_r(x_0)$. Again note that by (6.3.4) and (c), we have

$$\begin{aligned} \|y^\delta_{1,\alpha_k,s} - x_0\| &\leq \|y^\delta_{1,\alpha_k,s} - x^\delta_{1,\alpha_k,s}\| + \|x^\delta_{1,\alpha_k,s} - x_0\| \\ &\leq (1+q_s+q_s^2)e^\delta_{0,\alpha_k,s} \\ &\leq \frac{\gamma_\rho}{1-q_s} \\ &< r, \end{aligned}$$

i.e., $y^\delta_{1,\alpha_k,s} \in B_r(x_0)$. Further let us assume that $x^\delta_{m,\alpha_k,s}, y^\delta_{m,\alpha_k,s} \in B_r(x_0)$, for some $m \geq 0$. Then, using (b), (6.3.4) and Lemma 6.3.1, we have

$$\begin{aligned} \|x^\delta_{m+1,\alpha_k,s} - x_0\| &\leq \|x^\delta_{m+1,\alpha_k} - x^\delta_{m,\alpha_k,s}\| + \cdots + \|x^{h,\delta}_{1,\alpha_k,s} - x_0\| \\ &\leq (q_s+1)(q_s^{2m} + q_s^{2(m-1)} + \cdots + 1)e^\delta_{0,\alpha_k,s} \\ &\leq (q_s+1)\frac{1-(q_s^{2m+1})}{1-q_s^2} e^\delta_{0,\alpha_k,s} \\ &\leq \frac{\gamma_\rho}{1-q_s} \\ &< r, \end{aligned}$$

i.e., $x^\delta_{m+1,\alpha_k,s} \in B_r(x_0)$ and

$$\begin{aligned} \|y^\delta_{m+1,\alpha_k,s} - x_0\| &\leq \|y^\delta_{m+1,\alpha_k,s} - x^\delta_{m+1,\alpha_k,s}\| + \|x^\delta_{m+1,\alpha_k,s} - x_0\| \\ &\leq (q_s^{2(m+1)} + \cdots + q_s^3 + q_s^2 + q_s + 1)e^\delta_{0,\alpha_k,s} \\ &\leq \frac{\gamma_\rho}{1-q_s} \\ &< r, \end{aligned}$$

i.e., $y^\delta_{m+1,\alpha_k,s} \in B_r(x_0)$. Thus by induction $x^\delta_{n,\alpha_k,s}, y^\delta_{n,\alpha_k,s} \in B_r(x_0),\ \forall\ n \geq 0$. This completes the proof of the Theorem.

THEOREM 6.3.3 *Let $x^\delta_{n,\alpha_k,s}$ and $y^\delta_{n,\alpha_k,s}$ be as in (6.1.1) and (6.1.2) respectively with $\alpha = \alpha_k$ and $\delta \in [0, \delta_0]$, and assumptions of Theorem 6.3.2 hold. Then $(x^\delta_{n,\alpha_k,s})$ is a Cauchy sequence in $B_r(x_0)$ and converges to $x^\delta_{\alpha_k,s} \in \overline{B_r(x_0)}$. Further $F(x^\delta_{\alpha_k,s}) = z^\delta_{\alpha_k,s}$ and*

$$\|x^\delta_{n,\alpha_k,s} - x^\delta_{\alpha_k,s}\| \leq C_5 q_s^{2n}$$

where $C_5 = \frac{\gamma_\rho}{1-q_s}$.

Proof. The proof is analogous to the proof of Theorem 2.3.3 in Chapter 2.

Hereafter we assume that $\|\hat{x} - x_0\| < \rho \leq r$.

THEOREM 6.3.4 *Suppose that the hypothesis of Assumption 2.3.1 holds. Then*

$$\|\hat{x} - x^\delta_{\alpha_k,s}\| \leq \frac{\beta}{1-q_s}\|F(\hat{x}) - z^\delta_{\alpha_k,s}\|.$$

Proof. The proof is analogous to the proof of Theorem 2.3.4 in Chapter 2.

The following Theorem is a consequence of Theorem 6.3.3 and Theorem 6.3.4.

THEOREM 6.3.5 *Let $x^{\delta}_{n,\alpha_k,s}$ be as in (6.1.1) with $\alpha = \alpha_k$ and $\delta \in (0, \delta_0]$, assumptions in Theorem 6.3.3 and Theorem 6.3.4 hold. Then*

$$\|\hat{x} - x^{\delta}_{n,\alpha_k,s}\| \leq C_5 q_s^{2n} + \frac{\beta}{1-q_s}\|F(\hat{x}) - z^{\delta}_{\alpha_k,s}\|$$

where C_5 is as in Theorem 6.3.3.

THEOREM 6.3.6 *Let $x^{\delta}_{n,\alpha_k,s}$ be as in (6.1.1) with $\alpha = \alpha_k$ and $\delta \in [0, \delta_0]$, assumptions in Theorem 6.2.5 and Theorem 6.3.5 hold. Let*

$$n_k := \min\{n : q_s^{2n} \leq \alpha_k^{-a/2(s+a)}\delta\}.$$

Then

$$\|\hat{x} - x^{\delta}_{n_k,\alpha_k,s}\| = O(\psi^{-1}_{s,a}(\delta)).$$

6.3.2 Regularization of MFD Class

In this section, let X be a real Hilbert space. We consider the two step iterative method defined as (6.1.4) and (6.1.5) with α_k in place of α for approximating the zero $x^{\delta}_{c,\alpha_k,s}$ of the equation,

$$F(x) + \frac{\alpha_k}{c}L^{s/2}(x - x_0) = z^{\delta}_{\alpha_k,s} \tag{6.3.5}$$

and then we show that $x^{\delta}_{c,\alpha_k,s}$ is an approximation to the solution $\hat{x}$ of (2.1.1).

Let $F'(x_0) \in L(X)$ be a bounded positive self-adjoint operator on X and $B_s := L^{-s/4}F'(x_0)L^{-s/4}$. Usually, for the analysis of regularization methods in Hilbert scales, an assumption of the form (cf. Egger and Neubauer (2005), Neubauer (2000))

$$\|F'(.)x\| \sim \|x\|_{-b}, \;\; x \in X \tag{6.3.6}$$

is used. Here the number $b > 0$ can be interpreted as the degree of ill-posedness of (2.1.1). In this Chapter instead of (6.3.6) we use the following assumptions on the ill-posedness;

$$d_1\|x\|_{-b} \leq \|F'(x_0)x\| \leq d_2\|x\|_{-b}, \quad x \in D(F), \tag{6.3.7}$$

for some reals b, d_1, and d_2.

Note that (6.3.7) is simpler than that of (6.3.6). Now we define f_1 and g_1 by

$$f_1(t) = min\{d_1^t, d_2^t\}, \quad g_1(t) = max\{d_1^t, d_2^t\}, \quad t \in R, \ |t| \leq 1.$$

The following proposition is crucial for proving the further results in this Chapter.

PROPOSITION 6.3.7 *(see George and Nair (1997), Proposition 3.1) For $s > 0$ and $|\nu| \leq 1$,*

$$f_1(\nu/2)\|x\|_{\frac{-\nu(s+b)}{2}} \leq \|B_s^{\nu/2}x\| \leq g_1(\nu/2)\|x\|_{\frac{-\nu(s+b)}{2}}, \quad x \in H.$$

Let $\psi_2(s) := \frac{g_1(\frac{-s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}$, $\overline{\psi_2(s)} := \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}$ and let

$$\tilde{e}_{n,\alpha_k,s}^\delta := \|\tilde{y}_{n,\alpha_k,s}^\delta - \tilde{x}_{n,\alpha_k,s}^\delta\|, \qquad \forall n \geq 0. \tag{6.3.8}$$

Let $\delta_0 < \frac{1}{4k_0\psi(s)\psi_2(s)\overline{\psi_2(s)}}\alpha_0^{\frac{a}{2(s+a)}}$ and $\|\hat{x} - x_0\| \leq \rho$, with

$$\rho < \frac{1}{M\psi_1(s)}[\frac{1}{4k_0\psi_2(s)\overline{\psi_2(s)}} - \psi(s)\alpha_0^{\frac{-a}{2(s+a)}}\delta_0]$$

and

$$\tilde{\gamma}_\rho := \psi_2(s)[\psi_1(s)M\rho + \psi(s)\alpha_0^{\frac{-a}{2(s+a)}}\delta_0].$$

Further let

$$\tilde{r}_1 = \frac{1 - \sqrt{1 - 4k_0\overline{\psi_2(s)}\tilde{\gamma}_\rho}}{2\overline{\psi_2(s)}k_0}$$

and

$$\tilde{r}_2 = min\{\frac{1}{k_0}, \frac{1 + \sqrt{1 - 4k_0\overline{\psi_2(s)}\tilde{\gamma}_\rho}}{2\overline{\psi_2(s)}k_0}\}.$$

For $\tilde{r} \in (\tilde{r}_1, \tilde{r}_2)$, let

$$\tilde{q}_s = \overline{\psi_2(s)}k_0\tilde{r}, \tag{6.3.9}$$

then $\tilde{q}_s < 1$.

LEMMA 6.3.8 *Let $\tilde{e}_{0,\alpha_k,s}^\delta$ be as in (6.3.8) and let Proposition 6.3.7 holds. Then $\tilde{e}_{0,\alpha_k,s}^\delta < \tilde{\gamma}_\rho$.*

Proof. Observe that

$$\begin{aligned}
\tilde{e}^{\delta}_{0,\alpha_k,s} = \|\tilde{y}^{\delta}_{0,\alpha_k,s} - \tilde{x}^{\delta}_{0,\alpha_k,s}\| &= \|(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}(F(x_0) - z^{\delta}_{\alpha_k,s})\| \\
&\leq \|L^{-s/4}(L^{-s/4}F'(x_0)L^{-s/4} + \frac{\alpha_k}{c}I)^{-1}L^{-s/4} \\
&\quad (F(x_0) - z^{\delta}_{\alpha_k,s})\| \\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|B_s^{\frac{s}{2(s+b)}}(B_s + \frac{\alpha_k}{c}I)^{-1}L^{-s/4} \\
&\quad (F(x_0) - z^{\delta}_{\alpha_k,s})\| \\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|(B_s + \frac{\alpha_k}{c}I)^{-1}B_s^{\frac{s}{(s+b)}}B_s^{\frac{-s}{2(s+b)}} \\
&\quad L^{-s/4}(F(x_0) - z^{\delta}_{\alpha_k,s})\| \\
&\leq \frac{g_1(\frac{-s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}(\frac{\alpha_k}{c})^{\frac{-b}{(s+b)}}\|F(x_0) - z^{\delta}_{\alpha_k,s}\|. \\
&\leq \psi_2(s)[\|F(x_0) - z_{\alpha_k,s}\| + \|z_{\alpha_k,s} - z^{\delta}_{\alpha_k,s}\|]. \qquad (6.3.10)
\end{aligned}$$

Now using (6.2.8) and (6.2.10) in (6.3.10), one can see that

$$\begin{aligned}
\tilde{e}^{\delta}_{0,\alpha_k,s} &\leq \psi_2(s)[\psi_1(s)\|F(\hat{x}) - F(x_0)\| + \psi(s)\alpha^{\frac{-a}{2(s+a)}}\delta] \\
&\leq \psi_2(s)[\psi_1(s)M\rho + \psi(s)\alpha_0^{\frac{-a}{2(s+a)}}\delta_0] = \tilde{\gamma}_\rho.
\end{aligned}$$

LEMMA 6.3.9 *Let Proposition 6.3.7 hold. Then for all $h \in X$,*

$$\|(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}F'(x_0)h\| \leq \overline{\psi_2(s)}\|h\|.$$

Proof. Observe that by Proposition 6.3.7,

$$\begin{aligned}
\|(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}F'(x_0)h\| &= \|L^{-s/4}(L^{-s/4}F'(x_0)L^{-s/4} + \frac{\alpha_k}{c}I)^{-1}L^{-s/4} \\
&\quad F'(x_0)L^{-s/4}L^{s/4}h\| \\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|B_s^{\frac{s}{2(s+b)}}(B_s + \frac{\alpha_k}{c}I)^{-1}B_sL^{s/4}h\| \\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|(B_s + \frac{\alpha_k}{c}I)^{-1}B_s\|\|B_s^{\frac{s}{2(s+b)}}L^{s/4}h\| \\
&\leq \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}\|L^{s/4}h\|_{-s/2} \\
&\leq \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}\|h\|.
\end{aligned}$$

This completes the proof of the Lemma.

THEOREM 6.3.10 *Let $\tilde{e}^{\delta}_{n,\alpha_k,s}$ and $\tilde{q}_s$ be as in equation (6.3.8) and (6.3.9) respectively, $\tilde{y}^{\delta}_{n,\alpha_k,s}$ and $\tilde{x}^{\delta}_{n,\alpha_k,s}$ be as defined in (6.1.4) and (6.1.5) respectively with $\alpha = \alpha_k$ and $\delta \in (0, \delta_0]$. Then by Assumption 2.3.1 and Lemma 6.3.8, $\tilde{x}^{\delta}_{n,\alpha_k,s}, \ \tilde{y}^{\delta}_{n,\alpha_k,s} \in B_{\tilde{r}}(x_0)$, and the following estimates hold for all $n \geq 0$.*

(a) $\|\tilde{x}^{\delta}_{n,\alpha_k,s} - \tilde{y}^{\delta}_{n-1,\alpha_k,s}\| \leq \tilde{q}_s \|\tilde{y}^{\delta}_{n-1,\alpha_k,s} - \tilde{x}^{\delta}_{n-1,\alpha_k,s}\|$;

(b) $\|\tilde{x}^{\delta}_{n,\alpha_k,s} - \tilde{x}^{\delta}_{n-1,\alpha_k,s}\| \leq (1 + \tilde{q}_s) \|\tilde{y}^{\delta}_{n-1,\alpha_k,s} - \tilde{x}^{\delta}_{n-1,\alpha_k,s}\|$;

(c) $\|\tilde{y}^{\delta}_{n,\alpha_k,s} - \tilde{x}^{\delta}_{n,\alpha_k,s}\| \leq \tilde{q}_s{}^2 \|\tilde{y}^{\delta}_{n-1,\alpha_k,s} - \tilde{x}^{\delta}_{n-1,\alpha_k,s}\|$;

(d) $\tilde{e}^{\delta}_{n,\alpha_k,s} \leq \tilde{q}_s{}^{2n} \tilde{\gamma}_\rho, \quad \forall n \geq 0.$

Proof. If $\tilde{x}^{\delta}_{n,\alpha_k,s}, \ \tilde{y}^{\delta}_{n,\alpha_k,s} \in B_{\tilde{r}}(x_0)$, then by Assumption 2.3.1,

$$\begin{aligned}
\tilde{x}^{\delta}_{n+1,\alpha_k,s} - \tilde{y}^{\delta}_{n,\alpha_k,s} &= (F'(x_0) + \frac{\alpha_k}{c} L^{s/2})^{-1} [F'(x_0)(\tilde{y}^{\delta}_{n,\alpha_k,s} - \tilde{x}^{\delta}_{n,\alpha_k,s}) \\
&\quad - (F(\tilde{y}^{\delta}_{n,\alpha_k,s}) - F(\tilde{x}^{\delta}_{n,\alpha_k,s}))] \\
&= (F'(x_0) + \frac{\alpha_k}{c} L^{s/2})^{-1} \int_0^1 [F'(x_0) - F'(\tilde{x}^{\delta}_{n,\alpha_k,s} \\
&\quad + t(\tilde{y}^{\delta}_{n,\alpha_k,s} - \tilde{x}^{\delta}_{n,\alpha_k,s}))](\tilde{y}^{\delta}_{n,\alpha_k,s} - \tilde{x}^{\delta}_{n,\alpha_k,s}) dt \\
&= (F'(x_0) + \frac{\alpha_k}{c} L^{s/2})^{-1} F'(x_0) \int_0^1 \Phi(x_0, \tilde{x}^{\delta}_{n,\alpha_k,s} \\
&\quad + t(\tilde{y}^{\delta}_{n,\alpha_k,s} - \tilde{x}^{\delta}_{n,\alpha_k,s}), \tilde{y}^{\delta}_{n,\alpha_k} - \tilde{x}^{\delta}_{n,\alpha_k,s}) dt
\end{aligned}$$

and hence by Lemma 6.3.9 and Assumption 2.3.1, we have

$$\begin{aligned}
\|\tilde{x}^{\delta}_{n+1,\alpha_k,s} - \tilde{y}^{\delta}_{n,\alpha_k,s}\| &\leq \overline{\psi_2(s)} \| \int_0^1 \Phi(x_0, \tilde{x}^{\delta}_{n,\alpha_k,s} \\
&\quad + t(\tilde{y}^{\delta}_{n,\alpha_k,s} - \tilde{x}^{\delta}_{n,\alpha_k,s}), \tilde{y}^{\delta}_{n,\alpha_k,s} - \tilde{x}^{\delta}_{n,\alpha_k,s}) dt \| \\
&\leq \overline{\psi_2(s)} k_0 \tilde{r} \|\tilde{y}^{\delta}_{n,\alpha_k,s} - \tilde{x}^{\delta}_{n,\alpha_k,s}\|
\end{aligned}$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|\tilde{x}^{\delta}_{n,\alpha_k,s} - \tilde{x}^{\delta}_{n-1,\alpha_k,s}\| \leq \|\tilde{x}^{\delta}_{n,\alpha_k,s} - \tilde{y}^{\delta}_{n-1,\alpha_k,s}\| + \|\tilde{y}^{\delta}_{n-1,\alpha_k,s} - \tilde{x}^{\delta}_{n-1,\alpha_k,s}\|.$$

Again (c) follows from (a), Assumption 2.3.1, Lemma 6.3.9 and the following expression,

$$\begin{aligned}
\tilde{e}^{\delta}_{n,\alpha_k,s} &= \|(F'(x_0)+\frac{\alpha_k}{c}L^{s/2})^{-1}[F'(x_0)(\tilde{x}^{\delta}_{n,\alpha_k,s}-\tilde{y}^{\delta}_{n-1,\alpha_k,s}) \\
&\quad -(F(\tilde{x}^{\delta}_{n,\alpha_k,s})-F(\tilde{y}^{\delta}_{n-1,\alpha_k,s}))]\| \\
&= \|(F'(x_0)+\frac{\alpha_k}{c}L^{s/2})^{-1}\int_0^1[F'(x_0)-(F'(\tilde{y}^{\delta}_{n-1,\alpha_k,s}) \\
&\quad +t(\tilde{y}^{\delta}_{n-1,\alpha_k,s}-\tilde{x}^{\delta}_{n,\alpha_k,s}))](\tilde{x}^{\delta}_{n,\alpha_k,s}-\tilde{y}^{\delta}_{n-1,\alpha_k,s})dt\|.
\end{aligned}$$

Further (d) follows from (c). The remaining part of the proof is analogous to the proof of Theorem 6.3.2.

Next we shall go to the main result of this section.

THEOREM 6.3.11 *Let $\tilde{y}^{\delta}_{n,\alpha_k,s}$ and $\tilde{x}^{\delta}_{n,\alpha_k,s}$ be as in (6.1.4) and (6.1.5) respectively with $\alpha=\alpha_k$, $\delta\in[0,\delta_0]$ and assumptions of Theorem 6.3.10 hold. Then $(\tilde{x}^{\delta}_{n,\alpha_k,s})$ is a Cauchy sequence in $B_{\tilde{r}}(x_0)$ and converges, say to $x^{\delta}_{c,\alpha_k,s}\in\overline{B_{\tilde{r}}(x_0)}$. Further $F(x^{\delta}_{c,\alpha_k,s})-z^{\delta}_{\alpha_k,s}+\frac{\alpha_k}{c}L^{s/2}(x^{\delta}_{c,\alpha_k,s}-x_0)=0$ and $\|\tilde{x}^{\delta}_{n,\alpha_k,s}-x^{\delta}_{c,\alpha_k,s}\|\leq\tilde{C}_5\tilde{q_s}^{2n}$ where $\tilde{C}_5=\frac{\tilde{\gamma}_\rho}{1-\tilde{q_s}}$.*

Proof. Analogous to the proof of Theorem 2.3.3 of Chapter 2, one can see that $(\tilde{x}^{\delta}_{n,\alpha_k,s})$ is a Cauchy sequence in $B_{\tilde{r}}(x_0)$ and hence it converges, say to $x^{\delta}_{c,\alpha_k,s}\in\overline{B_{\tilde{r}}(x_0)}$. Observe that from (6.1.4)

$$\begin{aligned}
\|F(\tilde{x}^{\delta}_{n,\alpha_k,s})-z^{\delta}_{\alpha_k,s}+\frac{\alpha_k}{c}L^{s/2}(\tilde{x}^{\delta}_{n,\alpha_k,s}-x_0)\| &= \|(F'(x_0)+\frac{\alpha_k}{c}L^{s/2}) \\
&\quad (\tilde{y}^{\delta}_{n,\alpha_k,s}-\tilde{x}^{\delta}_{n,\alpha_k,s})\| \\
&\leq \|F'(x_0)+\frac{\alpha_k}{c}L^{s/2}\|_{X_s\to X} \\
&\quad \times\|\tilde{y}^{\delta}_{n,\alpha_k,s}-\tilde{x}^{\delta}_{n,\alpha_k,s}\| \\
&\leq \|F'(x_0)+\frac{\alpha_k}{c}L^{s/2}\|_{X_s\to X}\tilde{e}^{\delta}_{n,\alpha_k,s} \\
&\leq \|F'(x_0)+\frac{\alpha_k}{c}L^{s/2}\|_{X_s\to X} \\
&\quad \times\tilde{q_s}^{2n}\tilde{\gamma}_\rho. \qquad (6.3.11)
\end{aligned}$$

Now by letting $n\to\infty$ in (6.3.11) we obtain $F(x^{\delta}_{c,\alpha_k,s})+\frac{\alpha_k}{c}L^{s/2}(x^{\delta}_{c,\alpha_k,s}-x_0)=z^{\delta}_{\alpha_k,s}$. This completes the proof.

In addition to the Assumption 6.2.2, we use the following assumption to obtain the error estimate for $\|\hat{x} - \tilde{x}^{\delta}_{\alpha_k,s}\|$.

ASSUMPTION 6.3.12 *There exists a continuous, strictly monotonically increasing function* $\varphi_1 : (0, \|B_s\|] \to (0, \infty)$ *such that the following conditions hold:*

- $\lim_{\lambda \to 0} \varphi_1(\lambda) = 0,$

- $$\sup_{\lambda > 0} \frac{\alpha \varphi_1(\lambda)}{\lambda + \alpha} \leq \varphi_1(\alpha) \qquad \forall \lambda \in (0, \|B_s\|]$$

  and

- *there exists* $w \in X$ *with* $\|w\| \leq E_2$, *such that*

  $$B_s^{\frac{s}{2(s+b)}} L^{s/4}(x_0 - \hat{x}) = \varphi_1(B_s) w$$

- *for each* $x \in B_{\tilde{r}}(x_0)$ *there exists a bounded linear operator* $G(x, x_0)$ *(cf.Ramm et al. (2003)) such that*

  $$F'(x) = F'(x_0) G(x, x_0)$$

  with $\|G(x, x_0)\| \leq k_2$.

REMARK 6.3.13 *If* $x_0 - \hat{x} \in X_{t_1}$ *i.e.,* $\|x_0 - \hat{x}\|_{t_1} \leq E_1$ *for some positive constant* E_1 *and* $0 \leq t_1 \leq s + b$. *Then as in Remark 6.2.3, we have* $B_s^{\frac{s}{2(s+b)}} L^{s/4}(x_0 - \hat{x}) = \varphi_1(B_s) w$ *where* $\varphi_1(\lambda) = \lambda^{t_1/(s+b)}$, $w = B_s^{\frac{s-2t_1}{2(s+b)}} L^{s/4}(\hat{x} - x_0)$ *and* $\|w\| \leq g_1(\frac{s-2t_1}{2(s+b)}) E_1 := E_2$.

Assume that $k_2 < \frac{1}{1-c}[\frac{1}{\psi_2(s)} - k_0 \tilde{r}]$ with $c < 1$ and for the sake of simplicity assume that $\varphi_1(\alpha) \leq \varphi(\alpha)$, for $\alpha > 0$. Let $\psi_3(s) := \frac{E_2}{f_1(\frac{s}{2(s+b)})}$.

THEOREM 6.3.14 *Suppose* $x^{\delta}_{c,\alpha_k,s}$ *is the solution of (6.3.5) and Assumption 2.3.1 and 6.3.12 hold. Then*

$$\|\hat{x} - x^{\delta}_{c,\alpha_k,s}\| = O(\psi_{s,a}^{-1}(\delta)).$$

Proof. Note that $c(F(x^{\delta}_{c,\alpha_k,s}) - z^{\delta}_{\alpha_k,s}) + \alpha_k L^{s/2}(x^{\delta}_{c,\alpha_k,s} - x_0) = 0$, so

$$\begin{aligned}
(F'(x_0) + \alpha_k L^{s/2})(x^{\delta}_{c,\alpha_k,s} - \hat{x}) &= (F'(x_0) + \alpha_k L^{s/2})(x^{\delta}_{c,\alpha_k,s} - \hat{x}) \\
&\quad -c(F(x^{\delta}_{c,\alpha_k,s}) - z^{\delta}_{\alpha_k,s}) - \alpha_k L^{s/2}(x^{\delta}_{c,\alpha_k,s} - x_0) \\
&= \alpha_k L^{s/2}(x_0 - \hat{x}) + F'(x_0)(x^{\delta}_{c,\alpha_k,s} - \hat{x}) \\
&\quad -c[F(x^{\delta}_{c,\alpha_k,s}) - z^{\delta}_{\alpha_k,s}] \\
&= \alpha_k L^{s/2}(x_0 - \hat{x}) + F'(x_0)(x^{\delta}_{c,\alpha_k,s} - \hat{x}) \\
&\quad -c[F(x^{\delta}_{c,\alpha_k,s}) - F(\hat{x}) + F(\hat{x}) - z^{\delta}_{\alpha_k,s}] \\
&= \alpha_k L^{s/2}(x_0 - \hat{x}) - c(F(\hat{x}) - z^{\delta}_{\alpha_k,s}) \\
&\quad +F'(x_0)(x^{\delta}_{c,\alpha_k,s} - \hat{x}) - c[F(x^{\delta}_{c,\alpha_k,s}) - F(\hat{x})].
\end{aligned}$$

Thus, since $0 < c < \alpha_k$, we have

$$\begin{aligned}
\|x^{\delta}_{c,\alpha_k,s} - \hat{x}\| &\leq \|\alpha_k(F'(x_0) + \alpha_k L^{s/2})^{-1} L^{s/2}(x_0 - \hat{x})\| + \|(F'(x_0) + \alpha_k L^{s/2})^{-1} \\
&\quad c(F(\hat{x}) - z^{\delta}_{\alpha_k,s})\| + \|(F'(x_0) + \alpha_k L^{s/2})^{-1} \\
&\quad [F'(x_0)(x^{\delta}_{c,\alpha_k,s} - \hat{x}) - c(F(x^{\delta}_{c,\alpha_k,s}) - F(\hat{x}))]\| \\
&\leq \Gamma_1 + \overline{\psi_2(s)}\|F(\hat{x}) - z^{\delta}_{\alpha_k,s}\| + \Gamma_2 \qquad (6.3.12)
\end{aligned}$$

where

$$\Gamma_1 := \|\alpha_k(F'(x_0) + \alpha_k L^{s/2})^{-1} L^{s/2}(x_0 - \hat{x})\|,$$

$$\Gamma_2 := \|(F'(x_0) + \alpha_k L^{s/2})^{-1}[F'(x_0)(x^{\delta}_{c,\alpha_k,s} - \hat{x}) - c(F(x^{\delta}_{c,\alpha_k,s}) - F(\hat{x}))]\|.$$

Note that by Assumption 6.3.12

$$\begin{aligned}
\Gamma_1 &\leq \|\alpha_k L^{-s/4}(B_s+\alpha_k I)^{-1}L^{s/4}(x_0-\hat{x})\| \\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|\alpha_k(B_s+\alpha_k I)^{-1}B_s^{\frac{s}{2(s+b)}}L^{s/4}(x_0-\hat{x})\| \\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\varphi_1(\alpha_k)E_2 \qquad (6.3.13)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_2 &= \|(F'(x_0)+\alpha_k L^{s/2})^{-1}\int_0^1[F'(x_0)-cF'(\hat{x}+t(x^\delta_{c,\alpha_k,s}-\hat{x}))](x^\delta_{c,\alpha_k,s}-\hat{x})dt\| \\
&\leq \|(F'(x_0)+\alpha_k L^{s/2})^{-1}\int_0^1[F'(x_0)-F'(\hat{x}+t(x^\delta_{c,\alpha_k,s}-\hat{x}))](x^\delta_{c,\alpha_k,s}-\hat{x})dt\| \\
&\quad +(1-c)\|(F'(x_0)+\alpha_k L^{s/2})^{-1}F'(x_0)\int_0^1 G(\hat{x}+t(x^\delta_{c,\alpha_k,s}-\hat{x}),x_0) \\
&\quad (x^\delta_{c,\alpha_k,s}-\hat{x})dt\| \\
&\leq \overline{\psi_2(s)}k_0\tilde{r}\|x^\delta_{c,\alpha_k,s}-\hat{x}\|+\overline{\psi_2(s)}(1-c)k_3\|x^\delta_{c,\alpha_k,s}-\hat{x}\|. \qquad (6.3.14)
\end{aligned}$$

The last step follows from Lemma 6.3.9, Assumptions 6.3.12 and 2.3.1. Hence by (6.3.14), (6.3.13) and (6.3.12) we have

$$\begin{aligned}
\|x^\delta_{c,\alpha_k,s}-\hat{x}\| &\leq \frac{\psi_3(s)\varphi_1(\alpha_k)+\overline{\psi_2(s)}\|F(\hat{x})-z^\delta_{\alpha_k,s}\|}{1-[(1-c)k_2+k_0\tilde{r}]\overline{\psi_2(s)}} \\
&\leq \frac{\psi_3(s)\varphi_1(\alpha_k)+\overline{\psi_2(s)}C_s(2+\frac{4\eta}{\eta-1})\eta(\psi^{-1}_{s,a}(\delta))}{1-[(1-c)k_2-k_0\tilde{r}]\overline{\psi_2(s)}}=O(\psi^{-1}_{s,a}(\delta)).
\end{aligned}$$
(6.3.15)

This completes the proof of the Theorem.

The following Theorem is a consequence of Theorem 6.3.11 and Theorem 6.3.14.

THEOREM 6.3.15 *Let* $\tilde{x}^{\delta}_{n,\alpha_k,s}$ *be as in (6.1.5) with* $\alpha = \alpha_k$ *and* $\delta \in [0, \delta_0]$, *assumptions in Theorem 6.3.11 and Theorem 6.3.14 hold. Then*

$$\|\hat{x} - \tilde{x}^{\delta}_{n,\alpha_k,s}\| \leq \tilde{C}_5 \tilde{q}_s^{2n} + O(\psi^{-1}_{s,a}(\delta))$$

where $\tilde{C}_5$ *is as in Theorem 6.3.11.*

THEOREM 6.3.16 *Let* $\tilde{x}^{\delta}_{n,\alpha_k,s}$ *be as in (6.1.5) with* $\alpha = \alpha_k$ *and* $\delta \in [0, \delta_0]$, *and assumptions in Theorem 6.3.15 hold. Let*

$$n_k := \min\{n : \tilde{q_s}^{2n} \leq \alpha_k^{\frac{-a}{2(s+a)}} \delta\}.$$

Then

$$\|\hat{x} - \tilde{x}^{\delta}_{n_k,\alpha_k,s}\| = O(\psi^{-1}_{s,a}(\delta)).$$

Chapter 7

DYNAMICAL SYSTEM METHOD IN HILBERT SPACES

We present a new method for approximately solving an ill-posed Hammerstein operator equation in this Chapter. It is a combination of the Dynamical System Method considered by Ramm (2005) and Tikhonov regularization method. We present a detailed analysis for both IFD Class and MFD Class of the operator F. By choosing the regularization parameter according to an adaptive scheme considered by Pereverzev and Schock (2005) an order optimal error estimate has been obtained. The notations appearing in this Chapter are independent of the notations used in previous Chapters.

7.1 INTRODUCTION

In this Chapter we consider a Dynamical system method for approximately solving (2.1.1). We assume throughout that the solution $\hat{x}$ of (2.1.1) satisfies

$$\|\hat{x} - x_0\| = \min\{\|x - x_0\| : KF(x) = f, x \in D(F)\}$$

and that $f^\delta \in Y$ are the available noisy data with

$$\|f - f^\delta\| \leq \delta.$$

As in earlier chapters the solution x of (2.1.1) can be obtained by first solving

$$Kz = f \tag{7.1.1}$$

for z and then solving the nonlinear equation

$$F(x) = z. \tag{7.1.2}$$

In Ramm (2005) (cf. section 2.4.6, page 59), Ramm considered a method called Dynamical System Method (DSM), which avoids, inverting of the operator $F'(.)$. In this chapter we consider a method which is a combination of a modified form of DSM and the Tikhonov regularization instead of Newton type method and Tikhonov regularizationconsidered in earlier Chapters. The DSM consists of finding (cf. Ramm (2005), Nair and Ravishankar (2008)) a nonlinear locally Lipschitz operator $\Phi(u, t)$, such that the Cauchy problem:

$$u'(t) = \Phi(u, t), \qquad u(0) = u_0 \tag{7.1.3}$$

has the following three properties:

$$\exists u(t) \forall t \geq 0, \quad \exists u(\infty), \quad F(u(\infty)) = 0,$$

i.e., (7.1.3) is globally uniquely solvable, its unique solution has a limit at infinity $u(\infty)$, and this limit solves $F(x) = z^\delta_{\alpha_k}$ ($z^\delta_{\alpha_k}$ is the Tikhonov regularized solution of $Kz = f^\delta$ as given in (2.1.7)). We assume that $F(x) = z^\delta_{\alpha_k}$ is well posed, so $F(x) = z$ has a solution say $x^\delta_{\alpha_k}$, such that $x^\delta_{\alpha_k} \in B_R(x_0)$, where $B_R(x_0)$ denotes the ball of radius R with center at x_0.

The Chapter is organized as follows: In Section 7.2 we give the preparatory results, Section 7.3 discusses the Dynamical System Method for IFD and MFD Class with the error analysis.

7.2 DYNAMICAL SYSTEM METHOD(DSM)

We assume that $F \in C^2_{loc}$ i.e., $\forall x \in B_R(x_0)$,

$$\|F^{(j)}(x)\| \leq M_j, \qquad j = 1, 2. \tag{7.2.1}$$

The assumption on source condition which is based on a source function φ and a property of the source function φ is used as in Chapter 2 to obtain an error estimate for $\|F(\hat{x}) - z^\delta_{\alpha_k}\|$. As in Chapter 2, we use the adaptive choice scheme suggested by Perverzev and Schock (2005) for the selection of regularization parameter α.

7.2.1 DSM for IFD Class

Continuous schemes

Hereafter we assume that $\|F'(x_0)^{-1}\| =: \beta$, $\beta < \frac{1}{2}$ and

$$R < \frac{2(1-2\beta)}{\beta M_2 + 2k_0}. \tag{7.2.2}$$

In this section we consider the following Cauchy's problem

$$x'(t) = -(F'(x_0) + \varepsilon(t)I)^{-1}(F(x) - z_{\alpha_k}^{\delta}), \qquad x(0) = x_0 \tag{7.2.3}$$

where x_0 is an initial approximation for $x_{\alpha_k}^{\delta}$ and

$$\varepsilon : [0, \infty) \to [0, K] \tag{7.2.4}$$

is monotonic increasing function with $\varepsilon(0) = 0$ and

$$0 < K \leq \min\{\frac{1 - k_0 R}{2\beta}, 1\}. \tag{7.2.5}$$

REMARK 7.2.1 *Note that (7.2.2) implies $R < \frac{1}{k_0}$ and (7.2.5) implies $\beta\varepsilon(t) < 1$.*

In order to find a local solution for the Cauchy problem (7.2.3), we make use of the following theorem.

THEOREM 7.2.2 *(Nair and Ravishankar (2008), Theorem 2.1) Let X be a real Banach space, U be an open subset of X, and $x_0 \in U$. Let $\Phi : U \times R^+ \to X$ be of class C^1 that is bounded on bounded sets. Then the following hold.*

- *There exists a maximal interval J containing 0 such that the initial value problem*

  $$x'(t) = \Phi(x(t), t), \quad x(0) = x_0,$$

  has a unique solution $x(t) \in U$ for all $t \in J$.

- *If J has the right end point, say τ, and $x_\tau := \lim_{t \to \tau} x(t)$ exists, then x_τ is on the boundary of U.*

Now the following Proposition establishes the existence and uniqueness of the solution of the Cauchy problem (7.2.3).

PROPOSITION 7.2.3 *Let $\varepsilon(t)$ be as in (7.2.4) and F maps bounded sets onto bounded sets. Then there exists a maximal interval $J \subseteq [0,\infty)$ such that (7.2.3) has a unique solution $x(t)$ for all $t \in J$.*

Proof. Let

$$\Phi = -(F'(x_0)+\varepsilon(t)I)^{-1}(F(x)-z_{\alpha_k}^{\delta}), \qquad x \in B_R(x_0), \quad t \in R^+.$$

Then $\Phi : B_R(x_0) \times R^+ \to X$ is of class C^1. Because F is bounded on bounded sets and since $\beta\varepsilon(t) < 1$, we have

$$\begin{aligned} \|(F'(x_0)+\varepsilon(t)I)^{-1}\| &\leq \|F'(x_0)^{-1}\|\|(I+\varepsilon(t)F'(x_0)^{-1})^{-1}\| \\ &\leq \frac{\beta}{1-\beta\varepsilon(t)}. \end{aligned} \tag{7.2.6}$$

That is $(F'(x_0)+\varepsilon(t)I)$ has a bounded inverse for every $t \in R^+$. So Φ is bounded on bounded sets. Hence the conclusion follows by applying Theorem 7.2.2.

Let $x(t) - x_{\alpha_k}^{\delta} := w$ and $\|w\| := g_1(t)$. Then by Taylor Theorem (cf. Argyros (2008), Theorem 1.1.20)

$$F(x(t)) - z_{\alpha_k}^{\delta} = F(x(t)) - F(x_{\alpha_k}^{\delta}) = F'(x_{\alpha_k}^{\delta})(x(t)-x_{\alpha_k}^{\delta}) + T(x(t), x_{\alpha_k}^{\delta})$$

where $T(x(t), x_{\alpha_k}^{\delta}) = \int_0^1 F''(\lambda x(t) + (1-\lambda)x_{\alpha_k}^{\delta})(x(t)-x_{\alpha_k}^{\delta})^2(1-\lambda)d\lambda$.

Observe that

$$w'(t) = x'(t) = -(F'(x_0)+\varepsilon(t)I)^{-1}[F'(x_{\alpha_k}^{\delta})(x(t)-x_{\alpha_k}^{\delta}) + T(x(t), x_{\alpha_k}^{\delta})]$$

and hence

$$\begin{aligned} g_1 g_1' &= \frac{1}{2}\frac{dg_1^2}{dt} \\ &= \frac{1}{2}\frac{d}{dt}\langle w, w\rangle \\ &= \langle w, w'\rangle \\ &= \langle w, -(F'(x_0)+\varepsilon(t)I)^{-1}[F'(x_{\alpha_k}^{\delta})(x(t)-x_{\alpha_k}^{\delta}) + T(x(t), x_{\alpha_k}^{\delta})]\rangle \\ &= \langle w, -w\rangle + \langle w, \Lambda w\rangle + \langle w, -(F'(x_0)+\varepsilon(t)I)^{-1}T(x(t), x_{\alpha_k}^{\delta})\rangle \\ &\leq -\|w\|^2 + \|\Lambda\|\|w\|^2 + \|(F'(x_0)+\varepsilon(t)I)^{-1}T(x(t), x_{\alpha_k}^{\delta})\|\|w\| \\ &\leq -g_1^2 + \|\Lambda\|g_1^2 + \|(F'(x_0)+\varepsilon(t)I)^{-1}T(x(t), x_{\alpha_k}^{\delta})\|g_1 \end{aligned} \tag{7.2.7}$$

where $\Lambda = I - (F'(x_0) + \varepsilon(t)I)^{-1}F'(x^\delta_{\alpha_k})$. Note that

$$\begin{aligned}
\|\Lambda\| &\leq \sup_{\|v\| \leq 1} \|(F'(x_0) + \varepsilon(t)I)^{-1}[(F'(x_0) - F'(x^\delta_{\alpha_k})) + \varepsilon(t)I)]v\| \\
&\leq \|(F'(x_0) + \varepsilon(t)I)^{-1}(F'(x_0) - F'(x^\delta_{\alpha_k}))\| \\
&\quad + \|((F'(x_0) + \varepsilon(t)I)^{-1}\varepsilon(t)I)v\| \\
&\leq \|(F'(x_0) + \varepsilon(t)I)^{-1}F'(x_0)\Phi(x^\delta_{\alpha_k}, x_0, v)\| \\
&\quad + \|(F'(x_0) + \varepsilon(t)I)^{-1}\varepsilon(t)v\| \\
&\leq \frac{k_0R + \beta\varepsilon(t)}{1 - \beta\varepsilon(t)}, \qquad (7.2.8)
\end{aligned}$$

the last step follows from Assumption 2.3.1, (7.2.6) and the inequality $\|(I + \varepsilon(t)F'(x_0)^{-1})^{-1}\| \leq \frac{1}{1-\beta\varepsilon(t)}$. Again by (7.2.6) and (7.2.1)

$$\begin{aligned}
\|(F'(x_0) + \varepsilon(t)I)^{-1}T(x(t), x^\delta_{\alpha_k})\| &\leq \frac{\beta}{1 - \beta\varepsilon(t)}\|T(x(t), x^\delta_{\alpha_k})\| \\
&\leq \frac{\beta}{1 - \beta\varepsilon(t)}\frac{M_2\|x(t) - x^\delta_{\alpha_k}\|^2}{2} \\
&\leq \frac{\beta}{1 - \beta\varepsilon(t)}\frac{M_2g_1^2}{2}. \qquad (7.2.9)
\end{aligned}$$

Therefore by (7.2.7), (7.2.8) and (7.2.9) we have

$$g_1g_1' \leq -g_1^2 + (\frac{k_0R + \beta\varepsilon(t)}{1 - \beta\varepsilon(t)})g_1^2 + \frac{\beta}{1 - \beta\varepsilon(t)}\frac{M_2}{2}g_1^3$$

and hence

$$g_1' \leq -\gamma g_1 + c_0g_1^2 \qquad (7.2.10)$$

where $\gamma := 1 - (\frac{k_0R+\beta\varepsilon(t)}{1-\beta\varepsilon(t)}) > 0$ and $c_0 := \frac{\beta}{1-\beta\varepsilon(t)}\frac{M_2}{2}$. So by (7.2.10)

$$g_1(t) \leq \Upsilon e^{-\gamma t} \qquad (7.2.11)$$

where $\Upsilon = \frac{g_1(0)}{1-\frac{c_0g_1(0)}{\gamma}}$. Note that $g_1(0) = \|x_0 - x^\delta_{\alpha_k}\| \leq R$ and hence condition (7.2.2) implies $\frac{c_0g_1(0)}{\gamma} < 1$.

The above discussion leads to the following Theorem.

THEOREM 7.2.4 *If (7.2.1) and the assumptions of Proposition 7.2.3 hold, then (7.2.3) has a unique global solution $x(t)$ and $x(t)$ converges to $x^\delta_{\alpha_k}$. Further*

$$\|x(t) - x^\delta_{\alpha_k}\| \leq \Upsilon e^{-\gamma t}$$

where Υ is as in (7.2.11).

THEOREM 7.2.5 *(cf. George and Nair (2008), Theorem 3.3) Suppose (7.2.1) and (7.2.2) hold. If, in addition, $\|\hat{x} - x_0\| \leq R$ then*

$$\|\hat{x} - x^\delta_{\alpha_k}\| \leq \frac{\beta}{1 - k_0 R}\|F(\hat{x}) - z^\delta_{\alpha_k}\|.$$

Proof. Proof is as in Theorem 2.3.4 of Chapter 2.

Iterative Schemes

We present DSM for constructing convergent iterative schemes for the well-posed equations $F(x) - z^\delta_{\alpha_k} = 0$. In this section we assume that

$$\beta < \frac{2}{M_2}(1 - k_0 R). \tag{7.2.12}$$

For solving $F(x) - z^\delta_{\alpha_k} = 0$ we consider the following discretization scheme

$$x_{n+1} = x_n - hF'(x_0)^{-1}(F(x_n(t)) - z^\delta_{\alpha_k}). \tag{7.2.13}$$

We shall consider the DSM method for proving the convergence of (x_n) to the solution $x^\delta_{\alpha_k}$ of (7.1.2). We begin our analysis with the following Cauchy's problem:

$$w'_{n+1}(t) = -F'(x_0)^{-1}(F(w_{n+1}(t)) - z^\delta_{\alpha_k}), \qquad w_{n+1}(t_n) = x_n, \;\; t_n \leq t \leq t_{n+1} \tag{7.2.14}$$

where x_n is as in (7.2.13). The following Proposition establishes the existence and uniqueness of the solution of the Cauchy problem (7.2.14).

PROPOSITION 7.2.6 *Let F maps bounded sets onto bounded sets. Then there exists a maximal interval $J \subseteq [0, \infty)$ such that (7.2.14) has a unique solution $x(t)$ for all $t \in J$.*

Proof. Let

$$\Phi = -F'(x_0)^{-1}(F(w_{n+1}(t)) - z^\delta_{\alpha_k}), \qquad w_{n+1} \in B_R(x_0), \quad t \in R^+.$$

Then $\Phi : B_R(x_0) \times R^+ \to X$ is of class C^1. Because F is bounded on bounded sets, Φ is also bounded on bounded sets. Hence the conclusion follows by applying Theorem 7.2.2.

PROPOSITION 7.2.7 *If (7.2.1) and the Assumptions of Proposition 7.2.6 hold, then (7.2.14) has a unique global solution* $w_{n+1}(t)$ *and* $w_{n+1}(t)$ *converges to* $x^{\delta}_{\alpha_k}$*. Further*

$$\|w_{n+1}(t) - x^{\delta}_{\alpha_k}\| \leq \frac{e^{-\tilde{\gamma}nh}}{1-\frac{\tilde{c_0}}{\tilde{\gamma}}} e^{-\tilde{\gamma}(t-t_n)} \tag{7.2.15}$$

where $\tilde{c_0} = \frac{M_2\beta}{2}$ *and* $\tilde{\gamma} = 1 - k_0 R$*.*

Proof. We shall prove (7.2.15) by induction. Clearly for $n = 0$ the result is true. Suppose (7.2.15) is true for some n. Let $w_{n+1}(t) - x^{\delta}_{\alpha_k} := \tilde{w}$ and $\|\tilde{w}\| := \tilde{g}_1$. Then by Taylor Theorem (cf.Argyros (2008), Theorem 1.1.20)

$$F(w_{n+1}(t)) - z^{\delta}_{\alpha_k} = F(w_{n+1}(t)) - F(x^{\delta}_{\alpha_k}) = F'(x^{\delta}_{\alpha_k})(w_{n+1}(t) - x^{\delta}_{\alpha_k}) + T(w_{n+1}(t), x^{\delta}_{\alpha_k}) \tag{7.2.16}$$

where $T(w_{n+1}(t), x^{\delta}_{\alpha_k}) = \int_0^1 F''(\lambda w_{n+1}(t) + (1-\lambda)x^{\delta}_{\alpha_k})(w_{n+1}(t) - x^{\delta}_{\alpha_k})^2(1-\lambda)d\lambda$. Observe that

$$\tilde{w}'(t) = w'_{n+1}(t) = -F'(x_0)^{-1}[F'(x^{\delta}_{\alpha_k})(w_{n+1}(t) - x^{\delta}_{\alpha_k}) + T(w_{n+1}(t), x^{\delta}_{\alpha_k})] \tag{7.2.17}$$

and hence

$$\begin{aligned} \tilde{g_1}\tilde{g_1}' &= \frac{1}{2}\frac{d\tilde{g_1}^2}{dt} = \frac{1}{2}\frac{d}{dt}\langle \tilde{w}, \tilde{w}\rangle = \langle \tilde{w}, \tilde{w}'\rangle \\ &= \langle \tilde{w}, -F'(x_0)^{-1}[F'(x^{\delta}_{\alpha_k})(w_{n+1}(t) - x^{\delta}_{\alpha_k}) + T(w_{n+1}(t), x^{\delta}_{\alpha_k})]\rangle \\ &= \langle \tilde{w}, -\tilde{w}\rangle + \langle \tilde{w}, \tilde{\Lambda}\tilde{w}\rangle + \langle \tilde{w}, -F'(x_0)^{-1}T(w_{n+1}(t), x^{\delta}_{\alpha_k})\rangle \end{aligned} \tag{7.2.18}$$

where $\tilde{\Lambda} = I - F'(x_0)^{-1}F'(x^{\delta}_{\alpha_k})$. Note that

$$\begin{aligned} \|\langle \tilde{w}, \tilde{\Lambda}\tilde{w}\rangle\| &= \|w\|\|F'(x_0)^{-1}(F'(x_0) - F'(x^{\delta}_{\alpha_k}))\tilde{w}\| \\ &\leq k_0 R\|\tilde{w}\|^2 \end{aligned} \tag{7.2.19}$$

the last step follows from Assumption 2.3.1. Again by (7.2.6) and (7.2.1)

$$\begin{aligned} \|F'(x_0)^{-1}T(w_{n+1}(t), x^{\delta}_{\alpha_k})\| &\leq \beta\|T(w_{n+1}(t), x^{\delta}_{\alpha_k})\| \\ &\leq \beta\frac{M_2\|x(t) - x^{\delta}_{\alpha_k}\|^2}{2} \\ &\leq \beta\frac{M_2\tilde{g_1}^2}{2}. \end{aligned} \tag{7.2.20}$$

Therefore by (7.2.18), (7.2.19) and (7.2.20) we have

$$\tilde{g_1}\tilde{g_1}' \leq -\tilde{g_1}^2 + k_0 R\tilde{g_1}^2 + \beta\frac{M_2}{2}\tilde{g_1}^3 \tag{7.2.21}$$

i.e.,

$$\tilde{g_1}' \leq -\gamma\tilde{g_1} + \tilde{c_0}\tilde{g_1}^2, \tag{7.2.22}$$

and hence

$$\tilde{g_1}(t) \leq \tilde{\Upsilon}e^{-\gamma(t-t_n)} \tag{7.2.23}$$

where $\tilde{\Upsilon} = \frac{\tilde{g_1}(t_n)}{1-\frac{\tilde{c_0}\tilde{g_1}(t_n)}{\tilde{\gamma}}}$. Note that $\tilde{\Upsilon} = \frac{\tilde{g_1}(t_n)}{1-\frac{\tilde{c_0}\tilde{g_1}(t_n)}{\tilde{\gamma}}} \leq \frac{e^{-\tilde{\gamma}nh}}{1-\frac{\tilde{c_0}}{\tilde{\gamma}}}$, condition (7.2.12) implies $\frac{\tilde{c_0}}{\tilde{\gamma}} < 1$ and hence

$$\tilde{g_1}(t) \leq \frac{e^{-\tilde{\gamma}nh}}{1-\frac{\tilde{c_0}}{\tilde{\gamma}}}e^{-\tilde{\gamma}(t-t_n)}. \tag{7.2.24}$$

Analogous to the proof of the above proposition one can prove (by taking $\tilde{g_1} = \|F(w_{n+1}(t)) - z^\delta_{\alpha_k}\|$) the following Proposition.

PROPOSITION 7.2.8 *Let $w_{n+1}(t)$ be the solution of (7.2.14) and $z^\delta_{\alpha_k}$ be as in (2.1.7). Then*

$$\|F(w_{n+1}(t)) - z^\delta_{\alpha_k}\| \leq \|F(x_0) - z^\delta_{\alpha_k}\|e^{-\tilde{\gamma}(nh+t-t_n)}. \tag{7.2.25}$$

PROPOSITION 7.2.9 *Let $w_{n+1}(t)$ be the solution of (7.2.14) and x_{n+1} be as in (7.2.13). Then*

$$\|x_{n+1} - w_{n+1}(t_{n+1})\| \leq h^2\beta^2 M_1\|F(x_0) - z^\delta_{\alpha_k}\|e^{-\tilde{\gamma}nh}.$$

Proof. Observe that

$$\begin{aligned}
\|x_{n+1} - w_{n+1}(t_{n+1})\| &= \int_{t_n}^{t_{n+1}} \|\Phi(x_n) - \Phi(w_{n+1}(t))\|dt \\
&\leq \beta\int_{t_n}^{t_{n+1}} \|F(x_n) - F(w_{n+1}(t)\|dt \\
&\leq \beta M_1\int_{t_n}^{t_{n+1}} \|x_n - w_{n+1}(t)\|dt \\
&\leq \beta M_1 h\int_{t_n}^{t_{n+1}} \|\Phi(w_{n+1}(t))\|dt \\
&\leq \beta^2 M_1 h\int_{t_n}^{t_{n+1}} \|F(w_{n+1}(t)) - z^\delta_{\alpha_k}\|dt.
\end{aligned}$$

Now the result follows from (7.2.25).

Thus by triangle inequality, (7.2.24) and (7.2.26) we have the following

THEOREM 7.2.10 *If (7.2.1) and the assumptions of Proposition 7.2.6 hold, then* x_{n+1} *converges to* $x^\delta_{\alpha_k}$. *Further*

$$\|x_{n+1} - x^\delta_{\alpha_k}\| \leq \tilde{C}e^{-\tilde{\gamma}nh}$$

where $\tilde{C} = h^2\beta^2 M_1\|F(x_0) - z^\delta_{\alpha_k}\| + \frac{1}{1-\frac{\tilde{c}_0}{\tilde{\gamma}}}e^{-h\tilde{\gamma}}$.

Now we give the error analysis of both the schemes discussed above.

THEOREM 7.2.11 *Suppose (7.2.1), (7.2.2) and the assumptions in Theorem 7.2.4 and Theorem 7.2.5 hold. If, in addition,* $\|\hat{x} - x_0\| \leq R$ *then*

$$\|\hat{x} - x(t)\| \leq \frac{\beta}{1-\kappa_0 R}\|F(\hat{x}) - z^\delta_{\alpha_k}\| + re^{-\gamma t}.$$

Proof. The proof follows using Theorem 7.2.4, Theorem 7.2.5 and the triangle inequality:

$$\|\hat{x} - x(t)\| \leq \|\hat{x} - x^\delta_{\alpha_k}\| + \|x^\delta_{\alpha_k} - x(t)\|.$$

THEOREM 7.2.12 *Suppose (7.2.1) and the assumptions in Theorem 7.2.10 and Theorem 7.2.5 hold. If, in addition,* $\|\hat{x} - x_0\| \leq R$ *then*

$$\|\hat{x} - x_{n+1}\| \leq \frac{\beta}{1-\kappa_0 R}\|F(\hat{x}) - z^\delta_{\alpha_k}\| + \tilde{C}e^{-\tilde{\gamma}nh}.$$

Proof. The proof follows using Theorem7.2.10, Theorem 7.2.5 and the triangle inequality:

$$\|\hat{x} - x_{n+1}\| \leq \|\hat{x} - x^\delta_{\alpha_k}\| + \|x^\delta_{\alpha_k} - x_{n+1}\|.$$

THEOREM 7.2.13 *Let* $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$ *and the assumptions of Theorem 7.2.11 is satisfied. Let*

$$T := min\{t : e^{-\gamma t} < \frac{\delta}{\sqrt{\alpha_k}}\},$$

and $x(T)$ *be the solution of the Cauchy's problem (7.2.3). Then*

$$\|\hat{x} - x(T)\| = O(\psi^{-1}(\delta)).$$

THEOREM 7.2.14 *Let* $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$ *and the assumptions of Theorem 7.2.12 is satisfied. Let*

$$N := min\{n : e^{-\tilde{\gamma}nh} < \frac{\delta}{\sqrt{\alpha_\delta}}\}$$

and x_{N+1} *be as in (7.2.13). Then*

$$\|\hat{x} - x_{N+1}\| = O(\psi^{-1}(\delta)).$$

7.2.2 DSM for MFD Class

In this section we consider X to be a real Hilbert space. Here for approximately solving (7.1.2) with $z^\delta_{\alpha_k}$ in place of z we consider the Laverentiev regularization method, i.e., we consider the solution x^δ_{c,α_k} of the equation

$$F(x) + \frac{\alpha_k}{c}(x - x_0) = z^\delta_{\alpha_k}, \;\; c \leq \alpha_k \tag{7.2.26}$$

as an approximate solution of (7.1.2) with $z^\delta_{\alpha_k}$ in place of z.

Assumption 2.3.1 is used throughout the analysis.

Let $\delta_0 < \frac{2}{M_2+2k_0}\sqrt{\alpha_0}$ and

$$R_\rho := \frac{\delta_0}{\sqrt{\alpha_0}} + M\rho. \tag{7.2.27}$$

LEMMA 7.2.15 *Let R_ρ be as in (7.2.27). Let $z^\delta_{\alpha_k}$ be as in (2.1.7), and if x^δ_{c,α_k} is the solution of (7.2.26) with $\alpha := \alpha_k$ and $\delta \in [0, \delta_0]$, then $x^\delta_{c,\alpha_k} \in B_{R_\rho}(x_0)$.*

Proof. Observe that $F(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^\delta_{c,\alpha_k} - x_0) = z^\delta_{\alpha_k}$.

Let $M := \int_0^1 F'(x_0 + t(x^\delta_{c,\alpha_k} - x_0))dt$. Then

$$\begin{aligned} F(x^\delta_{c,\alpha_k}) - F(x_0) + \frac{\alpha_k}{c}(x^\delta_{c,\alpha_k} - x_0) &= z^\delta_{\alpha_k} - F(x_0) \\ (M + \frac{\alpha_k}{c}I)(x^\delta_{c,\alpha_k} - x_0) &= z^\delta_{\alpha_k} - F(x_0) \\ (x^\delta_{c,\alpha_k} - x_0) &= (M + \frac{\alpha_k}{c}I)^{-1}(z^\delta_{\alpha_k} - F(x_0)). \end{aligned}$$

Thus

$$\begin{aligned} \|x^\delta_{c,\alpha_k} - x_0\| &\leq \|z^\delta_{\alpha_k} - F(x_0)\| \\ &\leq \|(K^*K + \alpha_k I)^{-1}K^*(f^\delta - KF(x_0))\| \\ &\leq \|(K^*K + \alpha_k I)^{-1}K^*(f^\delta - f + f - KF(x_0))\| \\ &\leq \|(K^*K + \alpha_k I)^{-1}K^*(f^\delta - f)\| \\ &\quad + \|(K^*K + \alpha_k I)^{-1}K^*K(F(\hat{x}) - F(x_0))\| \\ &\leq \frac{\delta}{\sqrt{\alpha_k}} + M\rho < R_\rho. \end{aligned}$$

Hence the Lemma.

Continuous Schemes

In this section we consider the following Cauchy's problem for solving (7.2.26):

$$x'(t) = -(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}(F(x(t)) - z^{\delta}_{\alpha_k} + \frac{\alpha_k}{c}(x(t) - x_0)), \qquad x(0) = x_0 \quad (7.2.28)$$

where $c \leq \alpha_k$ and x_0 is an initial approximation. In this section we assume that

$$\rho < \frac{1}{M}[\frac{2}{M_2 + 2k_0} - \frac{\delta_0}{\sqrt{\alpha_0}}]. \quad (7.2.29)$$

Note that (7.2.29) implies that $R_\rho < \frac{1}{k_0}$.

The local solution for the Cauchy problem (7.2.28) is given by Theorem 7.2.2. The Proposition below establishes the existence and uniqueness of the solution of the Cauchy problem (7.2.28).

PROPOSITION 7.2.16 *Let F maps bounded sets onto bounded sets. Then there exists a maximal interval $J \subseteq [0, \infty)$ such that (7.2.28) has a unique solution $x(t)$ for all $t \in J$.*

Proof. Proof is analogous to the proof of Proposition 7.2.3.

THEOREM 7.2.17 *Let $\delta \in (0, \delta_0]$, Assumption 2.3.1 and Lemma 7.2.15 be satisfied with ρ as in (7.2.29). If (7.2.1) and Proposition 7.2.16 hold, then (7.2.28) has a unique global solution $x(t)$ and $x(t)$ converges to $x^{\delta}_{\alpha_k}$. Further*

$$\|x(t) - x^{\delta}_{c,\alpha_k}\| \leq c_3 e^{-c_1 t}$$

where $c_3 = \frac{g_2(0)}{1 - \frac{c_2 g_2(0)}{c_1}}$, $c_1 = 1 - k_0 R_\rho > 0$, $c_2 = \frac{M_2}{2}$ *and* $g_2(0) = \|x(0) - x^{\delta}_{c,\alpha_k}\|$.

Proof. Let $x(t) - x^{\delta}_{c,\alpha_k} := \vartheta$ and $\|\vartheta\| := g_2(t)$.

Then by Taylor Theorem (cf.Argyros and Hilout (2010), Theorem 1.1.20)

$$F(x(t)) - F(x^{\delta}_{c,\alpha_k}) = F'(x^{\delta}_{c,\alpha_k})(x(t) - x^{\delta}_{c,\alpha_k}) + T(x(t), x^{\delta}_{c,\alpha_k}) \quad (7.2.30)$$

where $T(x(t), x^{\delta}_{c,\alpha_k}) = \int_0^1 F''(\lambda x(t) + (1-\lambda)x^{\delta}_{c,\alpha_k})(x(t) - x^{\delta}_{c,\alpha_k})^2(1-\lambda)d\lambda$. Since $F(x^{\delta}_{c,\alpha_k}) - z^{\delta}_{\alpha_k} + \frac{\alpha_k}{c}(x^{\delta}_{c,\alpha_k} - x_0) = 0$, by (7.2.30) we have

$$F(x(t)) - z^{\delta}_{\alpha_k} + \frac{\alpha_k}{c}(x(t) - x_0) = (F'(x^{\delta}_{c,\alpha_k}) + \frac{\alpha_k}{c}I)(x(t) - x^{\delta}_{c,\alpha_k}) + T(x(t), x^{\delta}_{c,\alpha_k}).$$

Observe that

$$\vartheta'(t) = x'(t) = -(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}[(F'(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c}I)(x(t) - x^\delta_{c,\alpha_k}) + T(x(t), x^\delta_{c,\alpha_k})]$$

and hence

$$\begin{aligned} g_2g_2' &= \frac{1}{2}\frac{dg_2^2}{dt} = \frac{1}{2}\frac{d}{dt}\langle\vartheta,\vartheta\rangle = \langle\vartheta,\vartheta'\rangle \\ &= \langle\vartheta, -(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}[(F'(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c}I)(x(t) - x^\delta_{c,\alpha_k}) + T(x(t), x^\delta_{c,\alpha_k})]\rangle \\ &= \langle\vartheta, -\vartheta\rangle + \langle\vartheta, \Theta\vartheta\rangle + \langle\vartheta, -(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}T(x(t), x^\delta_{c,\alpha_k})\rangle \\ &\leq -\|\vartheta\|^2 + \|\Theta\|\|\vartheta\|^2 + \|(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}T(x(t), x^\delta_{c,\alpha_k})\|\|\vartheta\| \\ &\leq -g_2^2 + \|\Theta\|g_2^2 + \|T(x(t), x^\delta_{c,\alpha_k})\|g_2 \qquad (7.2.31) \end{aligned}$$

where $\Theta = -(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}(F'(x^\delta_{\alpha_k}) - F'(x_0))$. Note that

$$\begin{aligned} \|\Theta\| &\leq \sup_{\|v\|\leq 1}\|(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}[F'(x_0) - F'(x^\delta_{c,\alpha_k})]v\| \\ &\leq \sup_{\|v\|\leq 1}\|(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}F'(x_0)\Phi(x^\delta_{c,\alpha_k}, x_0, v)\| \\ &\leq k_0R_\rho\|v\|, \qquad (7.2.32) \end{aligned}$$

the last step follows from Assumption 2.3.1. Again by (7.2.1),

$$\begin{aligned} \|(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}T(x(t), x^\delta_{c,\alpha_k})\| &\leq \|T(x(t), x^\delta_{c,\alpha_k})\| \\ &\leq \frac{M_2\|x(t) - x^\delta_{c,\alpha_k}\|^2}{2} \\ &\leq \frac{M_2g_2^2}{2}. \qquad (7.2.33) \end{aligned}$$

Therefore by (7.2.31), (7.2.32) and (7.2.33) we have

$$g_2g_2' \leq -g_2^2 + k_0R_\rho g_2^2 + \frac{M_2}{2}g_2^3$$

and hence

$$g_2' \leq -c_1g_2 + c_2g_2^2 \qquad (7.2.34)$$

where $c_1 := 1 - k_0R_\rho > 0$ and $c_2 := \frac{M_2}{2}$. So by solving (7.2.34) we get,

$$g_2(t) \leq c_3e^{-c_1t}.$$

REMARK 7.2.18 *Note that by Lemma 7.2.15,* $g_2(0) = \|x_0 - x^\delta_{c,\alpha_k}\| \leq R_\rho$ *and hence condition (7.2.29) implies* $\frac{c_2 g_2(0)}{c_1} < 1$.

Assume that $k_2 < \frac{1-k_0R_\rho}{1-c}$ and for the sake of simplicity assume that $\varphi_1(\alpha) \leq \varphi(\alpha)$ for $\alpha > 0$.

THEOREM 7.2.19 *Suppose* x^δ_{c,α_k} *is the solution of (7.2.26) with* $\delta \in (0, \delta_0]$, *and Assumptions 2.3.1, 2.3.9 and 2.3.10 hold with* ρ *as in (7.2.29). Then*

$$\|\hat{x} - x^\delta_{c,\alpha_k}\| \leq \frac{\varphi_1(\alpha_k) + \|F(\hat{x}) - z^\delta_{\alpha_k}\|}{1-(1-c)k_2 - k_0R_\rho}.$$

In particular by Theorem 2.2.3,

$$\|\hat{x} - x^\delta_{c,\alpha_k}\| \leq \frac{\varphi_1(\alpha_k) + (2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1-(1-c)k_2 - k_0R_\rho}.$$

The following Theorem is a consequence of Theorem 7.2.17 and Theorem 7.2.19.

THEOREM 7.2.20 *Suppose (7.2.1), and assumptions in Theorem 7.2.17 and Theorem 7.2.19 hold with* ρ *as in (7.2.29), then*

$$\|\hat{x} - x(t)\| \leq \frac{\varphi_1(\alpha_k) + (2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1-(1-c)k_1 - k_0R_\rho} + c_3 e^{-c_1 t},$$

where c_1 *and* c_3 *are as in Theorem 7.2.17.*

THEOREM 7.2.21 *Let* $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$ *and the assumptions of Theorem 7.2.20 are satisfied. Let*

$$T := min\{t : e^{-c_1 t} < \frac{\delta}{\sqrt{\alpha_k}}\},$$

and $x(T)$ *be the solution of the Cauchy's problem (7.2.28), with* $\delta \in (0, \delta]$. *Then*

$$\|\hat{x} - x(T)\| = O(\psi^{-1}(\delta)).$$

Iterative Schemes

In this section we assume that $M_2 < 2$, $\delta_0 < \frac{2-M_2}{2k_0}\sqrt{\alpha_0}$ and

$$\rho < \frac{1}{M}[\frac{2-M_2}{2k_0} - \frac{\delta_0}{\sqrt{\alpha_0}}]. \tag{7.2.35}$$

Now we solve $F(x) = z$ with the following discretization scheme

$$x_{n+1} = x_n - h(F'(x_0) + \frac{\alpha_k}{c} I)^{-1}[F(x_n) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(x_n - x_0)], h = constant > 0, \quad (7.2.36)$$

with $c \leq \alpha_k$. Let us consider the following Cauchy's problem:

$$w'_{n+1}(t) = -(F'(x_0) + \frac{\alpha_k}{c} I)^{-1}[F(w_{n+1}(t)) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(w_{n+1}(t) - x_0)], \quad (7.2.37)$$

$w_{n+1}(t_n) = x_n,\ t_n \leq t \leq t_{n+1}$ where x_n is as in (7.2.36).

The existence and uniqueness of the solution of the Cauchy problem (7.2.37) can be established as in Proposition 7.2.16.

THEOREM 7.2.22 *If $\delta \in (0, \delta_0]$, (7.2.1), Assumption 2.3.1 and Lemma 7.2.15 hold with ρ as in (7.2.35), then (7.2.37) has a unique global solution $w_{n+1}(t)$ and $w_{n+1}(t)$ converges to x^δ_{c,α_k}. Further*

$$\|w_{n+1}(t) - x^\delta_{c,\alpha_k}\| \leq \frac{e^{-\tilde{c_1} nh}}{1 - \frac{\tilde{c_2}}{\tilde{c_1}}} e^{-\tilde{c_1}(t - t_n)} \quad (7.2.38)$$

where $\tilde{c_2} = \frac{M_2}{2}$ and $\tilde{c_1} = 1 - k_0 R_\rho > 0$.

Proof. We shall prove (7.2.38) by induction. Clearly for $n = 0$ the result is true, suppose (7.2.38) is true for some n. Let $w_{n+1}(t) - x^\delta_{\alpha_k} := \tilde{\vartheta}$ and $\|\tilde{\vartheta}\| := \tilde{g_2}(t)$. Then by Taylor Theorem (cf. Argyros and Hilout (2010), Theorem 1.1.20)

$$\begin{aligned} F(w_{n+1}(t)) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(w_{n+1}(t) - x_0) &= F(w_{n+1}(t)) - F(x^\delta_{c,\alpha_k}) \\ &\quad + \frac{\alpha_k}{c}(w_{n+1}(t) - x^\delta_{c,\alpha_k}) \\ &= F'(x^\delta_{c,\alpha_k})(w_{n+1}(t) - x^\delta_{c,\alpha_k}) \\ &\quad + T(w_{n+1}(t), x^\delta_{c,\alpha_k}) \\ &\quad + \frac{\alpha_k}{c}(w_{n+1}(t) - x^\delta_{c,\alpha_k}) \quad (7.2.39) \end{aligned}$$

where $T(w_{n+1}(t), x^\delta_{c,\alpha_k}) = \int_0^1 F''(\lambda w_{n+1}(t) + (1-\lambda)x^\delta_{c,\alpha_k})(w_{n+1}(t) - x^\delta_{c,\alpha_k})^2(1-\lambda)d\lambda$. Observe that

$$\begin{aligned} \tilde{\vartheta}'(t) = w'_{n+1}(t) &= -(F'(x_0) + \frac{\alpha_k}{c} I)^{-1}[(F'(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c} I)(w_{n+1}(t) - x^\delta_{c,\alpha_k}) \\ &\quad + T(w_{n+1}(t), x^\delta_{c,\alpha_k})] \end{aligned}$$

and hence

$$\begin{aligned}
\tilde{g_2}\tilde{g_2}' &= \frac{1}{2}\frac{d\tilde{g_2}^2}{dt} = \frac{1}{2}\frac{d}{dt}\langle\tilde{\vartheta},\tilde{\vartheta}\rangle = \langle\tilde{\vartheta},\tilde{\vartheta}'\rangle \\
&= \langle\tilde{\vartheta}, -(F'(x_0)+\frac{\alpha_k}{c}I)^{-1}[(F'(x^\delta_{c,\alpha_k})+\frac{\alpha_k}{c}I)\tilde{w}+T(w_{n+1}(t),x^\delta_{c,\alpha_k})]\rangle \\
&= \langle\tilde{\vartheta},-\tilde{\vartheta}\rangle + \langle\tilde{\vartheta}, -(F'(x_0)+\frac{\alpha_k}{c}I)^{-1}T(w_{n+1}(t),x^\delta_{c,\alpha_k})\rangle \\
&\quad +\langle\tilde{\vartheta}, -(F'(x_0)+\frac{\alpha_k}{c}I)^{-1}(F'(x^\delta_{c,\alpha_k})-F'(x_0))\tilde{\vartheta}\rangle \qquad (7.2.40)
\end{aligned}$$

Note that

$$\begin{aligned}
\langle\tilde{\vartheta}, -(F'(x_0)+\frac{\alpha_k}{c}I)^{-1}[F'(x^\delta_{c,\alpha_k})-F'(x_0)]\tilde{\vartheta}\rangle &\leq \|\tilde{\vartheta}\|\|(F'(x_0)+\frac{\alpha_k}{c}I)^{-1} \\
&\quad (F'(x_0)-F'(x^\delta_{c,\alpha_k}))\tilde{\vartheta}\| \\
&\leq \|\tilde{\vartheta}\|\|(F'(x_0)+\frac{\alpha_k}{c}I)^{-1} \\
&\quad F'(x_0)\Phi(x^\delta_{c,\alpha_k},x_0,\tilde{\vartheta})\| \\
&\leq k_0R_\rho\|\tilde{\vartheta}\|^2 \qquad (7.2.41)
\end{aligned}$$

the last step follows from Assumption 2.3.1. Again by (7.2.39) and (7.2.1)

$$\begin{aligned}
\langle\tilde{\vartheta}, -(F'(x_0)+\frac{\alpha_k}{c}I)^{-1}T(w_{n+1}(t),x^\delta_{c,\alpha_k})\rangle &\leq \|\tilde{\vartheta}\|\|(F'(x_0)+\frac{\alpha_k}{c}I)^{-1} \\
&\quad T(w_{n+1}(t),x^\delta_{c,\alpha_k})\| \\
&\leq \|\tilde{\vartheta}\|\|T(w_{n+1}(t),x^\delta_{c,\alpha_k})\| \\
&\leq \|\tilde{\vartheta}\|\frac{M_2\|x(t)-x^\delta_{c,\alpha_k}\|^2}{2} \\
&\leq \|\tilde{\vartheta}\|\frac{M_2\tilde{g_2}^2}{2}. \qquad (7.2.42)
\end{aligned}$$

Therefore by (7.2.40), (7.2.41) and (7.2.42) we have

$$\tilde{g_2}\tilde{g_2}' \leq -\tilde{g_2}^2 + k_0R_\rho\tilde{g_2}^2 + \frac{M_2}{2}\tilde{g_2}^3$$

i.e.,

$$\tilde{g_2}' \leq -\tilde{c_1}\tilde{g_2} + \tilde{c_2}\tilde{g_2}^2,$$

and hence

$$\tilde{g_2}(t) \leq \tilde{c_3}e^{-\tilde{c_1}(t-t_n)}$$

where $\tilde{c_3} = \frac{\tilde{g_2}(t_n)}{1-\frac{\tilde{c_2}\tilde{g_2}(t_n)}{\tilde{c_1}}}$. Note that $\tilde{c_3} = \frac{\tilde{g_2}(t_n)}{1-\frac{\tilde{c_2}\tilde{g_2}(t_n)}{\tilde{c_1}}} \leq \frac{e^{-\tilde{c_1}nh}}{1-\frac{\tilde{c_2}}{\tilde{c_1}}}$, condition (7.2.35) implies $\frac{\tilde{c_2}}{\tilde{c_1}} < 1$ and hence

$$\tilde{g_2}(t) \leq \frac{e^{-\tilde{c_1}nh}}{1-\frac{\tilde{c_2}}{\tilde{c_1}}} e^{-\tilde{c_1}(t-t_n)}.$$

This completes the proof of the Theorem.

THEOREM 7.2.23 *Let $w_{n+1}(t)$ be the solution of (7.2.37) and $z_{\alpha_k}^{\delta}$ be as in (2.1.7) with $\delta \in (0, \delta_0]$ and $\alpha = \alpha_k$. If Lemma 7.2.15 holds with ρ as in (7.2.35), then*

$$\|F(w_{n+1}(t)) - z_{\alpha_k}^{\delta} + \frac{\alpha_k}{c}(w_{n+1}(t) - x_0)\| \leq \|F(x_0) - z_{\alpha_k}^{\delta}\| e^{-\tilde{c_1}(nh+t-t_n)}. \qquad (7.2.43)$$

Proof. The proof follows as in proof of Theorem 7.2.22 by taking

$$\tilde{g_2}(t) = \|F(w_{n+1}(t)) - z_{\alpha_k}^{\delta} + \frac{\alpha_k}{c}(w_{n+1}(t) - x_0)\|.$$

PROPOSITION 7.2.24 *Let x_{n+1} be as in (7.2.36) with $\delta \in (0, \delta_0]$. If (7.2.1) and Theorem 7.2.23 hold, then*

$$\|x_{n+1} - w_{n+1}(t_{n+1})\| \leq h^2(M_1+1)R_\rho e^{-\tilde{c_1}nh}.$$

Proof. Observe that

$$\begin{aligned}
\|x_{n+1} - w_{n+1}(t_{n+1})\| &= \int_{t_n}^{t_{n+1}} \|\Phi(x_n) - \Phi(w_{n+1}(t))\| dt \\
&\leq \int_{t_n}^{t_{n+1}} \|(F'(x_0) + \frac{\alpha_k}{c} I)^{-1}[F(x_n) - F(w_{n+1}(t)) \\
&\quad + \frac{\alpha_k}{c}(x_n - w_{n+1}(t))]\| dt \\
&\leq (M_1+1) \int_{t_n}^{t_{n+1}} \|x_n - w_{n+1}(t)\| dt
\end{aligned}$$

$$
\begin{aligned}
&\leq (M_1+1)h\int_{t_n}^{t_{n+1}} \|\Phi(w_{n+1}(t))\| dt \\
&\leq (M_1+1)h\int_{t_n}^{t_{n+1}} \|(F'(x_0)+\frac{\alpha_k}{c}I)^{-1}[F(w_{n+1}(t))-z^{\delta}_{\alpha_k} \\
&\quad +\frac{\alpha_k}{c}(w_{n+1}(t)-x_0)]\| dt. \qquad (7.2.44)
\end{aligned}
$$

Now from (7.2.43), (7.2.44) and Lemma 7.2.15 we have,

$$
\begin{aligned}
\|x_{n+1}-w_{n+1}(t_{n+1})\| &\leq h^2(M_1+1)\|F(x_0)-z^{\delta}_{\alpha_k}\|e^{-\tilde{c_1}nh} \\
&\leq h^2(M_1+1)R_\rho e^{-\tilde{c_1}nh}.
\end{aligned}
$$

Hence the Proposition.

Thus by triangle inequality, (7.2.38) and (7.2.44) we have the following

THEOREM 7.2.25 *If the assumptions of Proposition 7.2.24 and Theorem 7.2.22 hold, then x_{n+1} converges to x^{δ}_{c,α_k}. Further*

$$\|x_{n+1}-x^{\delta}_{c,\alpha_k}\| \leq \tilde{C}e^{-\tilde{c_1}nh}$$

where $\tilde{C} = h^2(M_1+1)R_\rho + \frac{1}{1-\frac{\tilde{c_2}}{\tilde{c_1}}}e^{-\tilde{c_1}h}$.

THEOREM 7.2.26 *Let assumptions of Theorem 7.2.25 hold. Suppose $k_2 < \frac{1-k_0R_\rho}{1-c}$ and assumptions of Theorem 7.2.19 hold with ρ as in (7.2.35), then*

$$\|\hat{x}-x_{n+1}\| \leq \frac{\varphi_1(\alpha_k)+(2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1-(1-c)k_2-k_0R_\rho} + \tilde{C}e^{-\tilde{c_1}nh}.$$

Proof. The proof follows from Theorem 7.2.25, Theorem 7.2.19 (with ρ as in (7.2.35)) and the triangle inequality:

$$\|\hat{x}-x_{n+1}\| \leq \|\hat{x}-x^{\delta}_{c,\alpha_k}\| + \|x^{\delta}_{c,\alpha_k}-x_{n+1}\|.$$

THEOREM 7.2.27 *Let* $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$ *and the assumptions of Theorem 7.2.26 be satisfied. Let*

$$N := min\{n : e^{-\tilde{c_1}nh} < \frac{\delta}{\sqrt{\alpha_\delta}}\}$$

and x_{N+1} *be as in (7.2.36) with* $z_{\alpha_k}^{\delta}$ *in place of* $z_{\alpha_k}^{\delta}$, *with* $\delta \in [0, \delta]$. *Then*

$$\|\hat{x} - x_{N+1}\| = O(\psi^{-1}(\delta)).$$

Chapter 8

CONCLUDING REMARKS

In this thesis, we have considered the problem of approximately solving non-linear ill-posed Hammerstein type operator equation. The regularization procedure involes the splitting of given non-linear Hammerstein type equation into linear and non-linear ill-posed operator equations, thus giving rise to the scope of using a combination of Tikhonov regularization for solving linear ill-posed problem and Newton-type method for regularizing non-linear ill-posed problem. Regularization parameter α is chosen according to the adaptive method considered by Pereverzev and Schock(2005) for the linear ill-posed operator equations and the same parameter α is used for solving the non-linear operator equation, so the choice of the regularization parameter does not depend on the non-linear operator.

The thesis comprises of seven chapters. A brief introduction and preliminaries are given in Chapter 1.

In Chapter 2 we presented an iterative method for obtaining an approximate solution for a nonlinear ill-posed Hammerstein type operator equation $KF(x) = f$, here $F : D(F) \subseteq X \to X$ is nonlinear operator, $K : X \to Y$ is a bounded linear operator. Throughout this thesis we assumed that the available data is f^δ with $\|f - f^\delta\| \leq \delta$. The proposed method combines the Tikhonov regularization and Guass Newton method. As the iterations involve the Fr$\acute{e}$chet derivative only at the initial approximation of the exact solution $\hat{x}$ of $KF(x) = f$, the method becomes simpler. In each chapter of this thesis we considered two cases of F (IFD Class and MFD Class), in the IFD Class it is assumed that $F'(x_0)^{-1}$ exist and in the MFD Class it is assumed that $F'(x_0)^{-1}$ does not exist but F is monotone. In both the cases, the derived error estimate using an a priori and balancing principle are of optimal order with respect to the general source condition.

In Chapter 3, we considered a finite dimensional realization of the method considered in Chapter 2. We have chosen the regularization parameter according to balancing principle of Pereverzev and Schock (2005). The error estimate is of optimal order and the method leads to local linear convergence. Numerical examples provided confirm the efficiency of the method.

Chapter 4 is a modified form of the method considered in Chapter 2 and Chapter 3. In Chapter 2, Frechet derivative of the non-linear operator F was considered only at the initial guess. But in this Chapter we have taken into consideration, the Frechet derivative at all points $x_n,\ n \geq 0$. This has improved the rate of convergence(cubic convergence). Also, we have presented a finite dimensional realization of the method. We have chosen the regularization parameter according to balancing principle of Pereverzev and Schock (2005). The derived error bounds are of optimal order. Numerical examples are given, which proves the efficiency of the proposed method.

And in Chapter 5 we further modified the method analyzed in Chapter 4 and obtained semi-local quartic convergence.

In Chapter 6, we considered an iterative regularization method for obtaining an approximate solution of an ill-posed Hammerstein type operator equation $KF(x) = f$ in the Hilbert scale setting. We considered the Hilbert scale $(X_t)_{t\in R}$ generated by L for the analysis where $L : D(L) \to X$ is a linear, unbounded, self-adjoint, densely defined and strictly positive operator on X. The derived error estimates under the general source conditions are of optimal order.

In Chapter 7, we presented a method, which is a combination of Dynamical System Method(DSM) and Tikhonov regularization method for approximately solving ill-posed Hammerstein type operator equation $KF(x) = f$. We analyzed DSM for IFD Class and MFD Class of the operator F. Infact we considered continuous and iterative schemes of DSM studied extensively by Ramm (see Ramm (2007),Ramm (2005)) and his collaborators.

In this Chapter also we obtained order optimal error bounds by choosing the regularization parameter α according to the adaptive method considered by Pereverzev and Schock(2005).

In future works, we would like to analyze the case when F is non-invertible and non-monotone operator. We have already obtained results in this direction and a paper (see George and Shobha (2014) (This work is not included in this thesis)). Further work is under progress.

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(a) **International Refereed Journals:**

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(c) **Papers presented in the conferences**

1. Regularized Gauss Newton Methods for Hammerstein Type Equations at 78th Annual conference of Indian Mathematical Society held at BHU, Varanasi during 22-25, January, 2013.(Won the AMU Prize for best paper in functional analysis)

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Publications not part of the thesis

- **International Refereed Journals:**

1. S. George and M. E. Shobha, Newton type iteration for Tikhonov regularization of non-linear ill-posed Hammerstein type equations, Journal of Applied Mathematics and Computing, 44, 69-82, doi: 10.1007/s12190-013-0681-1, 2014.

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