NEWTON TYPE METHODS FOR LAVRENTIEV REGULARIZATION OF NONLINEAR ILL-POSED OPERATOR EQUATIONS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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JULY, 2013

In Memory of my Father

DECLARATION

I hereby ***declare*** that the Research Thesis entitled **"NEWTON TYPE METHODS FOR LAVRENTIEV REGULARIZATION OF NONLINEAR ILL-POSED OPERATOR EQUATIONS"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a ***bonafide report of the research work carried out by me.*** The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to ***certify*** that the Research Thesis entitled **"NEWTON TYPE METHODS FOR LAVRENTIEV REGULARIZATION OF NONLINEAR ILL-POSED OPERATOR EQUATIONS"** submitted by **SURESAN PARETH**, (Register Number: **MA10F04**) as the record of the research work carried out by him, is ***accepted as the Research Thesis submission*** in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy**.

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ACKNOWLEDGEMENTS

I bow before the Almighty who has showered his tremendous grace and abundant blessings on me and has gifted me with the perseverance and strength of mind to complete this thesis successfully.

It is indeed a happy moment to start my acknowledgements in thanking the guide, Dr. Santhosh George, Professor, Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka for his guidance and suggestions.

It's my pleasure to thank the RPAC members Dr. R. K. Kadoli, Dr. B. R. Shankar and Dr. D. Sukumar, IIT Hyderabad and the DTAC members Dr. A. Mahesha and Dr. Sam Johnson for their valuable suggestions. I express my sincere gratitude to the reviewers Dr. Christine Bockmann, Department of Mathematics, University of Potsdam, Germany and Dr. P. Rekha Kulkarni, Department of Mathematics, IIT Bombay.

I would like to thank Dr. N. N. Murulidhar, Professor and Head, Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka for support and cooperation.

I express my sincere thanks to Prof. Keshava Prasad Halemane and Prof. S. M. Hegde, former Head, Department of Mathematical and Computational Sciences for introducing me to the field of research. Also my special thanks are extended to Prof. Frank Natterer, Department of Mathematics, Institute of Computational Mathematics and Instrumental, Germany and Prof. Jari P Kaipio, Department of Mathematics, The University of Auckland, New Zealand for providing me their research articles.

I extend my thanks to everyone in the Department of Mathematical and Computational Sciences and for everyone at the National Institute of Technology Karnataka for unlimited cooperation during the study.

Finally, I wish to extend my very special thanks to my wife Haseena, my son Aaron and all in my family. I dedicate this thesis to all of them.

I thank National Institute of Technology Karnataka for the financial support.

Place: NITK, Surathkal, **SURESAN PARETH**

Date: 29^{th} July 2013.

ABSTRACT

In this thesis we consider nonlinear ill-posed operator equations of the form $F(x) = f$, that arise from the study of nonlinear inverse problems, where $F : X \to X$ is a nonlinear monotone operator defined on a real Hilbert space X. In applications, instead of f, usually only noisy data f^δ are available. Then the problem of recovery of the exact solution $\hat{x}$ from noisy equation $F(x) = f^\delta$ is ill-posed, in the sense that a small perturbation in the data can cause large deviation in the solution. Thus the computation of a stable approximation for $\hat{x}$ from the solution of $F(x) = f^\delta$, becomes an important issue in ill-posed problems, and the regularization techniques have to be taken into account. Approximation methods are an attractive choice since they are straightforward to implement, for getting the numerical solution of nonlinear ill-posed problems. Thus in the last few years more emphasis was put on the investigation of iterative regularization methods.

We consider Newton type iterative regularization methods and their finite dimensional realizations, for obtaining approximation for $\hat{x}$ in the Hilbert space and Hilbert scales settings. We use the adaptive scheme of Pereverzyev and Schock (2005), for choosing the regularization parameter.

Keywords: Ill-posed nonlinear equations, Regularization, Hilbert scales, Monotone operator, Newton-Lavrentiev method, Adaptive parameter choice.

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Chapter 1

INTRODUCTION

The field of inverse problems is a wide and important area in applied mathematics and other sciences that has been rapidly growing over the last decades. The reason for the increasing interest is due to the wide variety of applications in sciences and engineering. This thesis is devoted to the mathematical theory of iterative regularization methods for nonlinear ill-posed problems that arise from the study of nonlinear inverse problems, with special emphasis on the development of parameter choice and stopping rules which leads to optimal convergence rates. We specifically present Newton-type iterative regularization method for approximately solving the ill-posed equation $F(x) = f$, when the operator F is nonlinear and monotone, from the perspective of our research program.

1.1 GENERAL INTRODUCTION

In this thesis we consider the nonlinear equation

$$F(x) = f \tag{1.1.1}$$

that arise from the study of nonlinear inverse problems, where $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator defined on X. Throughout the thesis $\langle ., . \rangle$ and $\|.\|$ respectively stand for the inner product and the corresponding norm in the real Hilbert space X.

Inverse problems are problems where causes for a desired or an observed effect are to be determined. They have, always driven by applications, been studied for nearly a century now. An important feature, both theoretically and numerically, of inverse problems is their ill-posedness. Many problems which appear in science and engineering

(eg: Signal and Image Processing, Computerized Tomography and Heat conduction, etc.) can be formulated as an equation of the form (1.1.1). In general the equation (1.1.1) is not well-posed in the sense proposed by Hadamard (1923) given in Definition 1.3.1 below. We study the operator equation (1.1.1) with a noisy data f^δ in place of the exact data f satisfying $\|f - f^\delta\| \leq \delta$ with the known noise level δ.

The ill-posed problems are generally handled using the regularization techniques. The process of obtaining a stable approximate solution for (1.1.1) is called a regularization method. In a regularization method, the ill-posed equation is replaced by a family of well-posed equations based on a regularization parameter. A regularization method for (1.1.1) with f^δ in place of f is said to be convergent, if the regularized solutions converge in the norm to a solution of (1.1.1) as δ tends to zero.

We will first set up the definitions and notations and then introduce the formal notion and difficulties encountered when one tries to solve (1.1.1).

1.2 DEFINITIONS AND NOTATIONS

Throughout this thesis X is a real Hilbert space, $\langle .,. \rangle$, $\|.\|$, $D(F)$ and $BL(X,Y)$ stand respectively for the inner product, the corresponding norm, the domain of F and the set of all bounded linear operators from X to Y. Also, δ_0, ρ, γ, $\gamma_\rho, \varepsilon_h, \varepsilon_0$ and q are generic constants which may take different values at different occasions.

Definition 1.2.1. *(Fréchet derivative) Let F be an operator mapping a Hilbert space X into a Hilbert space Y. If there exists a bounded linear operator $L : X \to Y$ such that*

$$\lim_{\|h\| \to 0} \frac{\|F(x_0 + h) - F(x_0) - L(h)\|}{\|h\|} = 0,$$

then F is said to be Fréchet differentiable at x_0 and the bounded linear operator $F'(x_0) := L$ is called the first Fréchet derivative of F at x_0.

Definition 1.2.2. *(Monotone operators) Let $F : D(F) \subseteq X \to X$ be an operator defined on a real Hilbert space X. Then F is said to be monotone if $\langle F(x) - F(y), x - y \rangle \geq 0, \forall x, y \in D(F)$.*

1.3 ILL-POSED PROBLEMS

Hadamard's concept of a well posed problem, reflected the idea that any mathematical model of a physical phenomena must have the properties of existence, uniqueness and stability of the solution.

Now we shall formally define the concept of well-posedness.

Definition 1.3.1. *(Well-posed) Let $F : D(F) \subseteq X \to Y$ be an operator (linear or nonlinear) between Hilbert spaces X and Y. The equation (1.1.1) is said to be well-posed if the following three conditions hold.*

1. *(1.1.1) has a solution*

2. *(1.1.1) cannot have more than one solution*

3. *the solution x of (1.1.1) depends continuously on the data f.*

In the operator theoretic language the above conditions together means that F is a bijection and F^{-1} is a continuous operator.

The equation (1.1.1) is said to be ill-posed if it is not well-posed.

We give examples for ill-posed problems.

Example 1.3.1. *Exponential growth model (see, Groetsch (1993)): For a given $c > 0$, consider the problem of determining $x(t), t \in (0,1)$, in the initial value problem*

$$\frac{dy}{dt} = x(t)y(t), \quad y(0) = c, \tag{1.3.2}$$

where $y \in L^2[0,1]$. This problem can be written as an operator equation of the form (1.1.1), where $F : L^2[0,1] \to L^2[0,1]$ is defined by

$$F(x)(t) = ce^{\int_0^t x(s)ds}, \quad x \in L^2[0,1], \quad t \in (0,1).$$

It can be seen from the following argument that the problem is ill-posed. Suppose, in place of an exact data y, we have a perturbed data

$$y^\delta(t) := y(t)e^{\delta sin(\frac{t}{\delta^2})}, \quad t \in (0,1).$$

Then, from (1.3.2), the solution corresponding to $y^\delta(t)$ is given by

$$x^\delta(t) := \frac{d}{dt}\log(y^\delta(t)), \quad t \in (0,1).$$

Note that,

$$\|y - y^\delta\|_2 \to 0 \quad as \quad \delta \to 0.$$

But

$$x^\delta(t) - x(t) = \frac{d}{dt}\log(e^{\delta sin\frac{t}{\delta^2}}) = \frac{d}{dt}(\delta sin\frac{t}{\delta^2}),$$

so that

$$\|x^\delta - x\|_2^2 = \frac{1}{4}sin\frac{2}{\delta^2} + \frac{1}{2\delta^2} \to \infty \quad as \quad \delta \to 0.$$

Hence, the solution does not depend continuously on the given data and thus the problem is ill-posed.

Example 1.3.2. *The Vibrating String (see, Groetsch (1993)): The free vibration of a nonhomogeneous string of unit length and density distribution* $\rho(x) > 0, 0 < x < 1$, *is modeled by the partial differential equation*

$$\rho(x)u_{tt} = u_{xx};$$

where $u(x,t)$ *is the position of the particle* x *at time* t. *Assume that the ends of the string are fixed and* $u(x,t)$ *satisfies the boundary conditions*

$$u(0,t) = 0, u(1,t) = 0.$$

Assuming the solution $u(x,t)$ *is of the form*

$$u(x,t) = y(x)r(t),$$

one observes that $y(x)$ satisfies the ordinary differential equation

$$y'' + \omega^2 \rho(x) y = 0 \tag{1.3.3}$$

with boundary conditions

$$y(0) = 0, y(1) = 0.$$

Suppose the value of y at certain frequency ω is known, then by integrating equation (1.3.3) twice, first from zero to s and then from zero to one, we obtain

$$\int_0^1 y'(s;\omega)ds - y'(0;\omega) + \omega^2 \int_0^1 \int_0^s \rho(x)y(x;\omega)dxds = 0.$$

$$\int_0^1 (1-s)\rho(s)y(s;\omega)ds = \frac{y'(0;\omega)}{\omega^2}. \tag{1.3.4}$$

The inverse problem here is to determine the variable density ρ of the string, satisfying (1.3.4) for all allowable frequencies ω.

Example 1.3.3. *Simplified Tomography (see, Groetsch (1993)): Consider a two dimensional object contained within a circle of radius R. The object is illuminated with a radiation of density I_0. As the radiation beams pass through the object it absorbs some radiation. Assume that the radiation absorption coefficient $f(x, y)$ of the object varies from point to point of the object. The absorption coefficient satisfies the law*

$$\frac{dI}{dy} = -fI$$

where I is the intensity of the radiation. By taking the above equation as the definition of the absorption coefficient, we have

$$I_x = I_0 e^{-\int_{-y(x)}^{y(x)} f(x,y)dy}$$

where $y = \sqrt{R^2 - x^2}$. *Let* $p(x) = \ln(\frac{I_0}{I_x})$, *i.e.,*

$$p(x) = \int_{-y(x)}^{y(x)} f(x, y)dy.$$

Suppose that f *is circularly symmetric, i.e.,* $f(x, y) = f(r)$ *with* $r = \sqrt{x^2 + y^2}$, *then*

$$p(x) = \int_x^R \frac{2r}{\sqrt{r^2 - x^2}} f(r)dr. \tag{1.3.5}$$

The inverse problem is to find the absorption coefficient f *satisfying the equation (1.3.5).*

Example 1.3.4. *Nonlinear singular integral equation (see, Buong (1998)): Consider the nonlinear singular integral equation in the form*

$$\int_0^t (t - s)^{-\lambda} x(s)ds + K(x(t)) = f_0(t), \quad 0 < \lambda < 1, \tag{1.3.6}$$

where $f_0 \in L^2[0, 1]$ *and the nonlinear function* $K(t)$ *satisfies the following conditions:*

- $|K(t)| \leq a_1 + a_2|t|, \quad a_1, \ a_2 > 0,$
- $K(t_1) \leq K(t_2) \Longleftrightarrow t_1 \leq t_2$, *and*
- K *is differentiable.*

Thus, K *is a monotone operator from* $X = L^2[0; 1]$ *into* $X^* = L^2[0; 1]$. *In addition, assume that* K *is a compact operator. Then the equation (1.3.6) is an ill-posed problem, since the operator* F *defined by*

$$Fx(t) = \int_0^t (t - s)^{-\lambda} x(s)ds,$$

is compact.

It is clear from the definition of ill-posed problems, that if (1.1.1) is ill-posed then (1.1.1) need not have a solution in the usual sense. So one has to modify the notion of a solution.

1.4 GENERALIZED INVERSE AND GENERALIZED SOLUTION

If $f \notin R(F)$, the range of F, then clearly (1.1.1) has no solution and hence the equation (1.1.1) is ill-posed. In such a case we may broaden the notion of a solution in a meaningful sense. For $F \in BL(X,Y)$ and $f \in Y$, an element $u \in X$ is said to be a least square solution of (1.1.1) if

$$\|F(u) - f\| = \inf\{\|F(x) - f\| : x \in X\}.$$

Observe that if F is not one-one, then the least square solution (cf. Groetsch (1984)) u, if exists, is not unique since $u + v$ is also a least square solution for every $v \in N(F)$, the null space of F. The following theorem provides a characterization of least square solutions.

Theorem 1.4.1. *(see, Groetsch (1993), Theorem 1.3.1) For $F \in BL(X,Y)$ and $f \in Y$, the following are equivalent.*

(i) $\|F(u) - f\| = \inf\{\|F(x) - f\| : x \in X\}$

(ii) $F^*F(u) = F^*f$

(iii) $F(u) = Pf$

where $P : Y \to Y$ is the orthogonal projection onto $\overline{R(F)}$.

From (iii) it is clear that (1.1.1) has a least square solution if and only if $Pf \in R(F)$. i.e., if and only if f belongs to the dense subset $R(F) + R(F)^\perp$ of Y. By Theorem 1.4.1 it is clear that the set of all least square solutions is a closed convex set and hence by Theorem 1.1.4 in Groetsch (1977), there is a unique least square solution of smallest norm. For $f \in R(F) + R(F)^\perp$, the unique least square solution of minimal norm of (1.1.1) is called the generalized solution or the pseudo solution of (1.1.1). It can be easily seen that the generalized solution belongs to the subspace $N(F)^\perp$ of X. The map $F^\dagger : D(F^\dagger) := R(F) + R(F)^\perp \to X$ which assigns each $f \in D(F^\dagger)$ with the unique least square solution of minimal norm is called the generalized inverse or Moore-Penrose inverse of F. Note that if $f \in R(F)$ and if F is injective then the generalized solution of (1.1.1) is nothing but the solution of (1.1.1). If F is bijective then it follows that $F^\dagger = F^{-1}$.

Theorem 1.4.2. *(see, Nair (2009), Theorem 4.4) Let $F \in BL(X,Y)$. Then $F^\dagger : D(F^\dagger) := R(F) + R(F)^\perp \to X$ is closed densely defined operator and $F^\dagger$ is bounded if and only if $R(F)$ is closed.*

If the equation (1.1.1) is ill-posed then one would like to obtain the generalized solution of (1.1.1). But by Theorem 1.4.2, the problem of finding the generalized solution of (1.1.1) is also ill-posed, i.e., $F^\dagger$ is discontinuous if $R(F)$ is not closed. This observation is important since a wide class of operators of practical importance, especially compact operators of infinite rank falls into this category (see, Groetsch (1993)). Further in application the data f may not be available exactly. So one has to work with an approximation f^δ of f. If $F^\dagger$ is discontinuous then for f^δ close to f, the generalized solution $F^\dagger f^\delta$, even when it is defined need not be close to $F^\dagger f$. To manage this situation the so called regularization procedures have to be employed in order to obtain approximations for $F^\dagger f$.

1.5 REGULARIZATION METHOD

Definition 1.5.1. *A family of operators $\{R_\alpha : 0 < \alpha \le \alpha_0\}$ is called a regularization method for the problem (1.1.1) with f in range of F, if there exists a parameter choice rule $\alpha = \alpha(\delta, f^\delta)$ such that*

$$\lim_{\delta \to 0} \sup \left\{ \|R_{\alpha(\delta,f^\delta)} f^\delta - F^\dagger f\| : f^\delta \in Y, \|f - f^\delta\| \le \delta \right\} = 0.$$

1.5.1 Regularization principle and Tikhonov regularization

Let us first consider the problem of finding the generalized solution of (1.1.1) with $F \in BL(X,Y)$ and $f \in D(F^\dagger)$. For $\delta > 0, f^\delta \in Y$ be an inexact data such that

$\|f - f^\delta\| \le \delta$. By a regularization of equation (1.1.1) with f^δ in place of f we mean a procedure of obtaining a family (x_α^δ) of vectors in X such that each $x_\alpha^\delta, \alpha > 0$ is a solution of a well posed equation and $x_\alpha^\delta \to F^\dagger f$ as $\alpha \to 0, \delta \to 0$.

A regularization method which has been studied most extensively is the so called Tikohonov regularization (see, Groetsch (1984)) introduced in the early sixties, where x_α^δ is taken as the minimizer of the functional $J_\alpha^\delta(x)$, where

$$J_\alpha^\delta(x) = \|F(x) - f^\delta\|^2 + \alpha\|x\|^2. \tag{1.5.7}$$

The fact that x_α^δ is the unique solution of the well-posed equation

$$(F^*F + \alpha I)x_\alpha^\delta = F^* f^\delta$$

is included in the following well known result (see, Nair (2009)).

Theorem 1.5.1. *(cf. Nair (2009), Theorem 4.9) Let $F \in BL(X,Y)$. For each $\alpha > 0$ there exists unique $x_\alpha^\delta \in X$ which minimizes the functional $J_\alpha^\delta(x)$ in (1.5.7). Moreover the map $f^\delta \to x_\alpha^\delta$ is continuous for each $\alpha > 0$ and*

$$x_\alpha^\delta = (F^*F + \alpha I)^{-1} F^* f^\delta.$$

1.5.2 Lavrentiev regularization method

If $X = Y$ and F is a positive self-adjoint operator on X, then one may consider (see, Bakushinskii (1965)) a simpler regularization method to solve equation (1.1.1), where the family of vectors w_α^δ, satisfying

$$(F + \alpha I)w_\alpha^\delta = f^\delta, \tag{1.5.8}$$

is considered to obtain approximations for $F^\dagger f$. Note that for positive self-adjoint operator F, the ordinary Tikhonov regularization applied to (1.1.1) results in a more complicated equation $(F^2 + \alpha I)x_\alpha^\delta = Ff^\delta$ than (1.5.8). Moreover it is known (see, Schock (1985)) that the approximation obtained by regularization procedure (1.5.8) has better convergence properties than the approximation obtained by Tikhonov regularization. As in Groetsch and Guacaneme (1987), we call the above regularization procedure which gives the family of vectors w_α^δ in (1.5.8), the simplified regularization of (1.1.1).

One of the prime concerns of regularization methods is the convergence of x_α^δ (w_α^δ in the case of simplified regularization) to $F^\dagger f$, as $\alpha \to 0$ and $\delta \to 0$. It is known that (see, Groetsch (1984)) if $R(F)$ is not closed then there exist sequences (δ_n) and $\alpha_n = \alpha(\delta_n)$ such that $\delta_n \to 0$ and $\alpha_n \to 0$ as $n \to \infty$ but the sequence $(x_{\alpha_n}^{\delta_n})$ diverges as $\delta_n \to 0$. Therefore it is important to choose the regularization parameter α depending on the error level δ and also possibly on f^δ, say $\alpha := \alpha(\delta, f^\delta)$ such that $\alpha(\delta, f^\delta) \to 0$ and $x_\alpha^\delta \to F^\dagger f$ as $\delta \to 0$. Practical considerations suggest that it is desirable to choose the regularization parameter at the time of solving x_α^δ using a so called a posteriori method which depends on f^δ as well as on δ (see, Pereverzyev and Schock (2005)). For our work we have used the adaptive selection of parameter proposed by Pereverzyev and Schock (2005). Before explaining this procedure in detail we shall briefly refer to the topic of Tikhonov regularization for a nonlinear ill-posed operator equation.

For the equation (1.1.1) with F a nonlinear operator, the least square solution $\hat{x}$ is defined by the requirement

$$\|F(\hat{x}) - f\| = \inf_{x \in D(F)} \|F(x) - f\| \tag{1.5.9}$$

and an x_0 minimum norm solution should satisfy (1.5.9) (see, Engl et al. (1989)) and also

$$\|\hat{x} - x_0\| = \min\{\|x - x_0\| : F(x) = f, x \in D(F)\}$$

here x_0 is some initial guess. Such a solution:

- need not exist
- need not be unique, even when it exists (see, Scherzer et al. (1993)).

(Let $S := \{x : F(x) = f\}$. Then S is closed and convex if F is monotone and continuous (see, e.g., Ramm (2007)) and hence has a unique element of minimal norm, denoted by $\hat{x}$ such that $F(\hat{x}) = f$.)

Tikhonov regularization for nonlinear ill-posed problem (1.1.1) provides approximate solutions as solutions of the minimization problem $J_F^\delta(x)$, where

$$J_F^\delta(x) = \|F(x) - f^\delta\|^2 + \alpha\|x - x_0\|^2, \quad \alpha > 0.$$

If x_α^δ is an interior point of $D(F)$, then the regularized approximation x_α^δ satisfies the normal operator equation

$$F'^*(x)[F(x) - f^\delta] + \alpha(x - x_0) = 0$$

of the Tikhonov functional $J_F^\delta(x)$. Here $F'^*(.)$ is the adjoint of the Fréchet derivative $F'(.)$ of F. For the special case when F is a monotone operator the least squares minimization (and hence the use of adjoint) can be avoided and one can use the simpler regularized equation

$$F(x) + \alpha(x - x_0) = f^\delta. \tag{1.5.10}$$

The method in which the regularized approximation x_α^δ is obtained by solving the singularly perturbed operator equation (1.5.10) is called the method of Lavrentiev regularization (see, Lavrentiev (1967)), or sometimes the method of singular perturbation (see, Liu and Nashed (1996)).

1.5.3 Regularization parameter selection

In general a regularized solution x_α^δ can be written as $x_\alpha^\delta = R_\alpha f^\delta$, where R_α is a regularization function. A regularization method consists not only of a choice of regularization functions R_α but also of a choice of the regularization parameter α. A choice $\alpha = \alpha_\delta$ of the regularization parameter may be made in either an a priori or a posteriori way (see, Groetsch (1993)).

Suppose there exists a function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(\hat{x})\|$ and $v \in X$ such that

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v, \tag{1.5.11}$$

where x_0 is an initial guess, $\hat{x}$ is the solution of (1.1.1) and $F'(\hat{x})$ is the Fréchet derivative of F at $\hat{x}$ and

$$\|\hat{x} - R_\alpha f\| \leq \varphi(\alpha),$$

then φ is called a source function and the condition (1.5.11) is called source condition.

Note that (see, Groetsch (1993)) the choice of the parameter α_δ depends on the unknown source conditions. In applications, it is desirable that α is chosen independent of the source function φ, but may depend on the data (δ, f^δ), and consequently on the regularized solutions. For linear ill-posed problems there exist many such a posteriori parameter choice strategies. These strategies include the ones proposed by Archangeli (see, Groetsch and Guacaneme (1987), Guacaneme (1990), George and Nair (1993) and Tautenhahn (2002)).

Pereverzyev and Schock (2005), considered an adaptive selection of the parameter which does not involve even the regularization method in an explicit manner. Let us briefly discuss this adaptive method in a general context of approximating an element $\hat{x} \in X$ by elements from a set $\{x_\alpha^\delta : \alpha > 0, \delta > 0\}$.

Suppose $\hat{x} \in X$ is to be approximated by using elements x_α^δ for $\alpha > 0, \delta > 0$. Assume that there exist increasing functions $\varphi(t)$ and $\psi(t)$ for $t > 0$ such that

$$\lim_{t \to o} \varphi(t) = 0 = \lim_{t \to o} \psi(t)$$

and

$$\|\hat{x} - x_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\psi(\alpha)},$$

for all $\alpha > 0, \delta > 0$. Here, the function φ may be associated with the unknown element $\hat{x}$, whereas the function ψ may be related to the method involved in obtaining x_α^δ. Note that the quantity $\varphi(\alpha) + \frac{\delta}{\psi(\alpha)}$ attains its minimum for the choice $\alpha := \alpha_\delta$ such that $\varphi(\alpha_\delta) = \frac{\delta}{\psi(\alpha_\delta)}$, that is for

$$\alpha_\delta = (\varphi\psi)^{-1}(\delta)$$

and in that case

$$\|\hat{x} - x_\alpha^\delta\| \leq 2\varphi(\alpha_\delta).$$

The above choice of the parameter is a priori in the sense that it depends on the unknown functions φ and ψ.

In an a posteriori choice, one finds a parameter α_δ without making use of the unknown source function φ such that one obtains an error estimate of the form

$$\|\hat{x} - x_{\alpha_\delta}^\delta\| \leq c\varphi(\alpha_\delta),$$

for some $c > 0$ with $\alpha_\delta = (\varphi\psi)^{-1}(\delta)$. The procedure in Pereverzyev and Schock (2005) starts with a finite number of positive real numbers, $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_N$, such that

$$\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N.$$

The following theorem is essentially a reformulation of a theorem proved in Pereverzyev and Schock (2005).

Theorem 1.5.2. *(cf. George and Nair (2008), Theorem 4.3, Semenova (2010), Theorem 3.1) Assume that there exists $i \in \{0, 1, 2, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\psi(\alpha_i)}$ and for some $\mu > 1$,*

$$\psi(\alpha_i) \leq \mu\psi(\alpha_{i-1}) \quad \forall i \in \{0, 1, 2, \cdots, N\}.$$

Let

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta}{\psi(\alpha_i)}\right\} < N,$$

$$k := \max\left\{i : \|x^\delta_{\alpha_i} - x^\delta_{\alpha_j}\| \leq 4\frac{\delta}{\psi(\alpha_j)}, \quad \forall j = 0, 1, \cdots, i\right\}.$$

Then $l \leq k$ *and*

$$\|\hat{x} - x^\delta_{\alpha_k}\| \leq 6\mu\varphi(\alpha_\delta), \quad \alpha_\delta := (\varphi\psi)^{-1}(\delta).$$

1.5.4 Iterative regularization methods and convergence rate

The last few years, many authors (see, Bakushinskii (1992), Hanke et al. (1995), Hanke (1997a), Hanke (1997b), Blaschke et al. (1997), Kaltenbacher (1997), Ramlau (1997), Ramlau (2003) and George (2010)) considered iterative methods for solving (1.1.1). An iterative method with iterations defined by

$$x^\delta_{n+1} = \Phi(x^\delta_n, \cdots, x^\delta_0, f^\delta), x^\delta_0 = x_0$$

for a known function Φ together with a stopping rule which determines a stopping index $k_\delta \in N$ is called an iterative regularization method if $\|x^\delta_{k_\delta} - \hat{x}\| \to 0, \delta \to 0$ (see, Mahale and Nair (2009)). Here $x_0 \in D(F)$ is a known initial approximation of the solution $\hat{x}$.

A sequence (x_n) in X with $\lim x_n = x^*$ is said to be convergent of order $p > 1$, if there exist positive reals β, γ, such that for all $n \in N$

$$\|x_n - x^*\| \leq \beta e^{-\gamma p^n}. \tag{1.5.12}$$

If the sequence (x_n) has the property that $\|x_n - x^*\| \leq \beta q^n$, $0 < q < 1$ then (x_n) is said to be linearly convergent. For an extensive discussion of convergence rate (see, Ortega and Rheinboldt (1970), Kelley (1995)).

1.6 OUTLINE OF THE THESIS

Chapter 1: In this chapter we present a general introduction to the problem, some known examples of nonlinear ill-posed problems and some well known results and background materials related to the thesis.

Chapter 2: We consider an iterative regularization method TSMNLM (Two Step Modified Newton Lavrentiev Method) for approximating the zero x_α^δ of (1.5.10) defined by:

$$y_{n,\alpha}^\delta = x_{n,\alpha}^\delta - R_\alpha^{-1}(x_0)[F(x_{n,\alpha}^\delta) - f^\delta + \alpha(x_{n,\alpha}^\delta - x_0)] \tag{1.6.13}$$

and

$$x_{n+1,\alpha}^\delta = y_{n,\alpha}^\delta - R_\alpha^{-1}(x_0)[F(y_{n,\alpha}^\delta) - f^\delta + \alpha(y_{n,\alpha}^\delta - x_0)] \tag{1.6.14}$$

where $x_{0,\alpha}^\delta := x_0$ and $R_\alpha(x_0) = F'(x_0) + \alpha I$.

We prove that under a general source condition on $x_0 - \hat{x}$ the error $\|\hat{x} - x_{n,\alpha}^\delta\|$ between the regularized approximation $x_{n,\alpha}^\delta$ and the solution $\hat{x}$ is of optimal order.

The regularization parameter α in this chapter as well as in the other chapters are selected from a finite set

$$D_N(\alpha) := \{\alpha_i = \mu^i \alpha_0, i = 0, 1, \cdots, N\} \tag{1.6.15}$$

where $\mu > 1$ and $\alpha_0 > 0$.

We prove that

$$\|x_{n_\delta,\alpha_k}^\delta - \hat{x}\| = O(\psi^{-1}(\delta)),$$

where the stopping index n_δ and the regularization parameter α_k are selected according to the adaptive method.

Also in this chapter we consider the finite dimensional approximation of TSMNLM. Analogous to the iterative scheme (1.6.13) and (1.6.14), we define the iterative sequence to obtain an approximate solution for the equation (1.5.10), in the finite dimensional subspace of X as:

$$y_{n,\alpha}^{h,\delta} = x_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(x_{0,\alpha}^{h,\delta})P_h[F(x_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(x_{n,\alpha}^{h,\delta} - x_0)]$$

and

$$x_{n+1,\alpha}^{h,\delta} = y_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(x_{0,\alpha}^{h,\delta})P_h[F(y_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(y_{n,\alpha}^{h,\delta} - x_0)]$$

where $x_{0,\alpha}^{h,\delta} := P_h x_0$ is the projection of the initial iterate x_0 on to $R(P_h)$, the range of P_h and $R_\alpha(x) := P_h F'(x) P_h + \alpha P_h$ with $\alpha > \alpha_0 > 0$. Here $\{P_h\}_{h>0}$ be a family of orthogonal projections on X and the regularization parameter α is chosen from the finite set defined in (1.6.15), according to the adaptive method. A numerical example and the corresponding computational results are exhibited to confirm the reliability and effectiveness of our method.

Chapter 3: In this chapter, we present Cubically converging two step Newton Lavrentiev Method (CNLM) and its finite dimensional realization for finding an approximate solution for equation (1.1.1).

Numerical example and the corresponding computational results are presented.

Chapter 4: In this chapter, we suggest and analyze another iterative method and its finite dimensional realization for obtaining an approximate solution for nonlinear ill-posed operator equation (1.1.1), and prove that the methods converge locally quarticaly to x_α^δ. We also obtain an optimal order error estimate by choosing the regularization parameter α according to the adaptive method considered by Perverzev and Schock (2005). Numerical results were provided.

Chapter 5: In this chapter, we consider the variant of the method (1.6.13) and (1.6.14) defined by

$$y_{n,\alpha,s}^\delta = x_{n,\alpha,s}^\delta - (F'(x_0) + \alpha L^s)^{-1}[F(x_{n,\alpha,s}^\delta) - f^\delta + \alpha L^s(x_{n,\alpha,s}^\delta - x_0)]$$

and

$$x_{n+1,\alpha,s}^\delta = y_{n,\alpha,s}^\delta - (F'(x_0) + \alpha L^s)^{-1}[F(y_{n,\alpha,s}^\delta) - f^\delta + \alpha L^s(y_{n,\alpha,s}^\delta - x_0)]$$

in the setting of Hilbert scales $\{X_r\}_{r\in R}$ generated by a densely defined, linear, unbounded, strictly positive self adjoint operator $L : D(L) \subset X \to X$. Where $x_{0,\alpha,s}^\delta := x_0$, is the initial approximation for the solution $\hat{x}$ of (1.1.1). We selected the regularization parameter α using adaptive method and obtained an optimal order error estimate. The sequence in this chapter converges linearly to the solution $x_{\alpha,s}^\delta$ of the equation $F(x_{\alpha,s}^\delta) + \alpha L^s(x_{\alpha,s}^\delta - x_0) = f^\delta$.

Chapter 6: We conclude the thesis in this chapter, by highlighting the scope for some future works.

■ □ ■

Chapter 2

MODIFIED NEWTON METHOD FOR NONLINEAR LAVRENTIEV REGULARIZATION

In this chapter a Two Step Modified Newton Lavrentiev Method (TSMNLM) is considered for obtaining an approximate solution for the nonlinear ill-posed equation $F(x) = f$ when the available data are f^δ with $\|f - f^\delta\| \leq \delta$ and the operator F is monotone. The derived error estimate under a general source condition on $x_0 - \hat{x}$ is of optimal order, here x_0 is the initial guess and $\hat{x}$ is the actual solution. The regularization parameter is chosen according to the adaptive method considered by Pereverzyev and Schock (2005). We consider also the finite dimensional approximation of the TSMNLM. A numerical example and the corresponding computational results are exhibited to confirm the reliability and effectiveness of our method.

2.1 INTRODUCTION

For monotone operators one usually uses the Lavrentiev regularization method (see, Jaan and Tautenhahn (2003); Pereverzyev and Schock (2005); Semenova (2010)) for solving (1.1.1). In this method the regularized approximation x_α^δ is obtained by solving the operator equation (1.5.10). It is known (cf. Tautenhahn (2002), Theorem 1.1) that the equation (1.5.10) has a unique solution $x_\alpha^\delta \in B_r(\hat{x}) := \{x \in X : \|x - \hat{x}\| < r\} \subset D(F)$ for any $\alpha > 0$ provided $r = ||x_0 - \hat{x}\| + \delta/\alpha$. The optimality of the Lavrentiev method was proved in Tautenhahn (2002) under a general source condition on $x_0 - \hat{x}$. However the main drawback here is that, the regularized equation (1.5.10) remains nonlinear and one may have difficulties in solving them numerically. Thus in the last few years more emphasis was put on the investigation of iterative regularization methods (see, Blaschke et al. (1997); Deuflhard et al. (1998); Jin.Qi-Nian (2000a,b); Mahale and Nair (2003); George (2006); George and Nair (2008)). In this chapter we consider a modified form of the method considered in George and Elmahdy (2012), but we analyze the method as a Two Step Modified Newton Lavrentiev Method (TSMNLM). The proposed analysis is motivated by the Two Step Directional Newton Method (TSDNM) considered in Argyros and Hilout (2010), for approximating a zero x^* of a differentiable function F defined on a convex subset $\mathcal{D}$ of a Hilbert space H with values in $\mathbb{R}$. The TSMNLM for approximating the zero x_α^δ of (1.5.10) is defined by:

$$y_{n,\alpha}^\delta = x_{n,\alpha}^\delta - R_\alpha^{-1}(x_0)[F(x_{n,\alpha}^\delta) - f^\delta + \alpha(x_{n,\alpha}^\delta - x_0)] \tag{2.1.1}$$

and

$$x_{n+1,\alpha}^\delta = y_{n,\alpha}^\delta - R_\alpha^{-1}(x_0)[F(y_{n,\alpha}^\delta) - f^\delta + \alpha(y_{n,\alpha}^\delta - x_0)] \tag{2.1.2}$$

where $x_{0,\alpha}^\delta := x_0$ and $R_\alpha(x_0) = F'(x_0) + \alpha I$. Here the regularization parameter α is chosen from the finite set

$$D_N(\alpha) = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\}. \tag{2.1.3}$$

The plan of this chapter is as follows. In section 2.1, we introduce the Two Step Modified Newton Lavrentiev Method (TSMNLM) and prove that the method converges to a solution of the equation

$$F(x) + \alpha(x - x_0) = f^\delta \tag{2.1.4}$$

in section 2.2. The error analysis under a general source condition is considered in section 2.3, precisely we consider an a priori parameter choice in section 2.3.1 and the balancing principle (adaptive method) considered by Pereverzyev and Schock (2005), in section 2.3.2. Section 2.4 deals with projection method and its convergence. In section 2.5 we consider the error analysis in finite dimensional case. Section 2.6 deals with the implementation of adaptive parameter choice strategy. Finally an example and the computational results are given in section 2.7.

2.2 CONVERGENCE ANALYSIS FOR TSMNLM

We need the following assumptions for the convergence analysis of TSMNLM.

Assumption 2.2.1. *(see, Semenova (2010)) F possesses a locally uniformly bounded Fréchet derivative $F'(.)$ at all x in the domain $D(F)$ i.e., $\|F'(x)\| \leq C_F, \quad x \in D(F)$ for some constant C_F.*

Assumption 2.2.2. *(cf. Semenova (2010), Assumption 3) There exists a constant $k_0 > 0, r > 0$ such that for every $x, u \in B_r(x_0) \cup B_r(\hat{x}) \subset D(F)$ and $v \in X$, there exists an element $\Phi(x, u, v) \in X$ such that*

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \quad \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|.$$

Hereafter we assume that $\delta_0 < \frac{\alpha_0}{4k_0}$, for some $\alpha_0 > 0$ and $\|x_0 - \hat{x}\| \leq \rho$ where

$$\rho \leq \frac{\sqrt{1 + (\frac{1}{2} - \frac{2k_0\delta_0}{\alpha_0})} - 1}{k_0}. \tag{2.2.5}$$

Let

$$e^\delta_{n,\alpha} \quad := \quad \|y^\delta_{n,\alpha} - x^\delta_{n,\alpha}\|, \forall n \geq 0 \tag{2.2.6}$$

and

$$\gamma_\rho \quad := \quad \frac{\delta_0}{\alpha_0} + \frac{k_0}{2}\rho^2 + \rho. \tag{2.2.7}$$

For convenience, we use the notation x_n, y_n and e_n for $x^\delta_{n,\alpha}, y^\delta_{n,\alpha}$ and $e^\delta_{n,\alpha}$ respectively.

Lemma 2.2.1. *Let $\delta \in (0, \delta_0]$, Assumption 2.2.2 hold and γ_ρ be as in (2.2.7). Then $e_0 \leq \gamma_\rho$.*

Proof. Observe that

$$\begin{aligned} e_0 &= \|y_0 - x_0\| \\ &= \|R_\alpha^{-1}(x_0)(F(x_0) - f^\delta)\| \\ &= \|R_\alpha^{-1}(x_0)[F(x_0) - F(\hat{x}) - F'(x_0)(x_0 - \hat{x}) + F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\ &= \|R_\alpha^{-1}(x_0)[\int_0^1 (F'(x_0 + t(\hat{x} - x_0)) - F'(x_0))(x_0 - \hat{x})dt + F'(x_0)(x_0 - \hat{x}) \\ &\quad +F(\hat{x}) - f^\delta]\|. \end{aligned}$$

Now, since $\|R_\alpha^{-1}(x_0)F'(x_0)\| \leq 1$, we have

$$\begin{aligned}
e_0 &\leq \|\int_0^1 \Phi(x_0 + t(\hat{x} - x_0), x_0, x_0 - \hat{x})\|dt + \|x_0 - \hat{x}\| + \|R_\alpha^{-1}(x_0)(F(\hat{x}) - f^\delta)\| \\
&\leq \frac{k_0}{2}\|x_0 - \hat{x}\|^2 + \|x_0 - \hat{x}\| + \frac{1}{\alpha}\|F(\hat{x}) - f^\delta\| \\
&\leq \frac{k_0}{2}\rho^2 + \rho + \frac{\delta}{\alpha} \\
&\leq \frac{k_0}{2}\rho^2 + \rho + \frac{\delta_0}{\alpha_0} \\
&= \gamma_\rho.
\end{aligned}$$

The last step follows from (2.2.5).
Let

$$q = k_0 r. \tag{2.2.8}$$

Then $\frac{\gamma_\rho}{1-q} < r$, if

$$r \in \left(\frac{1 - \sqrt{1 - 4k_0\gamma_\rho}}{2k_0}, \frac{1 + \sqrt{1 - 4k_0\gamma_\rho}}{2k_0}\right). \tag{2.2.9}$$

Theorem 2.2.2. *Let y_n, x_n and e_n be as in (2.1.1), (2.1.2) and (2.2.6) respectively with $\delta \in (0, \delta_0]$ and $\alpha \in D_N(\alpha)$. Let γ_ρ, q and r be as in (2.2.7), (2.2.8) and (2.2.9) respectively. Then*

(a) $\|x_n - y_{n-1}\| \leq q\|y_{n-1} - x_{n-1}\|$;

(b) $\|y_n - x_n\| \leq q^2\|y_{n-1} - x_{n-1}\|$;

(c) $e_n \leq q^{2n}\gamma_\rho$;

(d) $x_n, y_n \in B_r(x_0)$.

Proof. Observe that if $x_n, y_n \in B_r(x_0)$, then by Assumption 2.2.2 we have

$$\begin{aligned}
x_n - y_{n-1} &= y_{n-1} - x_{n-1} - R_\alpha^{-1}(x_0)[F(y_{n-1}) - F(x_{n-1}) + \alpha(y_{n-1} - x_{n-1})] \\
&= R_\alpha^{-1}(x_0)[R_\alpha(x_0)(y_{n-1} - x_{n-1}) - (F(y_{n-1}) - F(x_{n-1})) - \alpha(y_{n-1} - x_{n-1})] \\
&= R_\alpha^{-1}(x_0)\int_0^1 [F'(x_0) - F'(x_{n-1} + t(y_{n-1} - x_{n-1}))](y_{n-1} - x_{n-1})dt \\
&= R_\alpha^{-1}(x_0)F'(x_0)\int_0^1 \Phi(x_0, x_{n-1} + t(y_{n-1} - x_{n-1}), y_{n-1} - x_{n-1})dt
\end{aligned}$$

and hence,

$$\|x_n - y_{n-1}\| \leq k_0 r \|y_{n-1} - x_{n-1}\|. \tag{2.2.10}$$

Again observe that if $x_n, y_n \in B_r(x_0)$, by Assumption 2.2.2 and (2.2.10) we have

$$\begin{aligned}
y_n - x_n &= x_n - y_{n-1} - R_\alpha^{-1}(x_0)[F(x_n) - F(y_{n-1}) + \alpha(x_n - y_{n-1})] \\
&= R_\alpha^{-1}(x_0)[R_\alpha(x_0)(x_n - y_{n-1}) - (F(x_n) - F(y_{n-1})) - \alpha(x_n - y_{n-1})] \\
&= R_\alpha^{-1}(x_0) \int_0^1 [F'(x_0) - F'(y_{n-1} + t(x_n - y_{n-1})](x_n - y_{n-1}) dt \\
&= R_\alpha^{-1}(x_0) F'(x_0) \int_0^1 \Phi(x_0, y_{n-1} + t(x_n - y_{n-1}), x_n - y_{n-1}) dt
\end{aligned}$$

and hence,

$$\|y_n - x_n\| \leq k_0 r \|x_n - y_{n-1}\| \leq q^2 \|y_{n-1} - x_{n-1}\|. \tag{2.2.11}$$

Thus if $x_n, y_n \in B_r(x_0)$ then (a) and (b) follows from (2.2.10) and (2.2.11) respectively. Now using induction we shall prove that $x_n, y_n \in B_r(x_0)$. Note that $x_0, y_0 \in B_r(x_0)$ and hence by (2.2.10)

$$\begin{aligned}
\|x_1 - x_0\| &\leq \|x_1 - y_0\| + \|y_0 - x_0\| \\
&\leq (1+q)e_0 \\
&\leq \frac{e_0}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r
\end{aligned}$$

i.e., $x_1 \in B_r(x_0)$, again by (2.2.11)

$$\begin{aligned}
\|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| \\
&\leq q^2 e_0 + (1+q)e_0 \\
&\leq \frac{e_0}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r
\end{aligned}$$

i.e., $y_1 \in B_r(x_0)$. Suppose $x_k, y_k \in B_r(x_0)$ for some $k > 1$. Then since

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\| \tag{2.2.12}$$

we shall first find an estimate for $\|x_{k+1}-x_k\|$. Note that by (a) and (b) we have

$$\begin{aligned}\|x_{k+1}-x_k\| &\leq \|x_{k+1}-y_k\|+\|y_k-x_k\| \\ &\leq (q+1)\|y_k-x_k\| \\ &\leq (1+q)q^{2k}e_0.\end{aligned}$$

Therefore by (2.2.12) we have

$$\begin{aligned}\|x_{k+1}-x_0\| &\leq (1+q)[q^{2k}+q^{2(k-1)}+\cdots+1]e_0 \\ &\leq (1+q)\left[\frac{1-q^{2k+1}}{1-q^2}\right]e_0 \\ &\leq \frac{e_0}{1-q} \\ &\leq \frac{\gamma_\rho}{1-q} \\ &< r.\end{aligned} \tag{2.2.13}$$

So by induction $x_n\in B_r(x_0)$ for all $n\geq 0$. Again by (a), (b) and (2.2.13) we have

$$\begin{aligned}\|y_{k+1}-x_0\| &\leq \|y_{k+1}-x_{k+1}\|+\|x_{k+1}-x_0\| \\ &\leq q^{2k+2}e_0+(1+q)[q^{2k}+q^{2(k-1)}+\cdots+1]e_0 \\ &\leq (1+q)\left[\frac{1-q^{2k+3}}{1-q^2}\right]e_0 \\ &\leq \frac{e_0}{1-q} \\ &\leq \frac{\gamma_\rho}{1-q} \\ &< r.\end{aligned}$$

Thus $y_{k+1}\in B_r(x_0)$ and hence by induction $y_n\in B_r(x_0)$ for all $n\geq 0$. This completes the proof of the theorem.

The main result of this section is the following theorem.

Theorem 2.2.3. *Let y_n and x_n be as in (2.1.1) and (2.1.2) respectively and assumptions of Theorem 2.2.2 hold. Then (x_n) is Cauchy sequence in $B_r(x_0)$ and converges to $x_\alpha^\delta\in\overline{B_r(x_0)}$. Further $F(x_\alpha^\delta)+\alpha(x_\alpha^\delta-x_0)=f^\delta$ and*

$$\|x_n-x_\alpha^\delta\|\leq\frac{q^{2n}\gamma_\rho}{1-q}.$$

Proof. Using the relation (b) and (c) of Theorem 2.2.2, we obtain

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \sum_{i=0}^{m-1} \|x_{n+i+1} - x_{n+i}\| \\
&\leq \sum_{i=0}^{m-1} (1+q) e_{n+i} \\
&\leq \sum_{i=0}^{m-1} (1+q) q^{2(n+i)} e_0 \\
&\leq (1+q) q^{2n} \left[\frac{1-q^{2m}}{1-q^2}\right] e_0 \\
&\leq \frac{q^{2n}}{1-q} e_0 \\
&\leq \frac{q^{2n}}{1-q} \gamma_\rho.
\end{aligned}$$

Thus x_n is a Cauchy sequence in $B_r(x_0)$ and hence it converges, say to $x_\alpha^\delta \in \overline{B_r(x_0)}$. Observe that

$$\begin{aligned}
\|F(x_n) - f^\delta + \alpha(x_n - x_0)\| &= \|R_\alpha(x_0)(x_n - y_n)\| \\
&\leq \|R_\alpha(x_0)\| \|x_n - y_n\| \\
&\leq (C_F + \alpha) q^{2n} \gamma_\rho. \qquad (2.2.14)
\end{aligned}$$

Now by letting $n \to \infty$ in (2.2.14) we obtain $F(x_\alpha^\delta) - f^\delta + \alpha(x_\alpha^\delta - x_0) = 0$. This completes the proof.

2.3 ERROR BOUNDS UNDER SOURCE CONDITIONS

The objective of this section is to obtain an error estimate for $\|x_n - \hat{x}\|$ under the following assumption on $x_0 - \hat{x}$.

Assumption 2.3.1. *(see, Semenova (2010)) There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(\hat{x})\|$ satisfying $\lim_{\lambda \to 0} \varphi(\lambda) =$*

0 *and* $v \in X$ *with* $\|v\| \le 1$ *such that*

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v$$

and

$$\sup_{\lambda \ge 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \le \varphi(\alpha), \quad \forall \lambda \in (0, a].$$

Remark 2.3.1. *It can be seen that functions*

$$\varphi(\lambda) = \lambda^{\nu}, \lambda > 0$$

for $0 < \nu \le 1$ *and*

$$\varphi(\lambda) = \begin{cases} (ln\frac{1}{\lambda})^{-p} & , 0 < \lambda \le e^{-(p+1)} \\ 0 & , otherwise \end{cases}$$

for $p \ge 0$ *satisfy the Assumption 2.3.1 (see, Nair and Ravishankar (2008)).*

We will be using the error estimates in the following Proposition, which can be found in Tautenhahn (2002) , for our error analysis.

Proposition 2.3.2. *(cf. Tautenhahn (2002), Proposition 3.1) Let* $F : D(F) \subseteq X \to X$ *be a monotone operator in* X *and let* $\hat{x} \in D(F)$ *be a solution of (1.1.1). Let* x_α *be the unique solution of (1.5.10) with* f *in place of* f^δ *and* x_α^δ *be the unique solution of (1.5.10). Then*

$$\|x_\alpha^\delta - x_\alpha\| \le \frac{\delta}{\alpha}$$

and

$$\|x_\alpha - \hat{x}\| \le \|x_0 - \hat{x}\|.$$

Theorem 2.3.3. *(cf. Tautenhahn (2002), Theorem 3.3 or Semenova (2010), Proposition 4.1) Let Assumption 2.2.1, Assumption 2.2.2, Assumption 2.3.1 and assumptions* *in Proposition 2.3.2 be satisfied. Then*

$$\|x_\alpha - \hat{x}\| \le (k_0 r + 1)\varphi(\alpha).$$

Combining the estimates in Proposition 2.3.2, Theorem 2.2.3 and Theorem 2.3.3 we obtain the following;

Theorem 2.3.4. *Let x_n be as in (2.1.2) and let assumptions in Theorem 2.2.3 and Theorem 2.3.3 be satisfied. Then*

$$\|x_n - \hat{x}\| \le \frac{q^{2n}\gamma_\rho}{1-q} + (k_0 r + 1)\left[\varphi(\alpha) + \frac{\delta}{\alpha}\right].$$

Let

$$\bar{C} := \max\left\{\frac{\gamma_\rho}{1-q}, k_0 r\right\} + 1, \tag{2.3.15}$$

and let

$$n_\delta := \min\left\{n : q^{2n} \le \frac{\delta}{\alpha}\right\}. \tag{2.3.16}$$

Theorem 2.3.5. *Let x_n be as in (2.1.2). Let assumptions in Theorem 2.3.4 be satisfied. Let $\bar{C}$ be as in (2.3.15) and n_δ be as in (2.3.16). Then*

$$\|x_{n_\delta} - \hat{x}\| \le \bar{C}\left[\varphi(\alpha) + \frac{\delta}{\alpha}\right]. \tag{2.3.17}$$

2.3.1 A priori choice of the parameter

Note that the error estimate $\varphi(\alpha) + \frac{\delta}{\alpha}$ in (2.3.17) is of optimal order if $\alpha := \alpha_\delta$ satisfies, $\varphi(\alpha_\delta)\alpha_\delta = \delta$.

Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \le a$ we have $\delta = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. In view of the above observations and (2.3.17) we have the following;

Theorem 2.3.6. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \le a$, and let assumptions in Theorem 2.3.5 hold. For $\delta > 0$, let $\alpha := \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Let n_δ be as in (2.3.16), then*

$$\|x_{n_\delta} - \hat{x}\| = O(\psi^{-1}(\delta)).$$

2.3.2 An adaptive choice of the parameter

In this subsection, we present a parameter choice rule based on the adaptive method studied in Pereverzyev and Schock (2005); George and Nair (2008). Let

$$D_N(\alpha) := \{\alpha_i = \mu^i \alpha_0, i = 0, 1, \cdots, N\}$$

where $\mu > 1$, $\alpha_0 > 0$ and let

$$n_i := \min\left\{n : q^{2n} \leq \frac{\delta}{\alpha_i}\right\}.$$

Then for $i = 0, 1, \cdots, N$, we have

$$\|x_{n_i} - x_{\alpha_i}^{\delta}\| \leq \frac{\delta}{\alpha_i}, \quad \forall i = 0, 1, \cdots N.$$

Let $x_i := x_{n_i,\alpha_i}^{\delta}$. We select the regularization parameter $\alpha = \alpha_i$ from the set $D_N(\alpha)$ and operate only with corresponding x_i, $\quad i = 0, 1, \cdots, N$.

The proof of the following theorem is analogous to the proof of the Theorem 4.3 in George and Nair (2008). But for the sake of completeness we provide the proof.

Theorem 2.3.7. *Assume that there exists $i \in \{0, 1, 2, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}$. Let assumptions of Theorem 2.3.5 and Theorem 2.3.6 hold and let*

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}\right\} < N,$$

$$k := \max\left\{i : \|x_i - x_j\| \leq 4\bar{C}\frac{\delta}{\alpha_j}, \quad j = 0, 1, 2, \cdots, i-1\right\}.$$

Then $l \leq k$ and

$$\|\hat{x} - x_k\| \leq c\psi^{-1}(\delta)$$

where $c = 6\bar{C}\mu$.

Proof. To see that $l \leq k$, it is enough to show that, for $i = 1, 2, \cdots, N$,

$$\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i} \Longrightarrow \|x_i - x_j\| \leq 4\bar{C}\frac{\delta}{\alpha_j}, \qquad \forall j = 0, 1, \cdots, i-1.$$

For $j \leq i$, by Theorem 2.3.5 we obtain,

$$\begin{aligned}
\|x_i - x_j\| &\leq \|x_i - \hat{x}\| + \|x_j - \hat{x}\| \\
&\leq \bar{C}\left[\varphi(\alpha_i) + \frac{\delta}{\alpha_i}\right] + \bar{C}\left[\varphi(\alpha_j) + \frac{\delta}{\alpha_j}\right] \\
&\leq \bar{C}\left[2\frac{\delta}{\alpha_i} + 2\frac{\delta}{\alpha_j}\right] \\
&\leq 4\bar{C}\frac{\delta}{\alpha_j}
\end{aligned}$$

so, $$l \leq k.$$

Now, $$\begin{aligned}
\|\hat{x} - x_k\| &\leq \|\hat{x} - x_l\| + \|x_l - x_k\| \\
&\leq \bar{C}\left[2\frac{\delta}{\alpha_l} + 4\frac{\delta}{\alpha_l}\right] \\
&\leq 6\bar{C}\frac{\delta}{\alpha_l} \\
&\leq 6\bar{C}\mu\frac{\delta}{\alpha_\delta} \qquad (\because \alpha_\delta < \alpha_{l+1} < \mu\alpha_l) \\
&\leq c\psi^{-1}(\delta).
\end{aligned}$$

This completes the proof.

2.4 PROJECTION METHOD AND ITS CONVERGENCE

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections on X. Let, $\forall x \in D(F)$, $\varepsilon_h := \|F'(x)(I - P_h)\|$. And $\{b_h : h > 0\}$ be such that $\lim_{h\to 0} \frac{\|(I-P_h)x_0\|}{b_h} = 0$ and $\lim_{h\to 0} b_h = 0$. We assume that $\varepsilon_h \to 0$ as $h \to 0$. The above assumption is satisfied if, $P_h \to I$ pointwise and if $F'(x)$ is a compact operator. Further we assume that $\varepsilon_h \leq \varepsilon_0$, $b_h \leq b_0$ and $\delta \in (0, \delta_0]$.

Analogous to the iterative scheme (2.1.1) and (2.1.2), we define the iterative sequence to obtain an approximate solution for the equation (1.5.10), in the finite dimensional subspace of X as:

$$y_{n,\alpha}^{h,\delta} = x_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(x_{0,\alpha}^{h,\delta})P_h[F(x_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(x_{n,\alpha}^{h,\delta} - x_0)] \tag{2.4.18}$$

and

$$x_{n+1,\alpha}^{h,\delta} = y_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(x_{0,\alpha}^{h,\delta})P_h[F(y_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(y_{n,\alpha}^{h,\delta} - x_0)] \tag{2.4.19}$$

where $x_{0,\alpha}^{h,\delta} := P_h x_0$ is the projection of the initial iterate x_0 on to $R(P_h)$, the range of P_h and $R_\alpha(x) := P_h F'(x) P_h + \alpha P_h$ with $\alpha > \alpha_0 > 0$. Here the regularization parameter α is chosen from the finite set defined in (2.1.3).

Note that, even though the proposed method has local linear convergence, it requires, for its merit, the computation of the Fréchet derivative $F'(.)$ only at $P_h x_0$.

We need the following assumptions for the convergence analysis.

Let

$$e_{n,\alpha}^{h,\delta} := \|y_{n,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}\|, \quad \forall n \geq 0. \tag{2.4.20}$$

Hereafter we assume that $b_0 < \frac{\sqrt{1+(\frac{1}{2(1+\frac{\varepsilon_0}{\alpha_0})^2} - \frac{2k_0\delta_0}{(1+\frac{\varepsilon_0}{\alpha_0})\alpha_0})} - 1}{k_0}$, $\delta_0 < \frac{\alpha_0}{4k_0(1+\frac{\varepsilon_0}{\alpha_0})}$ for some $\alpha_0 > 0$ and $\|x_0 - \hat{x}\| \leq \rho$ where

$$\rho \leq \frac{\sqrt{1 + (\frac{1}{2(1+\frac{\varepsilon_0}{\alpha_0})^2} - \frac{2k_0\delta_0}{(1+\frac{\varepsilon_0}{\alpha_0})\alpha_0})} - 1}{k_0} - b_0.$$

Let

$$\gamma_\rho := (1 + \frac{\varepsilon_0}{\alpha_0}) \left[\frac{k_0}{2}(\rho + b_0)^2 + (\rho + b_0) \right] + \frac{\delta_0}{\alpha_0}. \tag{2.4.21}$$

Lemma 2.4.1. *Let $x \in D(F)$. Then $\|R_\alpha^{-1}(x) P_h F'(x)\| \leq (1 + \frac{\varepsilon_0}{\alpha_0})$.*

Proof. Note that,

$$\begin{aligned}
\|R_\alpha^{-1}(x) P_h F'(x)\| &= \sup_{\|v\| \leq 1} \|(P_h F'(x) P_h + \alpha P_h)^{-1} P_h F'(x) v\| \\
&= \sup_{\|v\| \leq 1} \|(P_h F'(x) P_h + \alpha P_h)^{-1} P_h F'(x)(P_h + I - P_h) v\| \\
&\leq \sup_{\|v\| \leq 1} \|(P_h F'(x) P_h + \alpha P_h)^{-1} P_h F'(x) P_h v\| \\
&\quad + \sup_{\|v\| \leq 1} \|(P_h F'(x) P_h + \alpha P_h)^{-1} P_h F'(x)(I - P_h) v\| \\
&\leq (1 + \frac{\varepsilon_h}{\alpha}) \\
&\leq (1 + \frac{\varepsilon_0}{\alpha_0}).
\end{aligned}$$

Lemma 2.4.2. *Let $e_0 = e_{0,\alpha}^{h,\delta}$ and γ_ρ be as in (2.4.21). Then $e_0 \leq \gamma_\rho$.*

Proof. Observe that,

$$\begin{aligned}
e_0 &= \|y^{h,\delta}_{0,\alpha} - x^{h,\delta}_{0,\alpha}\| \\
&= \|R_\alpha^{-1}(P_hx_0)P_h[F(P_hx_0) - f^\delta]\| \\
&= \|R_\alpha^{-1}(P_hx_0)P_h[F(P_hx_0) - F(\hat{x}) - F'(P_hx_0)(P_hx_0 - \hat{x}) \\
&\quad +F'(P_hx_0)(P_hx_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\
&= \|R_\alpha^{-1}(P_hx_0)P_h[\int_0^1 (F'(\hat{x} + t(P_hx_0 - \hat{x})) - F'(P_hx_0))(P_hx_0 - \hat{x})dt \\
&\quad +F'(P_hx_0)(P_hx_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\
&= \|R_\alpha^{-1}(P_hx_0)P_hF'(P_hx_0)[\int_0^1 \Phi(\hat{x} + t(P_hx_0 - \hat{x}), P_hx_0, P_hx_0 - \hat{x})dt \\
&\quad +(P_hx_0 - \hat{x})] + R_\alpha^{-1}(P_hx_0)P_h(F(\hat{x}) - f^\delta)\|
\end{aligned}$$

and hence by Assumption 2.2.2, Lemma 2.4.1 and the relation $\|R_\alpha^{-1}(P_hx_0)\| \leq \frac{1}{\alpha}$, we have

$$\begin{aligned}
e_0 &\leq (1 + \frac{\varepsilon_0}{\alpha_0})\left[\frac{k_0}{2}\|P_hx_0 - \hat{x}\|^2 + \|P_hx_0 - \hat{x}\|\right] + \frac{\delta}{\alpha} \\
&= (1 + \frac{\varepsilon_0}{\alpha_0})\left[\frac{k_0}{2}\|P_hx_0 - x_0 + x_0 - \hat{x}\|^2 + \|P_hx_0 - x_0 + x_0 - \hat{x}\|\right] + \frac{\delta}{\alpha} \\
&\leq (1 + \frac{\varepsilon_0}{\alpha_0})\left[\frac{k_0}{2}(\rho + b_h)^2 + (\rho + b_h)\right] + \frac{\delta}{\alpha} \\
&\leq (1 + \frac{\varepsilon_0}{\alpha_0})\left[\frac{k_0}{2}(\rho + b_0)^2 + (\rho + b_0)\right] + \frac{\delta_0}{\alpha_0} \\
&= \gamma_\rho.
\end{aligned}$$

Let

$$q = (1 + \frac{\varepsilon_0}{\alpha_0})k_0 r. \tag{2.4.22}$$

Then $\frac{\gamma_\rho}{1-q} < r$, if

$$r \in \left(\frac{1 - \sqrt{1 - 4k_0(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho}}{2k_0(1 + \frac{\varepsilon_0}{\alpha_0})}, \frac{1 + \sqrt{1 - 4k_0(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho}}{2k_0(1 + \frac{\varepsilon_0}{\alpha_0})}\right). \tag{2.4.23}$$

Theorem 2.4.3. *Let $y^{h,\delta}_{n,\alpha}$, $x^{h,\delta}_{n,\alpha}$ and $e^{h,\delta}_{n,\alpha}$ be as in (2.4.18), (2.4.19) and (2.4.20) respectively with $\delta \in (0, \delta_0]$ and $\alpha \in D_N(\alpha)$. Let γ_ρ, q and r be as in (2.4.21), (2.4.22) and (2.4.23) respectively. Then*

(a) $\|x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n-1,\alpha}\| \leq q\|y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}\|$;

(b) $\|y_{n,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}\| \leq q^2 \|y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\|$;

(c) $e_{n,\alpha}^{h,\delta} \leq q^{2n}\gamma_\rho$;

(d) $x_{n,\alpha}^{h,\delta}, y_{n,\alpha}^{h,\delta} \in B_r(P_h x_0)$.

Proof. Observe that if $x_{n,\alpha}^{h,\delta}, y_{n,\alpha}^{h,\delta} \in B_r(P_h x_0)$, then by Assumption 2.2.2 we have

$$\begin{aligned}
x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta} &= y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta} - R_\alpha^{-1}(P_h x_0)P_h[F(y_{n-1,\alpha}^{h,\delta}) - F(x_{n-1,\alpha}^{h,\delta}) \\
&\quad +\alpha(y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta})] \\
&= R_\alpha^{-1}(P_h x_0)[(R_\alpha(P_h x_0) - \alpha P_h)(y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) \\
&\quad -P_h(F(y_{n-1,\alpha}^{h,\delta}) - F(x_{n-1,\alpha}^{h,\delta}))] \\
&= R_\alpha^{-1}(P_h x_0)P_h \int_0^1 [F'(P_h x_0) - F'(x_{n-1,\alpha}^{h,\delta} + t(y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}))] \\
&\quad \times(y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta})dt \\
&= R_\alpha^{-1}(P_h x_0)P_h F'(P_h x_0) \int_0^1 \Phi(P_h x_0, x_{n-1,\alpha}^{h,\delta} + t(y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}), \\
&\quad y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta})dt
\end{aligned}$$

and hence,

$$\|x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta}\| \leq (1 + \frac{\varepsilon_0}{\alpha_0})k_0 r \|y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\|. \tag{2.4.24}$$

Again observe that if $x_{n,\alpha}^{h,\delta}, y_{n,\alpha}^{h,\delta} \in B_r(P_h x_0)$, by Assumption 2.2.2 and (2.4.24) we have

$$\begin{aligned}
y_{n,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta} &= x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta} - R_\alpha^{-1}(P_h x_0)P_h[F(x_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(x_{n,\alpha}^{h,\delta} - P_h x_0)] \\
&= R_\alpha^{-1}(P_h x_0)[(R_\alpha(P_h x_0) - \alpha P_h)(x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta}) \\
&\quad -P_h(F(x_{n,\alpha}^{h,\delta}) - F(y_{n-1,\alpha}^{h,\delta}))] \\
&= R_\alpha^{-1}(P_h x_0)P_h \int_0^1 [F'(P_h x_0) - F'(y_{n-1,\alpha}^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta})] \\
&\quad \times(x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta})dt \\
&= R_\alpha^{-1}(P_h x_0)P_h F'(P_h x_0) \int_0^1 \Phi(P_h x_0, y_{n-1,\alpha}^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta}), \\
&\quad x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta})dt
\end{aligned}$$

and hence,

$$\begin{aligned}
\|y_{n,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}\| &\leq (1 + \frac{\varepsilon_0}{\alpha_0})k_0 r \|x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta}\| \\
&\leq q^2 \|y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\|. \qquad (2.4.25)
\end{aligned}$$

Thus if $x_{n,\alpha}^{h,\delta}, y_{n,\alpha}^{h,\delta} \in B_r(P_h x_0)$ then (a) and (b) follow from (2.4.24) and (2.4.25) respectively. And by Lemma 2.4.2 and (2.4.25), (c) follows. Now using induction we shall prove that $x_{n,\alpha}^{h,\delta}, y_{n,\alpha}^{h,\delta} \in B_r(P_h x_0)$. Note that $x_{0,\alpha}^{h,\delta}, y_{0,\alpha}^{h,\delta} \in B_r(P_h x_0)$ and hence by (2.4.24)

$$\begin{aligned}
\|x_{1,\alpha}^{h,\delta} - P_h x_0\| &\leq \|x_{1,\alpha}^{h,\delta} - y_{0,\alpha}^{h,\delta}\| + \|y_{0,\alpha}^{h,\delta} - x_{0,\alpha}^{h,\delta}\| \\
&\leq (1+q)e_0 \\
&\leq \frac{e_0}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r
\end{aligned}$$

i.e., $x_{1,\alpha}^{h,\delta} \in B_r(P_h x_0)$, again by (2.4.25)

$$\begin{aligned}
\|y_{1,\alpha}^{h,\delta} - P_h x_0\| &\leq \|y_{1,\alpha}^{h,\delta} - x_{1,\alpha}^{h,\delta}\| + \|x_{1,\alpha}^{h,\delta} - P_h x_0\| \\
&\leq q^2 e_0 + (1+q)e_0 \\
&\leq \frac{e_0}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r
\end{aligned}$$

i.e., $y_{1,\alpha}^{h,\delta} \in B_r(P_h x_0)$. Suppose $x_{k,\alpha}^{h,\delta}, y_{k,\alpha}^{h,\delta} \in B_r(P_h x_0)$ for some $k > 1$. Then since

$$\|x_{k+1,\alpha}^{h,\delta} - P_h x_0\| \leq \|x_{k+1,\alpha}^{h,\delta} - x_{k,\alpha}^{h,\delta}\| + \|x_{k,\alpha}^{h,\delta} - x_{k-1,\alpha}^{h,\delta}\| + \cdots + \|x_{1,\alpha}^{h,\delta} - P_h x_0\| \quad (2.4.26)$$

we shall first find an estimate for $\|x_{k+1,\alpha}^{h,\delta} - x_{k,\alpha}^{h,\delta}\|$. Note that by (a) and (b) we have

$$\begin{aligned}
\|x_{k+1,\alpha}^{h,\delta} - x_{k,\alpha}^{h,\delta}\| &\leq \|x_{k+1,\alpha}^{h,\delta} - y_{k,\alpha}^{h,\delta}\| + \|y_{k,\alpha}^{h,\delta} - x_{k,\alpha}^{h,\delta}\| \\
&\leq (q+1)\|y_{k,\alpha}^{h,\delta} - x_{k,\alpha}^{h,\delta}\| \\
&\leq (1+q)q^{2k} e_0.
\end{aligned}$$

Therefore by (2.4.26) we have

$$\begin{aligned}
\|x_{k+1,\alpha}^{h,\delta} - P_h x_0\| &\leq (1+q)[q^{2k} + q^{2(k-1)} + \cdots + 1]e_0 \qquad (2.4.27) \\
&\leq (1+q)\left[\frac{1-q^{2k+1}}{1-q^2}\right] e_0 \\
&\leq \frac{e_0}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r.
\end{aligned}$$

So by induction $x_{n,\alpha}^{h,\delta} \in B_r(P_h x_0)$ for all $n \geq 0$. Again by (a), (b) and (2.4.27) we have

$$\begin{aligned}
\|y_{k+1,\alpha}^{h,\delta} - P_h x_0\| &\leq \|y_{k+1,\alpha}^{h,\delta} - x_{k+1,\alpha}^{h,\delta}\| + \|x_{k+1,\alpha}^{h,\delta} - P_h x_0\| \\
&\leq q^{2k+2} e_0 + (1+q)[q^{2k} + q^{2(k-1)} + \cdots + 1] e_0 \\
&\leq (1+q)\left[\frac{1-q^{2k+3}}{1-q^2}\right] e_0 \\
&\leq \frac{e_0}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r.
\end{aligned}$$

Thus $y_{k+1,\alpha}^{h,\delta} \in B_r(P_h x_0)$ and hence by induction $y_{n,\alpha}^{h,\delta} \in B_r(P_h x_0)$ for all $n \geq 0$. This completes the proof of the theorem.

The main result of this section is the following theorem.

Theorem 2.4.4. *Let $y_{n,\alpha}^{h,\delta}$ and $x_{n,\alpha}^{h,\delta}$ be as in (2.4.18) and (2.4.19) respectively and assumptions of Theorem 2.4.3 hold. Then $(x_{n,\alpha}^{h,\delta})$ is Cauchy sequence in $B_r(P_h x_0)$ and converges to $x_\alpha^{h,\delta} \in \overline{B_r(P_h x_0)}$. Further $P_h[F(x_\alpha^{h,\delta}) + \alpha(x_\alpha^{h,\delta} - x_0)] = P_h f^\delta$ and*

$$\|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \leq C q^{2n}$$

where $C = \frac{\gamma_\rho}{(1-q)}$.

Proof. Using the relation (b) and (c) of Theorem 2.4.3, we obtain

$$\begin{aligned}
\|x_{n+m,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}\| &\leq \sum_{i=0}^{m-1} \|x_{n+i+1,\alpha}^{h,\delta} - x_{n+i,\alpha}^{h,\delta}\| \\
&\leq \sum_{i=0}^{m-1} (1+q) e_{n+i,\alpha}^{h,\delta} \\
&\leq \sum_{i=0}^{m-1} (1+q) q^{2(n+i)} e_0 \\
&\leq (1+q) q^{2n} \left[\frac{1-q^{2m}}{1-q^2}\right] \gamma_\rho \\
&\leq C q^{2n}.
\end{aligned}$$

Thus $(x_{n,\alpha}^{h,\delta})$ is a Cauchy sequence in $B_r(P_h x_0)$ and hence it converges, say to $x_\alpha^{h,\delta} \in$

$\overline{B_r(P_h x_0)}$. Observe that,

$$\begin{aligned}
\|P_h[F(x_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(x_{n,\alpha}^{h,\delta} - x_0)]\| &= \|R_\alpha(x_{0,\alpha}^{h,\delta})(x_{n,\alpha}^{h,\delta} - y_{n,\alpha}^{h,\delta})\| \\
&\leq \|R_\alpha(x_{0,\alpha}^{h,\delta})\|\|x_{n,\alpha}^{h,\delta} - y_{n,\alpha}^{h,\delta}\| \\
&= \|(P_h F'(x_{0,\alpha}^{h,\delta})P_h + \alpha P_h)\|e_{n,\alpha}^{h,\delta} \\
&\leq (C_F + \alpha)q^{2n}\gamma_\rho. \qquad (2.4.28)
\end{aligned}$$

Now by letting $n \to \infty$ in (2.4.28) we obtain

$$P_h[F(x_\alpha^{h,\delta}) + \alpha(x_\alpha^{h,\delta} - x_0)] = P_h f^\delta. \qquad (2.4.29)$$

This completes the proof.

2.5 ERROR BOUNDS UNDER SOURCE CONDITIONS FOR PROJECTION METHOD

The objective of this section is to obtain an error estimate for $\|x_{n,\alpha}^{h,\delta} - \hat{x}\|$ under the Assumption 2.2.1 and Assumption 2.2.2.

Proposition 2.5.1. *Let $F : D(F) \subseteq X \to X$ be a monotone operator in X. Let $x_\alpha^{h,\delta}$ be the solution of (2.4.29) and $x_\alpha^h := x_\alpha^{h,0}$. Then*

$$\|x_\alpha^{h,\delta} - x_\alpha^h\| \leq \frac{\delta}{\alpha}.$$

Proof. The result follows from the monotonicity of F and the relation;

$$P_h[F(x_\alpha^{h,\delta}) - F(x_\alpha^h) + \alpha(x_\alpha^{h,\delta} - x_\alpha^h)] \;=\; P_h(f^\delta - f).$$

Theorem 2.5.2. *Let $\rho < \frac{2}{k_0(1+\frac{\varepsilon_0}{\alpha_0})}$ and $\hat{x} \in D(F)$ be a solution of (1.1.1). And let Assumption 2.2.1, Assumption 2.2.2 and the assumptions in Proposition 2.5.1 be satisfied. Then*

$$\|x_\alpha^h - \hat{x}\| \leq \tilde{C}\left[\varphi(\alpha) + \frac{\varepsilon_h}{\alpha}\right]$$

where $\tilde{C} := \frac{\max\{1, \rho + \|\hat{x}\|\}}{1 - (1 + \frac{\varepsilon_0}{\alpha_0})\frac{k_0}{2}\rho}$.

Proof. Let $M := \int_0^1 F'(\hat{x}+t(x_\alpha^h-\hat{x}))dt$. Then from the relation

$$P_h[F(x_\alpha^h)-F(\hat{x})+\alpha(x_\alpha^h-x_0)]=0$$

we have

$$(P_hMP_h+\alpha P_h)(x_\alpha^h-\hat{x}) = P_h\alpha(x_0-\hat{x})+P_hM(I-P_h)\hat{x}.$$

Hence,

$$\begin{aligned} x_\alpha^h-\hat{x} &= [(P_hMP_h+\alpha P_h)^{-1}P_h-(F'(\hat{x})+\alpha I)^{-1}]\alpha(x_0-\hat{x}) \\ &\quad +(F'(\hat{x})+\alpha I)^{-1}\alpha(x_0-\hat{x})+(P_hMP_h+\alpha P_h)^{-1}P_hM(I-P_h)\hat{x} \\ &= (P_hMP_h+\alpha P_h)^{-1}P_h[F'(\hat{x})-M+M(I-P_h)] \\ &\quad \times(F'(\hat{x})+\alpha I)^{-1}\alpha(x_0-\hat{x})+(F'(\hat{x})+\alpha I)^{-1}\alpha(x_0-\hat{x}) \\ &\quad +(P_hMP_h+\alpha P_h)^{-1}P_hM(I-P_h)\hat{x} \\ &= \zeta_1+\zeta_2 \end{aligned} \tag{2.5.30}$$

where $\zeta_1 := (P_hMP_h+\alpha P_h)^{-1}P_h[F'(\hat{x})-M+M(I-P_h)](F'(\hat{x})+\alpha I)^{-1}\alpha(x_0-\hat{x})$ and $\zeta_2 := (F'(\hat{x})+\alpha I)^{-1}\alpha(x_0-\hat{x})+(P_hMP_h+\alpha P_h)^{-1}P_hM(I-P_h)\hat{x}$.
Observe that,

$$\begin{aligned} \|\zeta_1\| &= \|(P_hMP_h+\alpha P_h)^{-1}P_h\int_0^1[F'(\hat{x})-F'(\hat{x}+t(x_\alpha^h-\hat{x}))]dt \\ &\quad \times(F'(\hat{x})+\alpha I)^{-1}\alpha(x_0-\hat{x})\| \\ &\quad +\|(P_hMP_h+\alpha P_h)^{-1}P_hM(I-P_h)(F'(\hat{x})+\alpha I)^{-1}\alpha(x_0-\hat{x})\| \\ &\le \|(P_hMP_h+\alpha P_h)^{-1}P_h\int_0^1[F'(\hat{x}+t(x_\alpha^h-\hat{x}))(P_h+I-P_h) \\ &\quad \phi(\hat{x},\hat{x}+t(x_\alpha^h-\hat{x}),(F'(\hat{x})+\alpha I)^{-1}\alpha(x_0-\hat{x}))]dt\|+\frac{\varepsilon_h}{\alpha}\rho \\ &\le (1+\frac{\varepsilon_h}{\alpha})\frac{k_0}{2}\rho\|x_\alpha^h-\hat{x}\|+\frac{\varepsilon_h}{\alpha}\rho \\ &\le (1+\frac{\varepsilon_0}{\alpha_0})\frac{k_0}{2}\rho\|x_\alpha^h-\hat{x}\|+\frac{\varepsilon_h}{\alpha}\rho \end{aligned} \tag{2.5.31}$$

and

$$\|\zeta_2\| \le \phi(\alpha)+\frac{\varepsilon_h}{\alpha}\|\hat{x}\|. \tag{2.5.32}$$

The result now follows from (2.5.30), (2.5.31) and (2.5.32).

Theorem 2.5.3. *Let $x_{n,\alpha}^{h,\delta}$ be as in (2.4.19). And the assumptions in Theorem 2.4.4 and Theorem 2.5.2 hold. Then*

$$\|x_{n,\alpha}^{h,\delta}-\hat{x}\| \le Cq^{2n}+\max\{1,\tilde{C}\}\left[\varphi(\alpha)+\frac{\delta+\varepsilon_h}{\alpha}\right].$$

Proof. Observe that,

$$\|x_{n,\alpha}^{h,\delta} - \hat{x}\| \leq \|x_{n,\alpha}^{h,\delta} - x_{\alpha}^{h,\delta}\| + \|x_{\alpha}^{h,\delta} - x_{\alpha}^{h}\| + \|x_{\alpha}^{h} - \hat{x}\|$$

so, by Proposition 2.5.1, Theorem 2.4.4 and Theorem 2.5.2 we obtain,

$$\begin{aligned} \|x_{n,\alpha}^{h,\delta} - \hat{x}\| &\leq Cq^{2n} + \frac{\delta}{\alpha} + \tilde{C}\left[\varphi(\alpha) + \frac{\varepsilon_h}{\alpha}\right] \\ &\leq Cq^{2n} + \max\{1, \tilde{C}\}\left[\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}\right]. \end{aligned}$$

Let

$$n_\delta := \min\left\{n : q^{2n} \leq \frac{\delta + \varepsilon_h}{\alpha}\right\} \tag{2.5.33}$$

and

$$C_0 = C + \max\{1, \tilde{C}\}. \tag{2.5.34}$$

Theorem 2.5.4. *Let $x_{n_\delta,\alpha}^{h,\delta}$ be as in (2.4.19) and the assumptions in Theorem 2.5.3 be satisfied. And let n_δ and C_0 be as in (2.5.33) and (2.5.34) respectively. Then*

$$\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| \leq C_0\left[\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}\right]. \tag{2.5.35}$$

2.5.1 A priori choice of the parameter

Note that the error estimate $\varphi(\alpha) + \frac{\delta+\varepsilon_h}{\alpha}$ in (2.5.35) is of optimal order if $\alpha_\delta := \alpha(\delta, h)$ satisfies, $\varphi(\alpha_\delta)\alpha_\delta = \delta + \varepsilon_h$.

Now as in section 2.3.1, using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \leq a$ we have $\delta + \varepsilon_h = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h))$.

Theorem 2.5.5. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$ and the assumptions in Theorem 2.5.4 hold. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h))$ and let n_δ be as in (2.5.33). Then*

$$\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

2.5.2 An adaptive choice of the parameter

In this method, the regularization parameter α is selected from some finite set

$$D_N(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0, 1, \cdots, N\}$$

where $\mu > 1$, $\alpha_0 > 0$ and let

$$n_i := \min\left\{n : q^{2n} \leq \frac{\delta + \varepsilon_h}{\alpha_i}\right\}.$$

Then for $i = 0, 1, \cdots, N$, we have

$$\|x^{h,\delta}_{n_i,\alpha_i} - x^{h,\delta}_{\alpha_i}\| \leq C\frac{\delta + \varepsilon_h}{\alpha_i}, \quad \forall i = 0, 1, \cdots N.$$

Let $x_i := x^{h,\delta}_{n_i,\alpha_i}, \quad i = 0, 1, \cdots, N$. Proof of the following theorem is analogous to the proof of the Theorem 2.3.7.

Theorem 2.5.6. *Assume that* $\exists\, i \in \{0, 1, 2, \cdots, N\}$ *such that* $\varphi(\alpha_i) \leq \frac{\delta+\varepsilon_h}{\alpha_i}$. *Let the assumptions of Theorem 2.5.4 and Theorem 2.5.5 hold and let*

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta + \varepsilon_h}{\alpha_i}\right\} < N,$$

$$k := \max\{i : \|x_i - x_j\| \leq 4C_0\frac{\delta + \varepsilon_h}{\alpha_j}, \quad j = 0, 1, 2, \cdots, i\}.$$

Then $l \leq k$ *and*

$$\|\hat{x} - x_k\| \leq c\psi^{-1}(\delta + \varepsilon_h)$$

where $c = 6C_0\mu$.

2.6 IMPLEMENTATION OF ADAPTIVE CHOICE RULE

Following steps are involved in implementing the adaptive choice rule:

- Choose $\alpha_0 > 0$ such that $\delta_0 < \frac{\alpha_0}{4k_0(1+\frac{\varepsilon_0}{\alpha_0})}$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i\alpha_0, \ i = 0, 1, 2, \cdots, N$.

Finally the adaptive algorithm associated with the choice of the parameter specified in Theorem 2.5.6 involves the following steps:

2.6.1 Algorithm

1. Set $i = 0$.
2. Choose $n_i = \min\left\{n : q^{2n} \leq \frac{\delta+\varepsilon_h}{\alpha_i}\right\}$.
3. Solve $x_i := x^{h,\delta}_{n_i,\alpha_i}$ by using the iteration (2.4.18) and (2.4.19).
4. If $\|x_i - x_j\| > 4C_0\frac{\delta+\varepsilon_h}{\alpha_j}, j < i$, then take $k = i-1$ and return x_k.
5. Else set $i = i+1$ and go to Step 2.

2.7 NUMERICAL EXAMPLE

We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with dimension of $V_n = n+1$. Precisely we choose V_n as the linear span of $\{v_1, v_2, \cdots, v_{n+1}\}$ where $v_i, i = 1, 2, \cdots, n+1$ are the linear splines in a uniform grid of $n+1$ points in $[0,1]$. Note that $x^{h,\delta}_{n,\alpha}$, $y^{h,\delta}_{n,\alpha} \in V_n$. So $y^{h,\delta}_{n,\alpha} = \sum_{i=1}^{n+1} \xi_i^n v_i$ and $x^{h,\delta}_{n,\alpha} = \sum_{i=1}^{n+1} \eta_i^n v_i$, where ξ_i^n and $\eta_i^n, i = 1, 2, \cdots, n+1$ are some scalars. Then from (2.4.18) we have

$$(P_hF'(x^{h,\delta}_{0,\alpha})P_h + \alpha P_h)(y^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n,\alpha}) = P_h[f^\delta - F(x^{h,\delta}_{n,\alpha}) + \alpha(x^{h,\delta}_{0,\alpha} - x^{h,\delta}_{n,\alpha})]. \quad (2.7.36)$$

Observe that $(y^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n,\alpha})$ is a solution of (2.7.36) if and only if

$$(\overline{\xi^n - \eta^n}) = (\xi_1^n - \eta_1^n, \xi_2^n - \eta_2^n, \cdots, \xi_{n+1}^n - \eta_{n+1}^n)^T$$

is the unique solution of

$$(Q_n + \alpha B_n)(\overline{\xi^n - \eta^n}) = B_n[\bar{\mu^n} - F_{h1} + \alpha(X_0 - \bar{\eta^n})] \quad (2.7.37)$$

where

$$\begin{aligned}
Q_n &= [\langle F'(x^{h,\delta}_{0,\alpha})v_i, v_j\rangle], i, j = 1, 2\cdots, n+1\\
B_n &= [\langle v_i, v_j\rangle], i, j = 1, 2\cdots, n+1\\
\bar{\mu^n} &= [f^\delta(t_1), f^\delta(t_2), \cdots, f^\delta(t_{n+1})]^T\\
F_{h1} &= [F(x^{h,\delta}_{n,\alpha})(t_1), F(x^{h,\delta}_{n,\alpha})(t_2), \cdots, F(x^{h,\delta}_{n,\alpha})(t_{n+1})]^T\\
X_0 &= [x_0(t_1), x_0(t_2), \cdots, x_0(t_{n+1})]^T
\end{aligned}$$

and $t_1, t_2, \cdots, t_{n+1}$ are the grid points. Further from (2.4.19) it follows that

$$(P_h F'(x_{0,\alpha}^{h,\delta})P_h + \alpha P_h)(x_{n+1,\alpha}^{h,\delta} - y_{n,\alpha}^{h,\delta}) = P_h[f^\delta - F(y_{n,\alpha}^{h,\delta}) + \alpha(x_{0,\alpha}^{h,\delta} - y_{n,\alpha}^{h,\delta})] \quad (2.7.38)$$

and hence $(x_{n+1,\alpha}^{h,\delta} - y_{n,\alpha}^{h,\delta})$ is a solution of (2.7.38) if and only if

$$(\overline{\eta^{n+1} - \xi^n}) = (\eta_1^{n+1} - \xi_1^n, \eta_2^{n+1} - \xi_2^n, \cdots, \eta_{n+1}^{n+1} - \xi_{n+1}^n)^T$$

is the unique solution of

$$(Q_n + \alpha B_n)(\overline{\eta^{n+1} - \xi^n}) = B_n[\bar{\mu^n} - F_{h2} + \alpha(X_0 - \bar{\xi^n})] \quad (2.7.39)$$

where $F_{h2} = [F(y_{n,\alpha}^{h,\delta})(t_1), F(y_{n,\alpha}^{h,\delta})(t_2), \cdots, F(y_{n,\alpha}^{h,\delta})(t_{n+1})]^T$. Note that (2.7.37) and (2.7.39) are uniquely solvable as Q_n is positive definite matrix (i.e., $xQ_nx^T > 0$ for all non-zero vector x) and B_n is an invertible matrix.

Example 2.7.1. *(see Semenova (2010), section 4.3) Let $F : D(F) \subseteq H^1(0,1) \longrightarrow L^2(0,1)$ defined by*

$$F(u) := \int_0^1 k(t,s)u(s)^3 ds,$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1 \\ (1-s)t, & 0 \le t \le s \le 1 \end{cases}.$$

Then for all $x(t), y(t) : x(t) > y(t)$:

$$\langle F(x) - F(y), x - y\rangle = \int_0^1 \left[\int_0^1 k(t,s)(x(s)^3 - y(s)^3)ds\right](x-y)(t)dt \ge 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)u(s)^2 w(s)ds. \quad (2.7.40)$$

Note that for $u, v > 0$,

$$[F'(v) - F'(u)]w = 3\int_0^1 k(t,s)[v(s)^2 - u(s)^2]w(s)ds$$
$$:= F'(u)\Phi(v,u,w)$$
$$\text{where} \quad \Phi(v,u,w) = \left[\frac{v(s)^2}{u(s)^2} - 1\right] w(s).$$

So $\Phi(v,u,w)$ *satisfies Assumption 2.2.2 (see, Scherzer et al. (1993), Example 2.7).*

In our computation, we take $f(t) = \frac{t-t^{11}}{110}$ *and* $f^\delta = f + \delta$. *Then the exact solution*

$$\hat{x}(t) = t^3.$$

We use

$$x_0(t) = t^3 + \frac{3}{56}(t - t^8)$$

as our initial guess, so that the function $x_0 - \hat{x}$ *satisfies the source condition*

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))1$$

where $\varphi(\lambda) = \lambda$.

Observe that while performing numerical computation on finite dimensional subspace (V_n) *of* X, *one has to consider the operator* $P_nF'(.)P_n$ *instead of* $F'(.)$, *where* P_n *is the orthogonal projection on to* V_n. *Thus incurs an additional error* $\|P_nF'(.)P_n - F'(.)\| = O(\|F'(.)(I - P_n)\|)$.

Let $\|F'(.)(I-P_n)\| \le \varepsilon_n$. *For the operator* $F'(.)$ *defined in (2.7.40),* $\varepsilon_n = O(n^{-2})$ *(cf. Groetsch et al. (1982)). Thus we expect to obtain the rate of convergence* $O((\delta + \varepsilon_n)^{\frac{1}{2}})$.

We choose $\alpha_0 = (1.5)\delta$, $\mu = 1.5$ *and* $q = 0.51$. *The results of the computation (four decimal places) are presented in Table 2.1. The plots of the exact solution and the* *approximate solution* $x_{n,\alpha}^{h,\delta}$ *obtained for* $n = 2^i$, $i = 3, \cdots, 10$ *are given in Figures 2.1 through 2.8.*

n	k	n_k	$\delta + \varepsilon_n$	α_k	$\|x_k - \hat{x}\|$	$\frac{\|x_k - \hat{x}\|}{(\delta+\varepsilon_n)^{1/2}}$
8	2	3	0.1016	0.3428	0.2634	0.8266
16	2	3	0.1004	0.3388	0.1962	0.6191
32	2	3	0.1001	0.3378	0.1429	0.4518
64	2	3	0.1000	0.3376	0.1036	0.3275
128	2	3	0.1000	0.3375	0.0755	0.2387
256	2	3	0.1000	0.3375	0.0560	0.1772
512	2	3	0.1000	0.3375	0.0430	0.1360
1024	2	3	0.1000	0.3375	0.0347	0.1096

Table 2.1: Iterations and corresponding error estimates

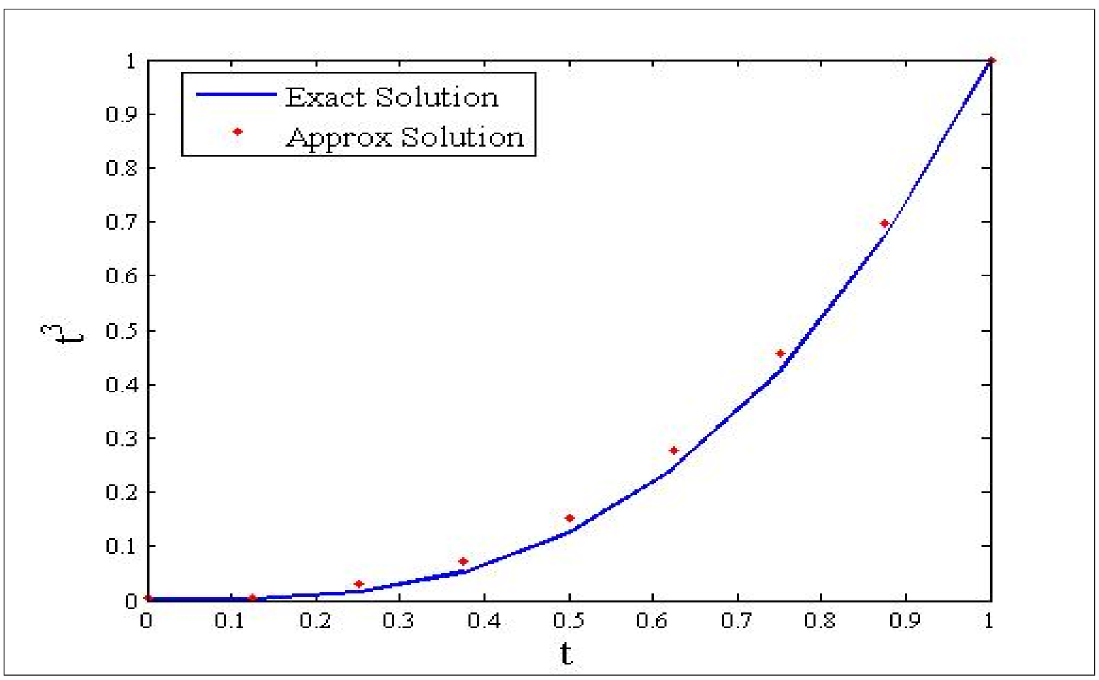

Figure 2.1: Curves of the exact and approximate solutions when n=8

The last column of the Table 2.1 shows that the error $\|x_k - \hat{x}\|$ is of $O((\delta + \varepsilon_n)^{\frac{1}{2}})$.

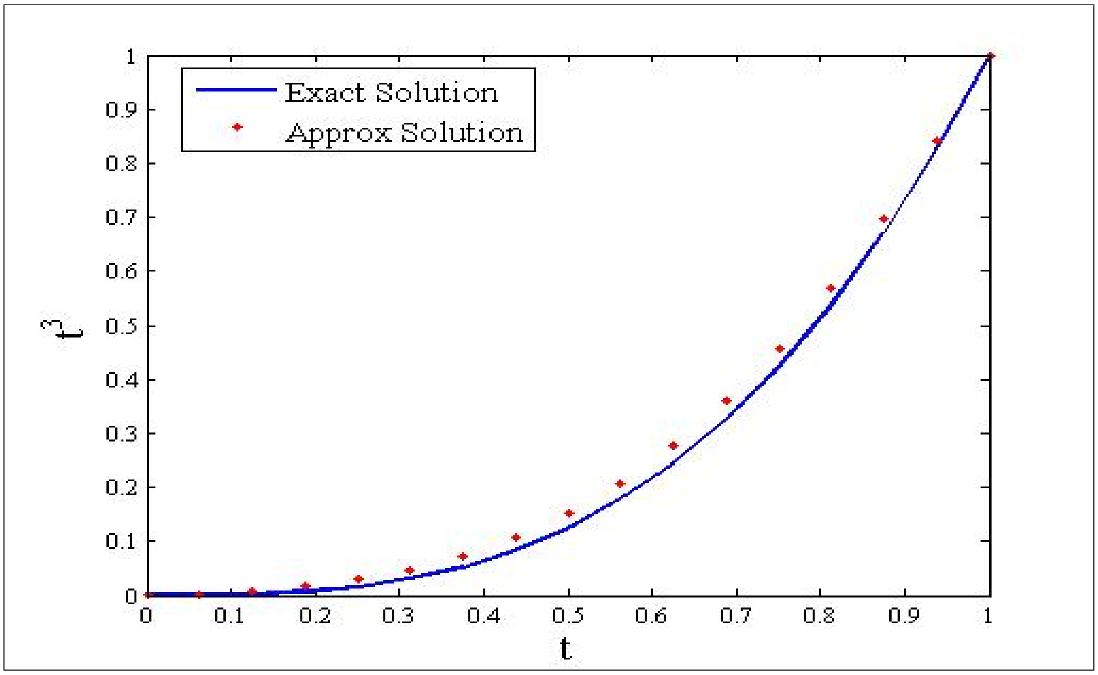

Figure 2.2: Curves of the exact and approximate solutions when n=16

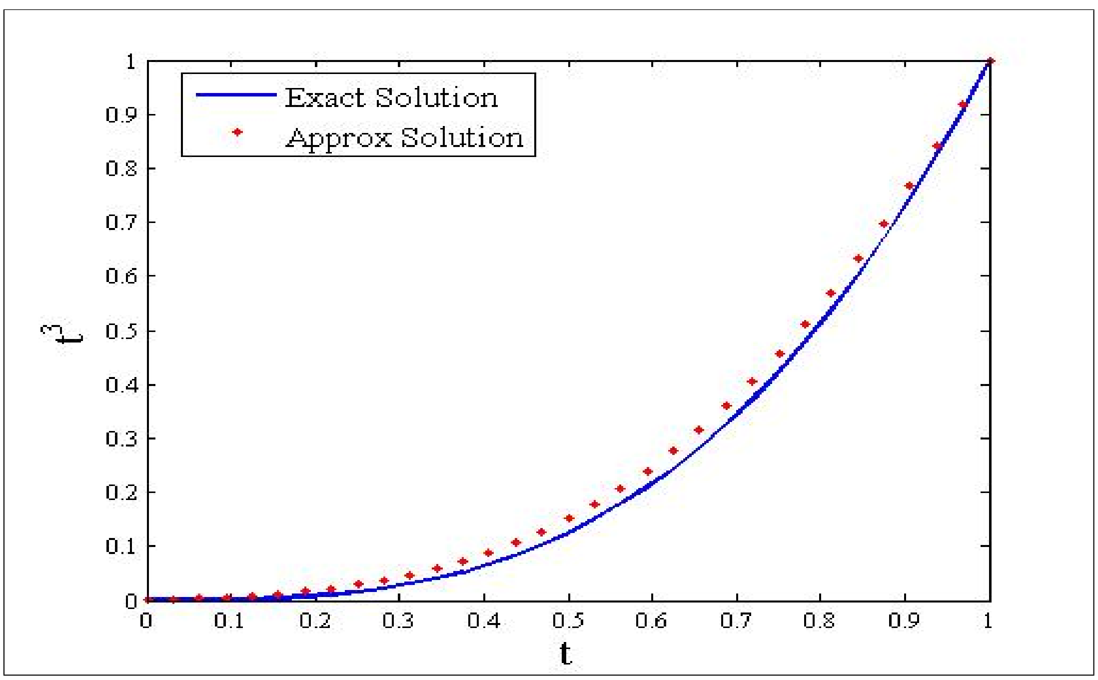

Figure 2.3: Curves of the exact and approximate solutions when n=32

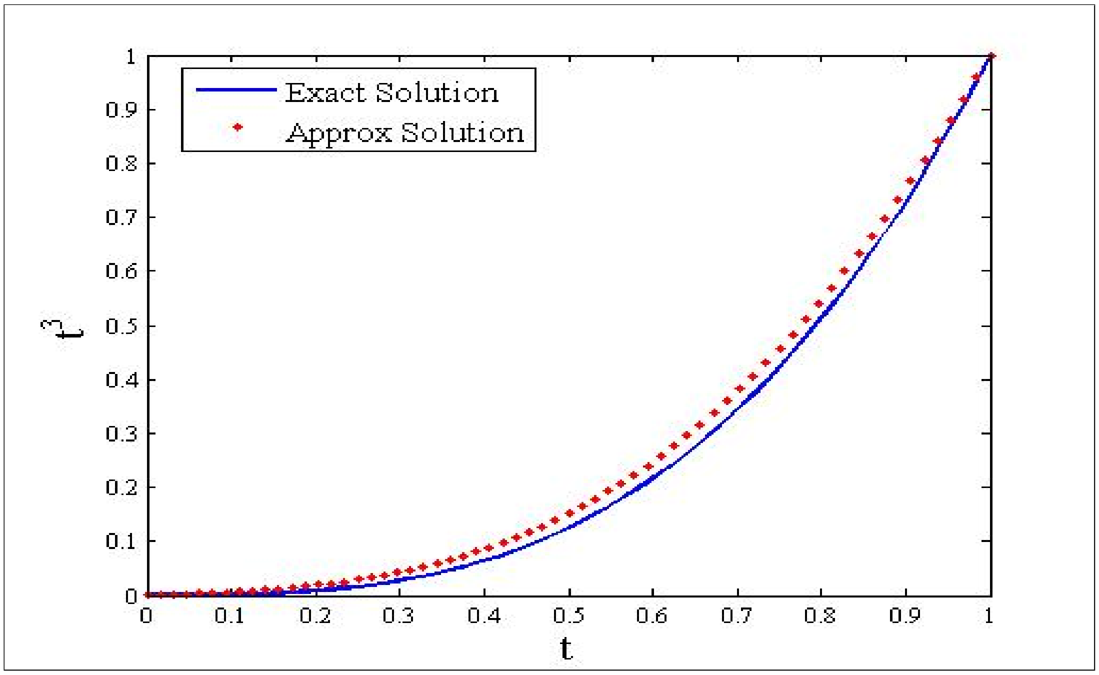

Figure 2.4: Curves of the exact and approximate solutions when n=64

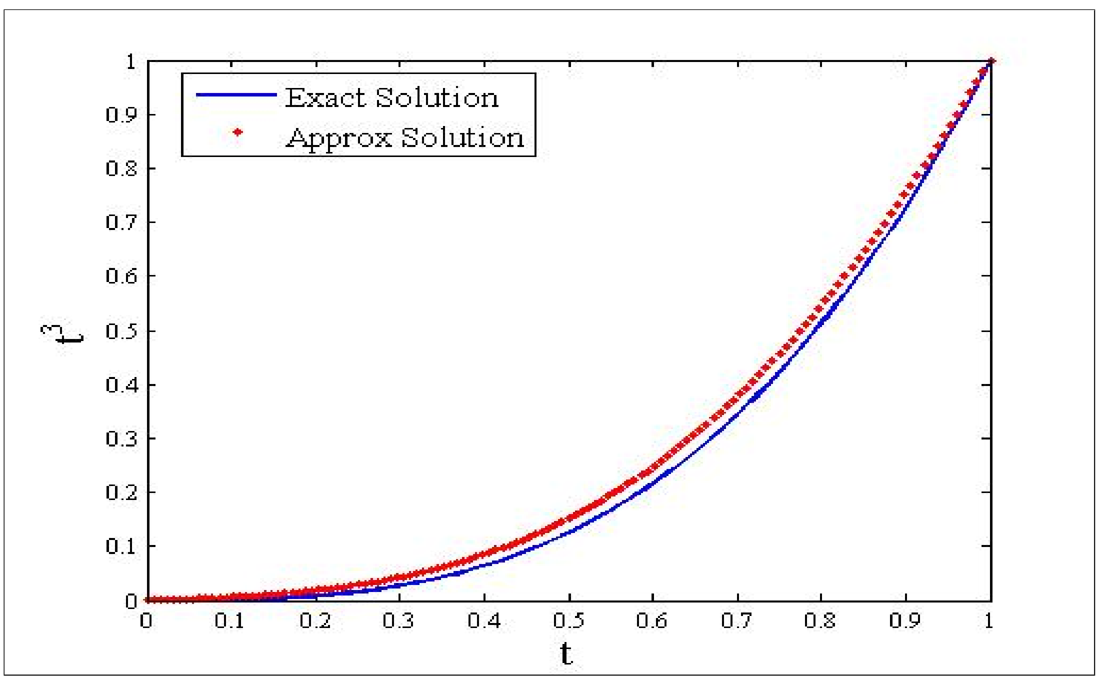

Figure 2.5: Curves of the exact and approximate solutions when n=128

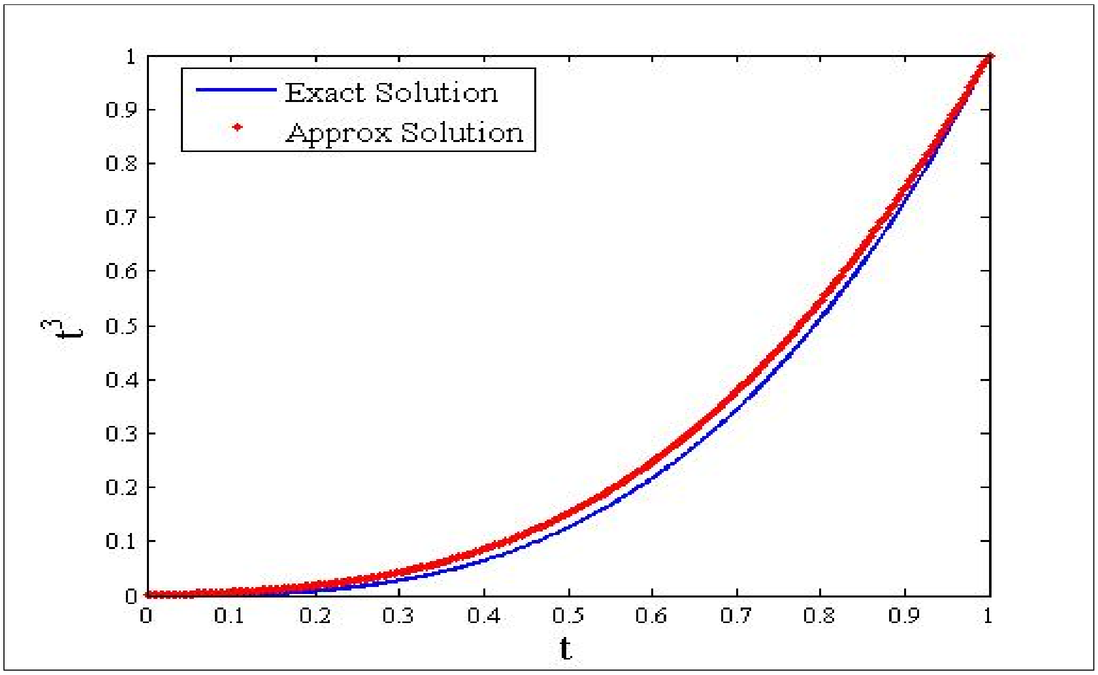

Figure 2.6: Curves of the exact and approximate solutions when n=256

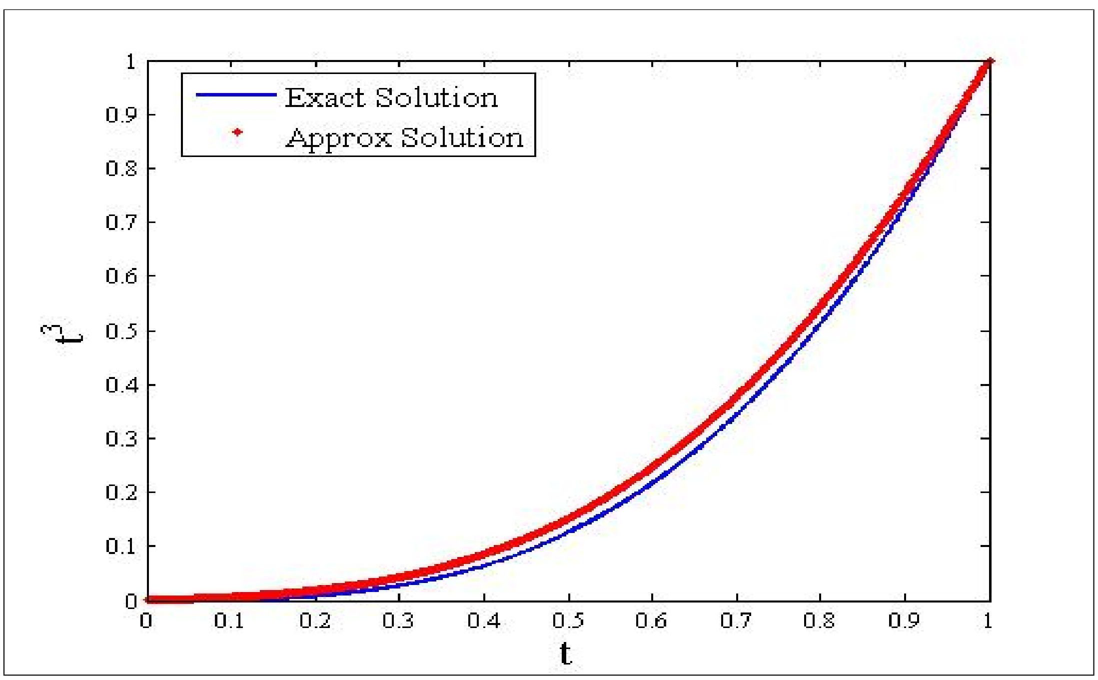

Figure 2.7: Curves of the exact and approximate solutions when n=512

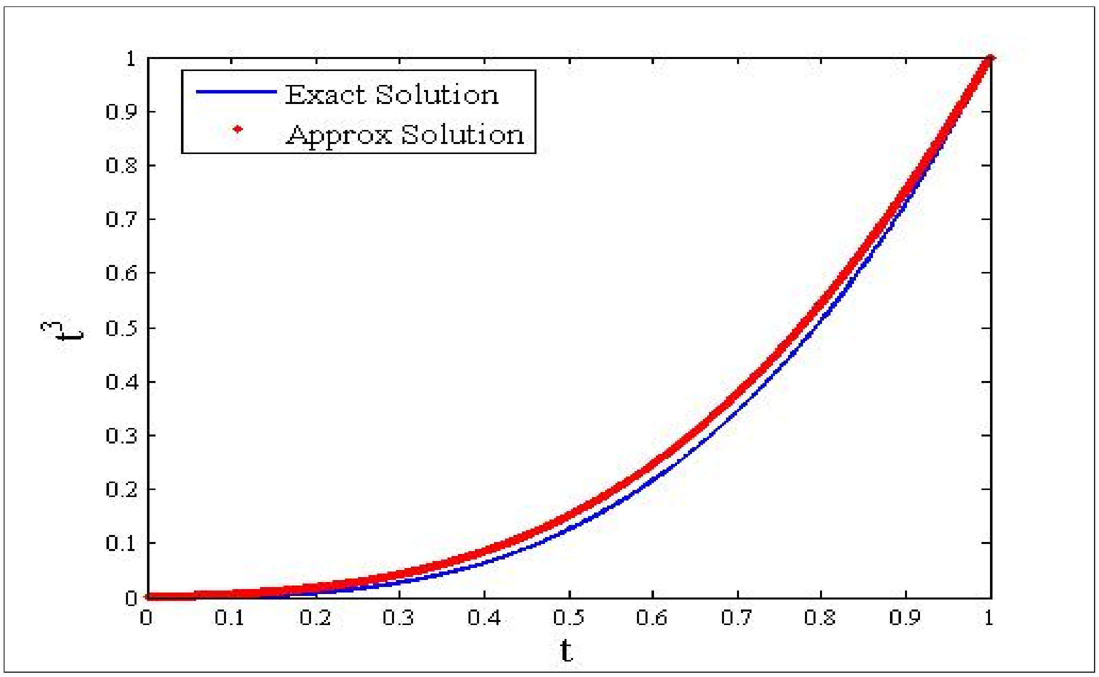

Figure 2.8: Curves of the exact and approximate solutions when n=1024

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Chapter 3

TWO STEP NEWTON TYPE ITERATIVE METHOD FOR THE APPROXIMATE IMPLEMENTATION OF LAVRENTIEV REGULARIZATION

In this chapter we present a semilocal convergence analysis of two step Newton Lavrentiev method for solving ill-posed operator equations in a Hilbert space setting. Using a two-step analysis we obtained local cubic convergence. We provide also the finite dimensional realization of the method considered. The test example provided endorses the reliability and effectiveness of our method.

3.1 INTRODUCTION

In this chapter, we consider a Cubic convergence yielding Newton Lavrentiev Method (CNLM) for approximately solving the nonlinear ill-posed operator equation (1.1.1). Instead of (2.1.1) and (2.1.2) we consider the following (CNLM):

$$\tilde{y}^{\delta}_{n,\alpha} = \tilde{x}^{\delta}_{n,\alpha} - R^{-1}_{\alpha}(\tilde{x}^{\delta}_{n,\alpha})[F(\tilde{x}^{\delta}_{n,\alpha}) - f^{\delta} + \alpha(\tilde{x}^{\delta}_{n,\alpha} - x_0)] \tag{3.1.1}$$

and

$$\tilde{x}^{\delta}_{n+1,\alpha} = \tilde{y}^{\delta}_{n,\alpha} - R^{-1}_{\alpha}(\tilde{x}^{\delta}_{n,\alpha})[F(\tilde{y}^{\delta}_{n,\alpha}) - f^{\delta} + \alpha(\tilde{y}^{\delta}_{n,\alpha} - x_0)] \tag{3.1.2}$$

where

$$R_\alpha(x) = F'(x) + \alpha I, \tag{3.1.3}$$

$\tilde{x}^\delta_{0,\alpha} := x_0$ and the regularization parameter α is chosen from (2.1.3) for approximating (2.1.4).

Note that with the above notation

$$\|R_\alpha^{-1}(x)F'(x)\| \leq 1. \tag{3.1.4}$$

The plan of this chapter is as follows. In section 3.1, we introduce (CNLM) and prove that the method converges cubically to a solution of the equation $F(x)+\alpha(x-x_0) = f^\delta$ in section 3.2. The error analysis under a general source condition is considered in section 3.3. Projection scheme of CNLM is considered in section 3.4 and the corresponding error analysis is given in section 3.5. Section 3.6 deals with the implementation of adaptive parameter choice strategy. An example and the computational results are given in section 3.7.

3.2 NEWTON LAVRENTIEV METHOD

Let

$$\tilde{\sigma}^\delta_{n,\alpha} := \|\tilde{y}^\delta_{n,\alpha} - \tilde{x}^\delta_{n,\alpha}\|, \qquad \forall n \geq 0 \tag{3.2.5}$$

and for $0 < k_0 \leq 1$, let $g : (0,1) \to (0,1)$ be the function defined by

$$g(t) = \frac{k_0^2}{8}(4 + 3k_0 t)t^2 \qquad \forall t \in (0,1). \tag{3.2.6}$$

Hereafter we assume that $\delta_0 < \alpha_0$ for some $\alpha_0 > 0$ and $\|x_0 - \hat{x}\| \leq \rho$ where

$$\rho \leq \frac{\sqrt{1 + 2k_0(1 - \frac{\delta_0}{\alpha_0})} - 1}{k_0}.$$

Let

$$\gamma_\rho := \frac{k_0}{2}\rho^2 + \rho + \frac{\delta_0}{\alpha_0}. \tag{3.2.7}$$

Remark 3.2.1. *Note that we use the condition $0 < k_0 \leq 1$, to ensure that $g(\gamma_\rho) < 1$ for $\gamma_\rho < 1$. Thus if $k_0 > 1$, then choose ρ, such that $g(\gamma_\rho) = \frac{k_0^2}{8}(4 + 3k_0\gamma_\rho)\gamma_\rho^2 < 1$, so that one can avoid the restriction $k_0 \leq 1$.*

For convenience, we use the notation $\tilde{x}_n, \tilde{y}_n$ and $\tilde{\sigma}_n$ for $\tilde{x}^\delta_{n,\alpha}, \tilde{y}^\delta_{n,\alpha}$ and $\tilde{\sigma}^\delta_{n,\alpha}$ respectively.

Theorem 3.2.2. *Let $\tilde{y}_n$ and $\tilde{x}_n$ be as in (3.1.1) and (3.1.2) respectively with $\delta \in (0, \delta_0]$ and let $\tilde{\sigma}_n$, g and γ_ρ be as in equation (3.2.5), (3.2.6) and (3.2.7) respectively. Let Assumption 2.2.2 hold. Then*

(a) $\|\tilde{x}_n - \tilde{y}_{n-1}\| \le \frac{k_0 \tilde{\sigma}_{n-1}}{2} \|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;

(b) $\|\tilde{x}_n - \tilde{x}_{n-1}\| \le (1 + \frac{k_0 \tilde{\sigma}_{n-1}}{2}) \|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;

(c) $\|\tilde{y}_n - \tilde{x}_n\| \le g(\tilde{\sigma}_{n-1}) \|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;

(d) $g(\tilde{\sigma}_n) \le g(\gamma_\rho)^{3^n}, \qquad \forall n \ge 0;$

(e) $\tilde{\sigma}_n \le g(\gamma_\rho)^{(3^n-1)/2} \gamma_\rho \qquad \forall n \ge 0.$

Proof. Observe that

$$\begin{aligned}
\tilde{x}_n - \tilde{y}_{n-1} &= \tilde{y}_{n-1} - \tilde{x}_{n-1} - R_\alpha^{-1}(\tilde{x}_{n-1})[F(\tilde{y}_{n-1}) - F(\tilde{x}_{n-1}) + \alpha(\tilde{y}_{n-1} - \tilde{x}_{n-1})] \\
&= R_\alpha^{-1}(\tilde{x}_{n-1})[(R_\alpha(\tilde{x}_{n-1}) - \alpha I)(\tilde{y}_{n-1} - \tilde{x}_{n-1}) - (F(\tilde{y}_{n-1}) - F(\tilde{x}_{n-1}))] \\
&= R_\alpha^{-1}(\tilde{x}_{n-1}) \int_0^1 [F'(\tilde{x}_{n-1}) - F'(\tilde{x}_{n-1} + t(\tilde{y}_{n-1} - \tilde{x}_{n-1}))](\tilde{y}_{n-1} - \tilde{x}_{n-1}) dt
\end{aligned}$$

and hence by Assumption 2.2.2 and (3.1.4), we have

$$\begin{aligned}
\|\tilde{x}_n - \tilde{y}_{n-1}\| &\le \| \int_0^1 \Phi(\tilde{x}_{n-1}, \tilde{x}_{n-1} + t(\tilde{y}_{n-1} - \tilde{x}_{n-1}), \tilde{y}_{n-1} - \tilde{x}_{n-1}) dt \| \\
&\le \frac{k_0}{2} \|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|^2.
\end{aligned}$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|\tilde{x}_n - \tilde{x}_{n-1}\| \le \|\tilde{x}_n - \tilde{y}_{n-1}\| + \|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|.$$

To prove (c) we observe that

$$\begin{aligned}
\tilde{y}_n - \tilde{x}_n &= \tilde{x}_n - \tilde{y}_{n-1} - R_\alpha^{-1}(\tilde{x}_n)[F(\tilde{x}_n) - f^\delta + \alpha(\tilde{x}_n - x_0)] \\
&\quad + R_\alpha^{-1}(\tilde{x}_{n-1})[F(\tilde{y}_{n-1}) - f^\delta + \alpha(\tilde{y}_{n-1} - x_0)] \\
&= \tilde{x}_n - \tilde{y}_{n-1} - R_\alpha^{-1}(\tilde{x}_n)[F(\tilde{x}_n) - F(\tilde{y}_{n-1}) + \alpha(\tilde{x}_n - \tilde{y}_{n-1})] \\
&\quad + [R_\alpha^{-1}(\tilde{x}_{n-1}) - R_\alpha^{-1}(\tilde{x}_n)][F(\tilde{y}_{n-1}) - f^\delta + \alpha(\tilde{y}_{n-1} - x_0)]
\end{aligned}$$

$$\begin{aligned} &= R_\alpha^{-1}(\tilde{x}_n)[R_\alpha(\tilde{x}_n)(\tilde{x}_n - \tilde{y}_{n-1}) - (F(\tilde{x}_n) - F(\tilde{y}_{n-1})) - \alpha(\tilde{x}_n - \tilde{y}_{n-1})] \\ &\quad +[R_\alpha^{-1}(\tilde{x}_{n-1}) - R_\alpha^{-1}(\tilde{x}_n)][F(\tilde{y}_{n-1}) - f^\delta + \alpha(\tilde{y}_{n-1} - x_0)] \end{aligned}$$

and hence

$$\begin{aligned} \tilde{\sigma}_n &\leq \|R_\alpha^{-1}(\tilde{x}_n)\int_0^1 [F'(\tilde{x}_n) - F'(\tilde{y}_{n-1} + t(\tilde{x}_n - \tilde{y}_{n-1})](\tilde{x}_n - \tilde{y}_{n-1})dt\| \\ &\quad +\|R_\alpha^{-1}(\tilde{x}_n)[F'(\tilde{x}_n) - F'(\tilde{x}_{n-1})]R_\alpha^{-1}(\tilde{x}_{n-1})[F(\tilde{y}_{n-1}) - f^\delta + \alpha(\tilde{y}_{n-1} - x_0)]\| \\ &\leq \|R_\alpha^{-1}(\tilde{x}_n)\int_0^1 [F'(\tilde{x}_n) - F'(\tilde{y}_{n-1} + t(\tilde{x}_n - \tilde{y}_{n-1}))](\tilde{x}_n - \tilde{y}_{n-1})dt\| \\ &\quad +\|R_\alpha^{-1}(\tilde{x}_n)(F'(\tilde{x}_n) - F'(\tilde{x}_{n-1}))(\tilde{y}_{n-1} - \tilde{x}_n)\| \\ &\leq \|\int_0^1 \Phi(\tilde{x}_n, \tilde{y}_{n-1} + t(\tilde{x}_n - \tilde{y}_{n-1}), \tilde{x}_n - \tilde{y}_{n-1})dt\| + \|\Phi(\tilde{x}_n, \tilde{x}_{n-1}, \tilde{y}_{n-1} - \tilde{x}_n)\| \\ &\leq \frac{k_0}{2}\|\tilde{x}_n - \tilde{y}_{n-1}\|^2 + k_0\|\tilde{x}_n - \tilde{x}_{n-1}\|\|\tilde{x}_n - \tilde{y}_{n-1}\|. \end{aligned}$$

The last but one step follows from the Assumption 2.2.2 and (3.1.4). Therefore by (a) and (b) we have (see, Argyros and Hilout (2010))

$$\begin{aligned} \tilde{\sigma}_n &\leq \left(\frac{k_0^2}{2} + \frac{3k_0^3}{8}\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|\right)\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|^3 \\ &\leq g(\tilde{\sigma}_{n-1})\tilde{\sigma}_{n-1}. \end{aligned} \tag{3.2.8}$$

This completes the proof of (c). Now since for $\mu \in (0,1)$, $g(\mu t) \leq \mu^2 g(t)$, for all $t \in (0,1)$, by (3.2.8) and Lemma 2.2.1 with γ_ρ as in (3.2.7) we have,

$$g(\tilde{\sigma}_n) \leq g(\tilde{\sigma}_0)^{3^n} \leq g(\gamma_\rho)^{3^n}$$

and

$$\begin{aligned} \tilde{\sigma}_n &\leq g(\tilde{\sigma}_{n-1})\tilde{\sigma}_{n-1} \\ &\leq g(\tilde{\sigma}_0)^{3^{n-1}}g(\tilde{\sigma}_{n-2})\tilde{\sigma}_{n-2} \\ &\leq g(\tilde{\sigma}_0)^{3^{n-1}}g(\tilde{\sigma}_0)^{3^{n-2}}g(\tilde{\sigma}_{n-3})\tilde{\sigma}_{n-3} \\ &\leq g(\tilde{\sigma}_0)^{3^{n-1}+3^{n-2}+\cdots+1}\tilde{\sigma}_0 \\ &\leq g(\tilde{\sigma}_0)^{(3^n-1)/2}\tilde{\sigma}_0 \\ &\leq g(\gamma_\rho)^{(3^n-1)/2}\gamma_\rho. \end{aligned}$$

The last step follows from the relation $\tilde{\sigma}_0 = e_0$ and Lemma 2.2.1 with γ_ρ as in (3.2.7).

This completes the proof of the theorem.

Theorem 3.2.3. *Let $0 < g(\gamma_\rho) < 1$, $r = [\frac{1}{1-g(\gamma_\rho)} + \frac{k_0}{2}\frac{\gamma_\rho}{1-g(\gamma_\rho)^2}]\gamma_\rho$ and the assumptions of Theorem 3.2.2 hold. Then $\tilde{x}_n, \tilde{y}_n \in B_r(x_0)$, for all $n \geq 0$.*

Proof. By induction we shall prove $\|\tilde{x}_n - x_0\| \leq \sum_{i=0}^{n-1}[1 + \frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^i]g(\tilde{\sigma}_0)^i\tilde{\sigma}_0$ and $\tilde{x}_n, \tilde{y}_n \in B_r(x_0)$, for all $n \geq 0$. Note that $\tilde{\sigma}_0 = e_0$, so by Lemma 2.2.1 with γ_ρ as in (3.2.7) and (b) of Theorem 3.2.2 we have

$$\begin{aligned}
\|\tilde{x}_1 - x_0\| &\leq [1 + \frac{k_0}{2}\tilde{\sigma}_0]\tilde{\sigma}_0 \qquad (3.2.9)\\
&\leq [1 + \frac{k_0}{2}\gamma_\rho]\gamma_\rho\\
&< r
\end{aligned}$$

i.e., $\|\tilde{x}_1 - x_0\| \leq [1 + \frac{k_0\tilde{\sigma}_0}{2}]\tilde{\sigma}_0$ and $\tilde{x}_1 \in B_r(x_0)$. Again note that by Lemma 2.2.1 with γ_ρ as in (3.2.7), (c) of Theorem 3.2.2 and (3.2.9), we have

$$\begin{aligned}
\|\tilde{y}_1 - x_0\| &\leq \|\tilde{y}_1 - \tilde{x}_1\| + \|\tilde{x}_1 - x_0\|\\
&\leq [1 + g(\tilde{\sigma}_0) + \frac{k_0}{2}\tilde{\sigma}_0]\tilde{\sigma}_0\\
&\leq [1 + g(\gamma_\rho) + \frac{k_0}{2}\gamma_\rho]\gamma_\rho\\
&< r
\end{aligned}$$

i.e., $\tilde{y}_1 \in B_r(x_0)$.

Suppose

$$\|\tilde{x}_k - x_0\| \leq \sum_{i=0}^{k-1}[1 + \frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^i]g(\tilde{\sigma}_0)^i\tilde{\sigma}_0 \qquad (3.2.10)$$

and $\tilde{x}_k, \tilde{y}_k \in B_r(x_0)$. Then by Lemma 2.2.1 with γ_ρ as in (3.2.7), (b) of Theorem 3.2.2 and (3.2.10), we have

$$\begin{aligned}
\|\tilde{x}_{k+1} - x_0\| &\leq \|\tilde{x}_{k+1} - \tilde{x}_k\| + \|\tilde{x}_k - x_0\|\\
&\leq (1 + \frac{k_0}{2}\tilde{\sigma}_k)\tilde{\sigma}_k + \sum_{i=0}^{k-1}[1 + \frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^i]g(\tilde{\sigma}_0)^i\tilde{\sigma}_0\\
&\leq [1 + \frac{k_0}{2}g(\tilde{\sigma}_0)^k\tilde{\sigma}_0]g(\tilde{\sigma}_0)^k\tilde{\sigma}_0 + \sum_{i=0}^{k-1}[1 + \frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^i]g(\tilde{\sigma}_0)^i\tilde{\sigma}_0\\
&\leq \sum_{i=0}^{k}[1 + \frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^i]g(\tilde{\sigma}_0)^i\tilde{\sigma}_0 \qquad (3.2.11)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{k}[1+\frac{k_0\gamma_\rho}{2}g(\gamma_\rho)^i]g(\gamma_\rho)^i\gamma_\rho \\
&\leq \left[\frac{1}{1-g(\gamma_\rho)}+\frac{k_0}{2}\frac{\gamma_\rho}{1-g(\gamma_\rho)^2}\right]\gamma_\rho \\
&< r.
\end{aligned}$$

So, by induction $\|\tilde{x}_n - x_0\| \leq \sum_{i=0}^{n-1}[1+\frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^i]g(\tilde{\sigma}_0)^i\tilde{\sigma}_0$ and $\tilde{x}_n \in B_r(x_0)$, for all $n \geq 0$. Again by Lemma 2.2.1 with γ_ρ as in (3.2.7), (c) of Theorem 3.2.2 and (3.2.11), we have

$$\begin{aligned}
\|\tilde{y}_{k+1} - x_0\| &\leq \|\tilde{y}_{k+1} - \tilde{x}_{k+1}\| + \|\tilde{x}_{k+1} - x_0\| \\
&\leq g(\tilde{\sigma}_k)\tilde{\sigma}_k + \sum_{i=0}^{k}[1+\frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^i]g(\tilde{\sigma}_0)^i\tilde{\sigma}_0 \\
&\leq g(\tilde{\sigma}_0)^{k+1}\tilde{\sigma}_0 + \sum_{i=0}^{k}[1+\frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^i]g(\tilde{\sigma}_0)^i\tilde{\sigma}_0 \\
&\leq \sum_{i=0}^{k+1} g(\tilde{\sigma}_0)^i\tilde{\sigma}_0 + \sum_{i=0}^{k}\frac{k_0}{2}g(\tilde{\sigma}_0)^{2i}(\tilde{\sigma}_0)^2 \\
&\leq \sum_{i=0}^{k+1} g(\gamma_\rho)^i\gamma_\rho + \sum_{i=0}^{k}\frac{k_0}{2}g(\gamma_\rho)^{2i}(\gamma_\rho)^2 \\
&\leq \left[\frac{1}{1-g(\gamma_\rho)}+\frac{k_0}{2}\frac{\gamma_\rho}{1-g(\gamma_\rho)^2}\right]\gamma_\rho \\
&< r.
\end{aligned}$$

Thus $\tilde{y}_{k+1} \in B_r(x_0)$ and hence by induction $\tilde{y}_n \in B_r(x_0)$, for all $n \geq 0$. This completes the proof.

The main result of this section is the following theorem.

Theorem 3.2.4. *Let $\tilde{y}_n$ and $\tilde{x}_n$ be as in (3.1.1) and (3.1.2) respectively with $\delta \in (0, \delta_0]$ and assumptions of Theorem 3.2.3 hold. Then $(\tilde{x}_n)$ is Cauchy sequence in $B_r(x_0)$ and converges to $x_\alpha^\delta \in \overline{B_r(x_0)}$. Further $F(x_\alpha^\delta) + \alpha(x_\alpha^\delta - x_0) = f^\delta$ and*

$$\|\tilde{x}_n - x_\alpha^\delta\| \leq Ce^{-\gamma 3^n}$$

where $C = \left[\frac{1}{1-g(\gamma_\rho)^3} + \frac{k_0\gamma_\rho}{2}\frac{1}{1-(g(\gamma_\rho)^2)^3}g(\gamma_\rho)^{3^n}\right]\gamma_\rho$ and $\gamma = -\log g(\gamma_\rho)$.

Proof. Using the relation (b) and (e) of Theorem 3.2.2, we obtain

$$\begin{aligned}
\|\tilde{x}_{n+m}-\tilde{x}_n\| &\leq \sum_{i=0}^{m-1}\|\tilde{x}_{n+i+1}-\tilde{x}_{n+i}\| \\
&\leq \sum_{i=0}^{m-1}[1+\frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^{3^{n+i}}]g(\tilde{\sigma}_0)^{3^{n+i}}\tilde{\sigma}_0 \\
&= [1+\frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^{3^n}]g(\tilde{\sigma}_0)^{3^n}\tilde{\sigma}_0+[1+\frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^{3^{n+1}}]g(\tilde{\sigma}_0)^{3^{n+1}}\tilde{\sigma}_0+\cdots \\
&\quad +[1+\frac{k_0\tilde{\sigma}_0}{2}g(\tilde{\sigma}_0)^{3^{n+m}}]g(\tilde{\sigma}_0)^{3^{n+m}}\tilde{\sigma}_0 \\
&\leq [(1+g(\tilde{\sigma}_0)^3+g(\tilde{\sigma}_0)^{3^2}+\cdots+g(\tilde{\sigma}_0)^{3^m})+\frac{k_0\tilde{\sigma}_0}{2}(1+ \\
&\quad (g(\tilde{\sigma}_0)^2)^3+(g(\tilde{\sigma}_0)^2)^{3^2}+\cdots+(g(\tilde{\sigma}_0)^2)^{3^m})g(\tilde{\sigma}_0)^{3^n}]g(\tilde{\sigma}_0)^{3^n}\tilde{\sigma}_0 \\
&\leq Cg(\tilde{\sigma}_0)^{3^n} \\
&\leq Cg(\gamma_\rho)^{3^n} \\
&\leq Ce^{-\gamma 3^n}.
\end{aligned}$$

Thus $\tilde{x}_n$ is a Cauchy sequence in $B_r(x_0)$ and hence it converges, say to $x_\alpha^\delta \in \overline{B_r(x_0)}$. Observe that

$$\begin{aligned}
\|F(\tilde{x}_n)-f^\delta+\alpha(\tilde{x}_n-x_0)\| &= \|R_\alpha(\tilde{x}_n)(\tilde{x}_n-\tilde{y}_n)\| \\
&\leq \|R_\alpha(\tilde{x}_n)\|\|\tilde{x}_n-\tilde{y}_n\| \\
&\leq (C_F+\alpha)g(\tilde{\sigma}_0)^{3^n}\tilde{\sigma}_0 \\
&\leq (C_F+\alpha)g(\gamma_\rho)^{3^n}\gamma_\rho. \qquad (3.2.12)
\end{aligned}$$

Now by letting $n\to\infty$ in (3.2.12) we obtain $F(x_\alpha^\delta)+\alpha(x_\alpha^\delta-x_0)=f^\delta$. This completes the proof.

Hereafter we assume that

$$\rho \leq r. \qquad (3.2.13)$$

Remark 3.2.5. *Note that (3.2.13) is satisfied if* $\frac{4(1-g(\gamma_\rho)^2)^2}{9\gamma_\rho^4}\left[2(1-\frac{\delta_0}{\alpha_0})-\frac{3\gamma_\rho^2}{1-g(\gamma_\rho)^2}\right] \leq k_0 \leq 1$. *Further observe that* $0<g(\gamma_\rho)<1$ *and hence* $\gamma>0$. *So by (1.5.12), sequence* $(\tilde{x}_n)$ *converges cubically to* x_α^δ.

3.3 ERROR BOUNDS UNDER SOURCE CONDITIONS

The objective of this section is to obtain an error estimate for $\|\tilde{x}^{\delta}_{n,\alpha} - \hat{x}\|$ under a source condition on $x_0 - \hat{x}$.

Combining the estimates in Proposition 2.3.2, Theorem 2.3.3 and Theorem 3.2.4 we obtain the following;

Theorem 3.3.1. *Let $\tilde{x}_n$ be as in (3.1.2) and let assumptions in Proposition 2.3.2, Theorem 2.3.3 and Theorem 3.2.4 be satisfied. Then*

$$\|\tilde{x}_n - \hat{x}\| \leq Ce^{-\gamma 3^n} + C_1 \left[\varphi(\alpha) + \frac{\delta}{\alpha}\right]$$

where $C_1 = k_0 r + 1$.

Let

$$\bar{C} := \max\{C, k_0 r\} + 1, \tag{3.3.14}$$

and let

$$n_\delta := \min\left\{n : e^{-\gamma 3^n} \leq \frac{\delta}{\alpha}\right\}. \tag{3.3.15}$$

Theorem 3.3.2. *Let x^{δ}_{α} be the unique solution of (2.1.4) and $\tilde{x}_n$ be as in (3.1.2). Let assumptions in Theorem 3.3.1 be satisfied. Let $\bar{C}$ and n_δ be as in (3.3.14) and (3.3.15) respectively. Then*

$$\|\tilde{x}_{n_\delta} - \hat{x}\| \leq \bar{C}\left[\varphi(\alpha) + \frac{\delta}{\alpha}\right]. \tag{3.3.16}$$

3.3.1 A priori choice of the parameter

In view of the observations in section 2.3.1 of Chapter 2 and (3.3.16) we have the following.

Theorem 3.3.3. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and the assumptions in Theorem 3.3.2 hold. For $\delta > 0$, let $\alpha := \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$ and let n_δ be as in (3.3.15). Then*

$$\|\tilde{x}_{n_\delta} - \hat{x}\| = O(\psi^{-1}(\delta)).$$

3.3.2 An adaptive choice of the parameter

Let

$$D_N(\alpha) := \{\alpha_i = \mu^i \alpha_0, i = 0, 1, \cdots, N\}$$

where $\mu > 1$, $\alpha_0 > 0$ and let

$$n_i := \min\left\{n : e^{-\gamma 3^n} \leq \frac{\delta}{\alpha_i}\right\}.$$

Then for $i = 0, 1, \cdots, N$, we have

$$\|\tilde{x}^\delta_{n_i,\alpha_i} - x^\delta_{\alpha_i}\| \leq c\frac{\delta}{\alpha_i}, \quad \forall i = 0, 1, \cdots N.$$

Let $\tilde{x}_i := \tilde{x}^\delta_{n_i,\alpha_i}$. We select the regularization parameter $\alpha = \alpha_i$ from the set $D_N(\alpha)$ and operate only with corresponding $\tilde{x}_i, \quad i = 0, 1, \cdots, N$. The proof of the following theorem is analogous to the proof of the Theorem 2.3.7.

Theorem 3.3.4. *Assume that there exists $i \in \{0, 1, 2, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}$. Let assumptions of Theorem 3.5.2 and Theorem 3.5.3 hold and let*

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}\right\} < N,$$

$$k := \max\left\{i : \|\tilde{x}_i - \tilde{x}_j\| \leq 4\bar{C}\frac{\delta}{\alpha_j}, \quad j = 0, 1, 2, \cdots, i\right\}.$$

Then $l \leq k$ and

$$\|\hat{x} - \tilde{x}_k\| \leq c\psi^{-1}(\delta)$$

where $c = 6\bar{C}\mu$.

3.4 PROJECTION METHOD AND ITS CONVERGENCE

The purpose of this section is to obtain an approximate solution for the equation (2.1.4), in the finite dimensional subspace of X.

Let $\tilde{x}_{0,\alpha}^{h,\delta} := P_h x_0$ be the projection of the initial guess x_0 on to $R(P_h)$, the range of P_h and let $R_\alpha(x) := P_h F'(x) P_h + \alpha P_h$ with $\alpha > \alpha_0 > 0$. We define the iterative sequence as:

$$\tilde{y}_{n,\alpha}^{h,\delta} = \tilde{x}_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta}) P_h [F(\tilde{x}_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(\tilde{x}_{n,\alpha}^{h,\delta} - x_0)] \tag{3.4.17}$$

and

$$\tilde{x}_{n+1,\alpha}^{h,\delta} = \tilde{y}_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta}) P_h [F(\tilde{y}_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(\tilde{y}_{n,\alpha}^{h,\delta} - x_0)]. \tag{3.4.18}$$

Note that the iteration (3.4.17) and (3.4.18) are the finite dimensional realization of the iteration (3.1.1) and (3.1.2). We will be selecting the parameter $\alpha = \alpha_i$ from some finite set defined in (2.1.3) using the adaptive method considered by Perverzev and Schock in Pereverzyev and Schock (2005).

Let

$$\tilde{\sigma}_{n,\alpha}^{h,\delta} := \|\tilde{y}_{n,\alpha}^{h,\delta} - \tilde{x}_{n,\alpha}^{h,\delta}\|, \qquad \forall n \geq 0 \tag{3.4.19}$$

For $0 < k_0 < min\left\{1, \frac{\sqrt{8}}{(1+\frac{\varepsilon_0}{\alpha_0})\sqrt{4+3(1+\frac{\varepsilon_0}{\alpha_0})}}\right\}$, let $\tilde{g} : (0,1) \to (0,1)$ be the function defined by

$$\tilde{g}(t) = \frac{k_0^2}{8}(1 + \frac{\varepsilon_0}{\alpha_0})^2 \left[4 + 3k_0(1 + \frac{\varepsilon_0}{\alpha_0})t\right] t^2 \qquad \forall t \in (0,1). \tag{3.4.20}$$

Hereafter we assume that $b_0 < \frac{\sqrt{1+\frac{2k_0}{(1+\frac{\varepsilon_0}{\alpha_0})}(1-\frac{\delta_0}{\alpha_0})}-1}{k_0}$, $\delta_0 < \alpha_0$ for some $\alpha_0 > 0$ and $\|x_0 - \hat{x}\| \leq \rho$ where

$$\rho \leq \frac{\sqrt{1 + \frac{2k_0}{(1+\frac{\varepsilon_0}{\alpha_0})}(1 - \frac{\delta_0}{\alpha_0})} - 1}{k_0} - b_0.$$

Let

$$\gamma_\rho := (1 + \frac{\varepsilon_0}{\alpha_0})[\frac{k_0}{2}(\rho + b_0)^2 + (\rho + b_0)] + \frac{\delta_0}{\alpha_0}. \tag{3.4.21}$$

Lemma 3.4.1. *Let $\tilde{y}_{n,\alpha}^{h,\delta}$, $\tilde{x}_{n,\alpha}^{h,\delta}$ and $\tilde{\sigma}_{n,\alpha}^{h,\delta}$ be as in (3.4.17), (3.4.18) and (3.4.19) respectively with $\delta \in (0, \delta_0]$. And let Assumption 2.2.2 hold. Then*

(a) $\|\tilde{x}^{h,\delta}_{n,\alpha} - \tilde{y}^{h,\delta}_{n-1,\alpha}\| \le \frac{k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}^{h,\delta}_{n-1,\alpha}\|\tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha}\|$;

(b) $\|\tilde{x}^{h,\delta}_{n,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha}\| \le (1+\frac{k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}^{h,\delta}_{n-1,\alpha})\|\tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha}\|$;

Proof. Observe that

$$\begin{aligned}
\tilde{x}^{h,\delta}_{n,\alpha} - \tilde{y}^{h,\delta}_{n-1,\alpha} &= \tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha} - R_\alpha^{-1}(\tilde{x}^{h,\delta}_{n-1,\alpha})P_h[F(\tilde{y}^{h,\delta}_{n-1,\alpha}) - F(\tilde{x}^{h,\delta}_{n-1,\alpha}) \\
&\quad +\alpha(\tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha})] \\
&= R_\alpha^{-1}(\tilde{x}^{h,\delta}_{n-1,\alpha})[(R_\alpha(\tilde{x}^{h,\delta}_{n-1,\alpha}) - \alpha P_h)(\tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha}) \\
&\quad -P_h(F(\tilde{y}^{h,\delta}_{n-1,\alpha}) - F(\tilde{x}^{h,\delta}_{n-1,\alpha}))] \\
&= R_\alpha^{-1}(\tilde{x}^{h,\delta}_{n-1,\alpha})P_h\int_0^1[F'(\tilde{x}^{h,\delta}_{n-1,\alpha}) - F'(\tilde{x}^{h,\delta}_{n-1,\alpha} + t(\tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha}))] \\
&\quad \times(\tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha})dt
\end{aligned}$$

and hence by Assumption 2.2.2 and Lemma 2.4.1, we have

$$\begin{aligned}
\|\tilde{x}^{h,\delta}_{n,\alpha} - \tilde{y}^{h,\delta}_{n-1,\alpha}\| &\le \|(1+\frac{\varepsilon_0}{\alpha_0}) \\
&\quad \times\int_0^1 \Phi(\tilde{x}^{h,\delta}_{n-1,\alpha}, \tilde{x}^{h,\delta}_{n-1,\alpha} + t(\tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha}), \tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha})dt\| \\
&\le \frac{k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\|\tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha}\|^2.
\end{aligned}$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|\tilde{x}^{h,\delta}_{n,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha}\| \le \|\tilde{x}^{h,\delta}_{n,\alpha} - \tilde{y}^{h,\delta}_{n-1,\alpha}\| + \|\tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha}\|.$$

Theorem 3.4.2. *Let Assumption 2.2.2 hold. Let $\tilde{y}^{h,\delta}_{n,\alpha}$ and $\tilde{x}^{h,\delta}_{n,\alpha}$ be as in (3.4.17) and (3.4.18) respectively with $\delta \in (0, \delta_0]$ and let $\tilde{\sigma}^{h,\delta}_{n,\alpha}$, $\tilde{g}$ and γ_ρ be as in equation (3.4.19), (3.4.20) and (3.4.21) respectively. Then*

(a) $\|\tilde{y}^{h,\delta}_{n,\alpha} - \tilde{x}^{h,\delta}_{n,\alpha}\| \le \tilde{g}(\tilde{\sigma}^{h,\delta}_{n-1,\alpha})\|\tilde{y}^{h,\delta}_{n-1,\alpha} - \tilde{x}^{h,\delta}_{n-1,\alpha}\|$;

(b) $\tilde{g}(\tilde{\sigma}^{h,\delta}_{n,\alpha}) \le \tilde{g}(\gamma_\rho)^{3^n}, \qquad \forall n \ge 0$;

(c) $\tilde{\sigma}^{h,\delta}_{n,\alpha} \le \tilde{g}(\gamma_\rho)^{(3^n-1)/2}\gamma_\rho \qquad \forall n \ge 0.$

Proof. Observe that

$$\begin{aligned}
\tilde{y}_{n,\alpha}^{h,\delta} - \tilde{x}_{n,\alpha}^{h,\delta} &= \tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta} - R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta})P_h[F(\tilde{x}_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(\tilde{x}_{n,\alpha}^{h,\delta} - x_0)] \\
&\quad + R_\alpha^{-1}(\tilde{x}_{n-1,\alpha}^{h,\delta})P_h[F(\tilde{y}_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(\tilde{y}_{n-1,\alpha}^{h,\delta} - x_0)] \\
&= \tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta} - R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta})P_h[F(\tilde{x}_{n,\alpha}^{h,\delta}) - F(\tilde{y}_{n-1,\alpha}^{h,\delta}) + \alpha(\tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta})] \\
&\quad + [R_\alpha^{-1}(\tilde{x}_{n-1,\alpha}^{h,\delta}) - R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta})]P_h[F(\tilde{y}_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(\tilde{y}_{n-1,\alpha}^{h,\delta} - x_0)] \\
&= R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta})[(R_\alpha(\tilde{x}_{n,\alpha}^{h,\delta}) - \alpha P_h)(\tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta}) - P_h(F(\tilde{x}_{n,\alpha}^{h,\delta}) - F(\tilde{y}_{n-1,\alpha}^{h,\delta}))] \\
&\quad + [R_\alpha^{-1}(\tilde{x}_{n-1,\alpha}^{h,\delta}) - R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta})]P_h[F(\tilde{y}_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(\tilde{y}_{n-1,\alpha}^{h,\delta} - x_0)]
\end{aligned}$$

and hence

$$\begin{aligned}
\tilde{\sigma}_{n,\alpha}^{h,\delta} &\leq \|R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta})P_h \int_0^1 [F'(\tilde{x}_{n,\alpha}^{h,\delta}) - F'(\tilde{y}_{n-1,\alpha}^{h,\delta} + t(\tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta})](\tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta})dt\| \\
&\quad + \|R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta})[F'(\tilde{x}_{n,\alpha}^{h,\delta}) - F'(\tilde{x}_{n-1,\alpha}^{h,\delta})]R_\alpha^{-1}(\tilde{x}_{n-1,\alpha}^{h,\delta}) \\
&\quad P_h[F(\tilde{y}_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(\tilde{y}_{n-1,\alpha}^{h,\delta} - x_0)]\| \\
&\leq \|R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta})P_h \int_0^1 [F'(\tilde{x}_{n,\alpha}^{h,\delta}) - F'(\tilde{y}_{n-1,\alpha}^{h,\delta} + t(\tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta}))](\tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta})dt\| \\
&\quad + \|R_\alpha^{-1}(\tilde{x}_{n,\alpha}^{h,\delta})P_h[F'(\tilde{x}_{n,\alpha}^{h,\delta}) - F'(\tilde{x}_{n-1,\alpha}^{h,\delta})](\tilde{y}_{n-1,\alpha}^{h,\delta} - \tilde{x}_{n,\alpha}^{h,\delta})\| \\
&\leq (1 + \frac{\varepsilon_0}{\alpha_0})\| \int_0^1 \Phi(\tilde{x}_{n,\alpha}^{h,\delta}, \tilde{y}_{n-1,\alpha}^{h,\delta} + t(\tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta}), \tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta})dt\| \\
&\quad + (1 + \frac{\varepsilon_0}{\alpha_0})\|\Phi(\tilde{x}_{n,\alpha}^{h,\delta}, \tilde{x}_{n-1,\alpha}^{h,\delta}, \tilde{y}_{n-1,\alpha}^{h,\delta} - \tilde{x}_{n,\alpha}^{h,\delta})\| \\
&\leq (1 + \frac{\varepsilon_0}{\alpha_0})\left[\frac{k_0}{2}\|\tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta}\|^2 + k_0\|\tilde{x}_{n,\alpha}^{h,\delta} - \tilde{x}_{n-1,\alpha}^{h,\delta}\|\|\tilde{x}_{n,\alpha}^{h,\delta} - \tilde{y}_{n-1,\alpha}^{h,\delta}\|\right].
\end{aligned}$$

The last but one step follows from the Assumption 2.2.2 and Lemma 2.4.1. Therefore by (a) and (b) of Lemma 3.4.1 we have,

$$\begin{aligned}
\tilde{\sigma}_{n,\alpha}^{h,\delta} &\leq (1 + \frac{\varepsilon_0}{\alpha_0})^2\left[\frac{k_0^2}{2} + \frac{3k_0^3}{8}(1 + \frac{\varepsilon_0}{\alpha_0})\|\tilde{y}_{n-1,\alpha}^{h,\delta} - \tilde{x}_{n-1,\alpha}^{h,\delta}\|\right]\|\tilde{y}_{n-1,\alpha}^{h,\delta} - \tilde{x}_{n-1,\alpha}^{h,\delta}\|^3 \\
&\leq \tilde{g}(\tilde{\sigma}_{n-1,\alpha}^{h,\delta})\tilde{\sigma}_{n-1,\alpha}^{h,\delta}. \qquad (3.4.22)
\end{aligned}$$

This completes the proof of (a). Now since for $\mu \in (0, 1)$, $\tilde{g}(\mu t) \leq \mu^2\tilde{g}(t)$, for all $t \in (0, 1)$, by Lemma 2.4.2 with γ_ρ as in (3.4.21) and (3.4.22) we have,

$$\tilde{g}(\tilde{\sigma}_{n,\alpha}^{h,\delta}) \leq \tilde{g}(\tilde{\sigma}_0)^{3^n} \leq \tilde{g}(\gamma_\rho)^{3^n}$$

and

$$\begin{aligned}
\tilde{\sigma}^{h,\delta}_{n,\alpha} &\leq \tilde{g}(\tilde{\sigma}^{h,\delta}_{n-1,\alpha})\tilde{\sigma}^{h,\delta}_{n-1,\alpha} \\
&\leq \tilde{g}(\tilde{\sigma}_0)^{3^{n-1}}\tilde{g}(\tilde{\sigma}^{h,\delta}_{n-2,\alpha})\tilde{\sigma}^{h,\delta}_{n-2,\alpha} \\
&\leq \tilde{g}(\tilde{\sigma}_0)^{3^{n-1}+3^{n-2}+\cdots+1}\tilde{\sigma}_0 \\
&\leq \tilde{g}(\tilde{\sigma}_0)^{(3^n-1)/2}\tilde{\sigma}_0 \\
&\leq \tilde{g}(\gamma_\rho)^{(3^n-1)/2}\gamma_\rho.
\end{aligned}$$

This completes the proof of the Theorem.

Theorem 3.4.3. *Let* $r = [\frac{1}{1-\tilde{g}(\gamma_\rho)} + \frac{k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\frac{\gamma_\rho}{1-\tilde{g}(\gamma_\rho)^2}]\gamma_\rho$ *and the assumptions of Theorem 3.4.2 hold. Then* $\tilde{x}^{h,\delta}_{n,\alpha}, \tilde{y}^{h,\delta}_{n,\alpha} \in B_r(P_h x_0)$, *for all* $n \geq 0$.

Proof. Note that by Lemma 2.4.2 with γ_ρ as in (3.4.21) and (b) of Lemma 3.4.1 we have

$$\begin{aligned}
\|\tilde{x}^{h,\delta}_{1,\alpha} - P_h x_0\| &= \|\tilde{x}^{h,\delta}_{1,\alpha} - \tilde{x}^{h,\delta}_{0,\alpha}\| \\
&\leq [1 + \frac{k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}_0]\tilde{\sigma}_0 \qquad (3.4.23) \\
&\leq [1 + \frac{k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\gamma_\rho]\gamma_\rho \\
&< r
\end{aligned}$$

i.e., $\tilde{x}^{h,\delta}_{1,\alpha} \in B_r(P_h x_0)$. Again note that by Lemma 2.4.2 with γ_ρ as in (3.4.21), (a) of Theorem 3.4.2 and (3.4.23) we have

$$\begin{aligned}
\|\tilde{y}^{h,\delta}_{1,\alpha} - P_h x_0\| &\leq \|\tilde{y}^{h,\delta}_{1,\alpha} - \tilde{x}^{h,\delta}_{1,\alpha}\| + \|\tilde{x}^{h,\delta}_{1,\alpha} - P_h x_0\| \\
&\leq [1 + \tilde{g}(\tilde{\sigma}_0) + \frac{k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}_0]\tilde{\sigma}_0 \\
&\leq [1 + \tilde{g}(\gamma_\rho) + \frac{k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\gamma_\rho]\gamma_\rho \\
&< r
\end{aligned}$$

i.e., $\tilde{y}^{h,\delta}_{1,\alpha} \in B_r(P_h x_0)$. Further by Lemma 2.4.2 with γ_ρ as in (3.4.21), (b) of Lemma 3.4.1

and (3.4.23) we have

$$
\begin{aligned}
\|\tilde{x}^{h,\delta}_{2,\alpha} - P_h x_0\| &\leq \|\tilde{x}^{h,\delta}_{2,\alpha} - \tilde{x}^{h,\delta}_{1,\alpha}\| + \|\tilde{x}^{h,\delta}_{1,\alpha} - P_h x_0\| \\
&\leq [1 + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}^{h,\delta}_{1,\alpha}]\tilde{\sigma}^{h,\delta}_{1,\alpha} + [1 + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}_0]\tilde{\sigma}_0 \\
&\leq [1 + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\tilde{g}(\tilde{\sigma}_0)\tilde{\sigma}_0]\tilde{g}(\tilde{\sigma}_0)\tilde{\sigma}_0 + [1 + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}_0]\tilde{\sigma}_0 \\
&\leq [1 + \tilde{g}(\tilde{\sigma}_0) + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}_0(1 + \tilde{g}(\tilde{\sigma}_0)^2)]\tilde{\sigma}_0 \qquad (3.4.24) \\
&\leq [1 + \tilde{g}(\gamma_\rho) + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho(1 + \tilde{g}(\gamma_\rho)^2)]\gamma_\rho \\
&< r
\end{aligned}
$$

and by Lemma 2.4.2 with γ_ρ as in (3.4.21), (a) of Theorem 3.4.2 and (3.4.24) we have

$$
\begin{aligned}
\|\tilde{y}^{h,\delta}_{2,\alpha} - P_h x_0\| &\leq \|\tilde{y}^{h,\delta}_{2,\alpha} - \tilde{x}^{h,\delta}_{2,\alpha}\| + \|\tilde{x}^{h,\delta}_{2,\alpha} - P_h x_0\| \\
&\leq \tilde{g}(\tilde{\sigma}^{h,\delta}_{1,\alpha})\tilde{\sigma}^{h,\delta}_{1,\alpha} + [1 + \tilde{g}(\tilde{\sigma}_0) + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}_0(1 + \tilde{g}(\tilde{\sigma}_0)^2)]\tilde{\sigma}_0 \\
&\leq \tilde{g}(\tilde{\sigma}_0)^4\tilde{\sigma}_0 + [1 + \tilde{g}(\tilde{\sigma}_0) + \frac{k_0}{2}1 + \frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}_0(1 + \tilde{g}(\tilde{\sigma}_0)^2)]\tilde{\sigma}_0 \\
&\leq [1 + \tilde{g}(\tilde{\sigma}_0) + \tilde{g}(\tilde{\sigma}_0)^2 + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}_0(1 + \tilde{g}(\tilde{\sigma}_0)^2)]\tilde{\sigma}_0 \\
&\leq [1 + \tilde{g}(\gamma_\rho) + \tilde{g}(\gamma_\rho)^2 + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho(1 + \tilde{g}(\gamma_\rho)^2)]\gamma_\rho \\
&< r
\end{aligned}
$$

i.e., $\tilde{x}^{h,\delta}_{2,\alpha}, \tilde{y}^{h,\delta}_{2,\alpha} \in B_r(P_h x_0)$. Continuing this way one can prove that $\tilde{x}^{h,\delta}_{n,\alpha}, \tilde{y}^{h,\delta}_{n,\alpha} \in B_r(P_h x_0)$, $\forall n \geq 0$. This completes the proof.

The main result of this section is the following theorem.

Theorem 3.4.4. *Let $\tilde{y}^{h,\delta}_{n,\alpha}$ and $\tilde{x}^{h,\delta}_{n,\alpha}$ be as in (3.4.17) and (3.4.18) respectively with $\delta \in (0, \delta_0]$ and assumptions of Theorem 3.4.3 hold. Then $(\tilde{x}^{h,\delta}_{n,\alpha})$ is Cauchy sequence in $B_r(P_h x_0)$ and converges to $x^{h,\delta}_\alpha \in \overline{B_r(P_h x_0)}$. Further $P_h[F(x^{h,\delta}_\alpha) + \alpha(x^{h,\delta}_\alpha - x_0)] = P_h f^\delta$ and*

$$\|\tilde{x}^{h,\delta}_{n,\alpha} - x^{h,\delta}_\alpha\| \leq Ce^{-\gamma 3^n}$$

where $C = [\frac{1}{1-\tilde{g}(\gamma_\rho)^3} + \frac{k_0}{2}\gamma_\rho(1 + \frac{\varepsilon_0}{\alpha_0})\frac{1}{1-(\tilde{g}(\gamma_\rho)^2)^3}\tilde{g}(\gamma_\rho)^{3^n}]\gamma_\rho$ and $\gamma = -\log \tilde{g}(\gamma_\rho)$.

Proof. Using the relation (b) of Lemma 3.4.1 and (c) of Theorem 3.4.2, we obtain

$$\begin{aligned}
\|\tilde{x}^{h,\delta}_{n+m,\alpha} - \tilde{x}^{h,\delta}_{n,\alpha}\| &\leq \sum_{i=0}^{m-1} \|\tilde{x}^{h,\delta}_{n+i+1,\alpha} - \tilde{x}^{h,\delta}_{n+i,\alpha}\| \\
&\leq \sum_{i=0}^{m-1} \left[1 + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}_0 \tilde{g}(\tilde{\sigma}_0)^{3^{n+i}}\right] \tilde{g}(\tilde{\sigma}_0)^{3^{n+i}} \tilde{\sigma}_0 \\
&\leq [(1 + \tilde{g}(\tilde{\sigma}_0)^3 + \tilde{g}(\tilde{\sigma}_0)^{3^2} + \cdots + \tilde{g}(\tilde{\sigma}_0)^{3^m}) + \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\tilde{\sigma}_0 (1 + \\
&\quad (\tilde{g}(\tilde{\sigma}_0)^2)^3 + (\tilde{g}(\tilde{\sigma}_0)^2)^{3^2} + \cdots + (\tilde{g}(\tilde{\sigma}_0)^2)^{3^m}) \tilde{g}(\tilde{\sigma}_0)^{3^n}] \tilde{g}(\tilde{\sigma}_0)^{3^n} \tilde{\sigma}_0 \\
&\leq C \tilde{g}(\gamma_\rho)^{3^n} \\
&\leq C e^{-\gamma 3^n}.
\end{aligned}$$

Thus $\tilde{x}^{h,\delta}_{n,\alpha}$ is a Cauchy sequence in $B_r(P_h x_0)$ and hence it converges cubically, say, to $x^{h,\delta}_\alpha \in \overline{B_r(P_h x_0)}$. Observe that

$$\begin{aligned}
\|P_h(F(\tilde{x}^{h,\delta}_{n,\alpha}) - f^\delta + \alpha(\tilde{x}^{h,\delta}_{n,\alpha} - x_0))\| &= \|R_\alpha(\tilde{x}^{h,\delta}_{n,\alpha})(\tilde{x}^{h,\delta}_{n,\alpha} - \tilde{y}^{h,\delta}_{n,\alpha})\| \\
&\leq \|R_\alpha(\tilde{x}^{h,\delta}_{n,\alpha})\| \|\tilde{x}^{h,\delta}_{n,\alpha} - \tilde{y}^{h,\delta}_{n,\alpha}\| \\
&= \|(P_h F'(\tilde{x}^{h,\delta}_{n,\alpha}) P_h + \alpha P_h)\| \tilde{\sigma}^{h,\delta}_{n,\alpha} \\
&\leq (C_F + \alpha) \tilde{g}(\gamma_\rho)^{\frac{3^n - 1}{2}} \gamma_\rho. \qquad (3.4.25)
\end{aligned}$$

Now by letting $n \to \infty$ in (3.4.25) we obtain

$$P_h[F(x^{h,\delta}_\alpha) + \alpha(x^{h,\delta}_\alpha - x_0)] = P_h f^\delta.$$

This completes the proof.

3.5 ERROR BOUNDS UNDER SOURCE CONDITIONS FOR PROJECTION METHOD

Theorem 3.5.1. *Let $\tilde{x}^{h,\delta}_{n,\alpha}$ be as in (3.4.18). And the assumptions in Theorem 2.5.2 and Theorem 3.4.4 hold. Then*

$$\|\tilde{x}^{h,\delta}_{n,\alpha} - \hat{x}\| \leq C e^{-\gamma 3^n} + \max\{1, \tilde{C}\} \left[\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}\right].$$

Proof. Observe that,

$$\|\tilde{x}_{n,\alpha}^{h,\delta} - \hat{x}\| \leq \|\tilde{x}_{n,\alpha}^{h,\delta} - x_{\alpha}^{h,\delta}\| + \|x_{\alpha}^{h,\delta} - \tilde{x}_{\alpha}^{h}\| + \|\tilde{x}_{\alpha}^{h} - \hat{x}\|$$

so, by Proposition 2.5.1, Theorem 2.5.2 and Theorem 3.4.4 we obtain,

$$\begin{aligned} \|\tilde{x}_{n,\alpha}^{h,\delta} - \hat{x}\| &\leq Ce^{-\gamma 3^n} + \frac{\delta}{\alpha} + \tilde{C}\left[\varphi(\alpha) + \frac{\varepsilon_h}{\alpha}\right] \\ &\leq Ce^{-\gamma 3^n} + \max\{1, \tilde{C}\}\left[\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}\right]. \end{aligned}$$

Let

$$n_\delta := \min\left\{n : e^{-\gamma 3^n} \leq \frac{\delta + \varepsilon_h}{\alpha}\right\} \tag{3.5.26}$$

and

$$C_0 = C + \max\{1, \tilde{C}\}. \tag{3.5.27}$$

Theorem 3.5.2. *Let $\tilde{x}_{n_\delta,\alpha}^{h,\delta}$ be as in (3.4.18) and the assumptions in Theorem 3.5.1 be satisfied. And let n_δ and C_0 be as in (3.5.26) and (3.5.27) respectively. Then*

$$\|\tilde{x}_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| \leq C_0\left[\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}\right].$$

3.5.1 A priori choice of the parameter

Theorem 3.5.3. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$ and the assumptions in Theorem 3.5.2 hold. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h))$ and let n_δ be as in (3.5.26). Then*

$$\|\tilde{x}_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

3.5.2 An adaptive choice of the parameter

Let $D_N(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0, 1, \cdots, N\}$ where $\mu > 1$, $\alpha_0 > 0$ and let $n_i := \min\left\{n : e^{-\gamma 3^n} \leq \frac{\delta+\varepsilon_h}{\alpha_i}\right\}$. Then for $i = 0, 1, \cdots, N$, we have

$$\|\tilde{x}_{n_i,\alpha_i}^{h,\delta} - \tilde{x}_{\alpha_i}^{h,\delta}\| \leq C\frac{\delta + \varepsilon_h}{\alpha_i}, \quad \forall i = 0, 1, \cdots N.$$

Let $\tilde{x}_i := \tilde{x}^{h,\delta}_{n_i,\alpha_i}, \quad i = 0, 1, \cdots, N$. We select the regularization parameter $\alpha = \alpha_i$ from the set $D_N(\alpha)$ and operate only with corresponding $\tilde{x}_i, \quad i = 0, 1, \cdots, N$.

Proof of the following theorem is analogous to the proof of Theorem 2.3.7.

Theorem 3.5.4. *Assume that there exists $i \in \{0, 1, 2, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta+\varepsilon_h}{\alpha_i}$. Let assumptions of Theorem 3.5.2 and Theorem 3.5.3 hold and let*

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta + \varepsilon_h}{\alpha_i}\right\} < N,$$

$k := \max\left\{i : \|\tilde{x}_i - \tilde{x}_j\| \leq 4C_0\frac{\delta+\varepsilon_h}{\alpha_j}, \quad j = 0, 1, 2, \cdots, i\right\}$. *Then $l \leq k$ and $\|\hat{x} - \tilde{x}_k\| \leq c\psi^{-1}(\delta + \varepsilon_h)$ where $c = 6C_0\mu$.*

3.6 IMPLEMENTATION OF ADAPTIVE CHOICE RULE

The balancing algorithm associated with the choice of the parameter specified in Theorem 3.4.2 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta_0 < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \cdots, N$.

3.6.1 Algorithm

1. Set $i = 0$.
2. Choose $n_i := \min\left\{n : e^{-\gamma 3^n} \leq \frac{\delta+\varepsilon_h}{\alpha_i}\right\}$.
3. Solve $\tilde{x}_i := \tilde{x}^{h,\delta}_{n_i,\alpha_i}$ by using the iteration (3.4.17) and (3.4.18).
4. If $\|\tilde{x}_i - \tilde{x}_j\| > 4C_0\frac{\delta+\varepsilon_h}{\alpha_j}, j < i$, then take $k = i - 1$ and return $\tilde{x}_k$.
5. Else set $i = i + 1$ and go to Step 2.

3.7 NUMERICAL EXAMPLE

In this section we consider the problem studied in Example 2.7.1 for illustrating the algorithm considered in section 3.6.1. We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X as in section 2.7.

Example 3.7.1. *In our computation, we take the kernel as in Example 2.7.1,*

$$f(t) = \frac{6sin(\pi t) + sin^3(\pi t)}{9\pi^2}$$

and $f^\delta = f + \delta$. *Then the exact solution*

$$\hat{x}(t) = sin(\pi t).$$

We use

$$x_0(t) = sin(\pi t) + \frac{3[t\pi^2 - t^2\pi^2 + sin^2(\pi t)]}{4\pi^2}$$

as our initial guess, so that the function $x_0 - \hat{x}$ *satisfies the source condition*

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))\frac{1}{4}$$

where $\varphi(\lambda) = \lambda$.

We choose $\alpha_0 = 1.1(\delta + \varepsilon_n)$, $\mu = 1.1, \rho = 0.1, \gamma_\rho = 0.766$ *and* $g(\gamma_\rho) = 0.461$. *The results of the computation are presented in Table 3.1. The plots of the exact solution and the approximate solution obtained are given in Figures 3.1 through 3.8.*

n	k	n_k	$\delta + \varepsilon_n$	α_k	$\|\tilde{x}_k - \hat{x}\|$	$\frac{\|\tilde{x}_k-\hat{x}\|}{(\delta+\varepsilon_n)^{1/2}}$
8	2	1	0.0135	0.0180	0.0356	0.3068
16	2	1	0.0134	0.0178	0.0432	0.3737
32	2	1	0.0133	0.0178	0.0450	0.3897
64	2	1	0.0133	0.0177	0.0455	0.3938
128	2	1	0.0133	0.0177	0.0456	0.3948
256	25	2	0.0133	0.1589	0.0456	0.3950
512	25	2	0.0133	0.1589	0.0456	0.3951
1024	25	2	0.0133	0.1589	0.0456	0.3951

Table 3.1: Iterations and corresponding error estimates

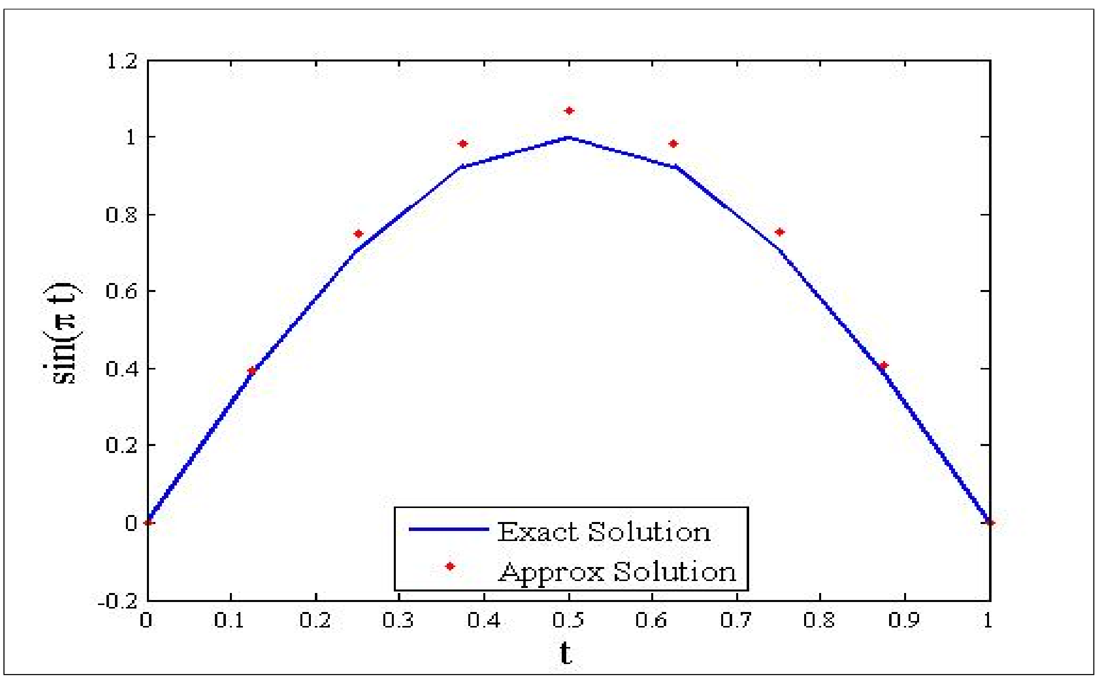

Figure 3.1: Curves of the exact and approximate solutions when n=8

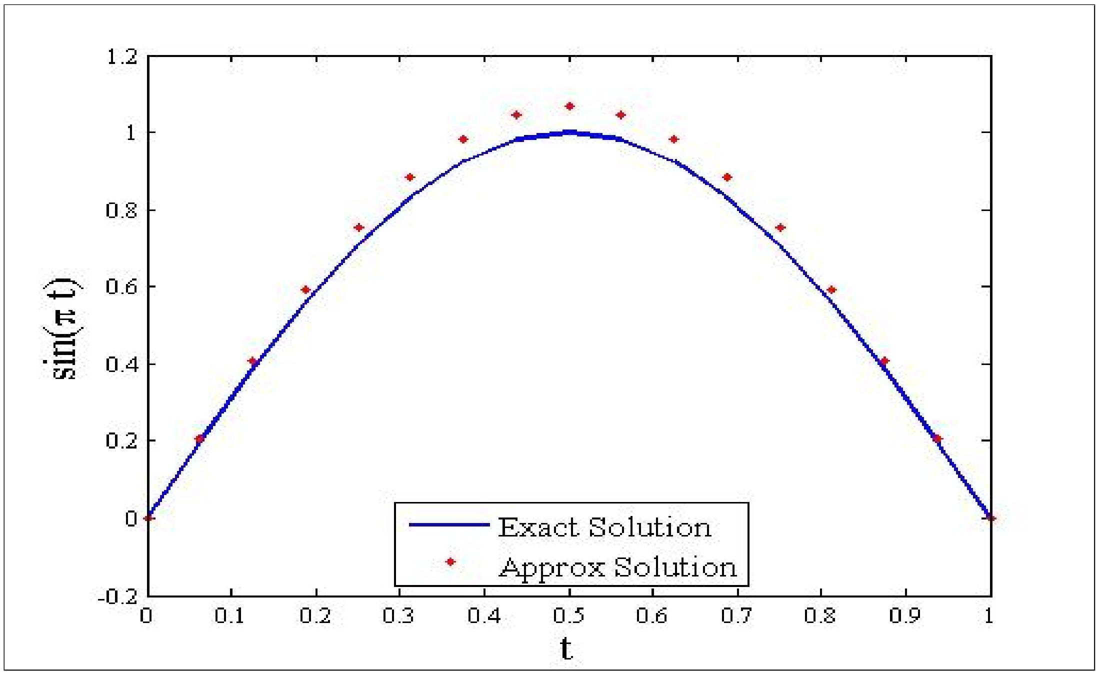

Figure 3.2: Curves of the exact and approximate solutions when n=16

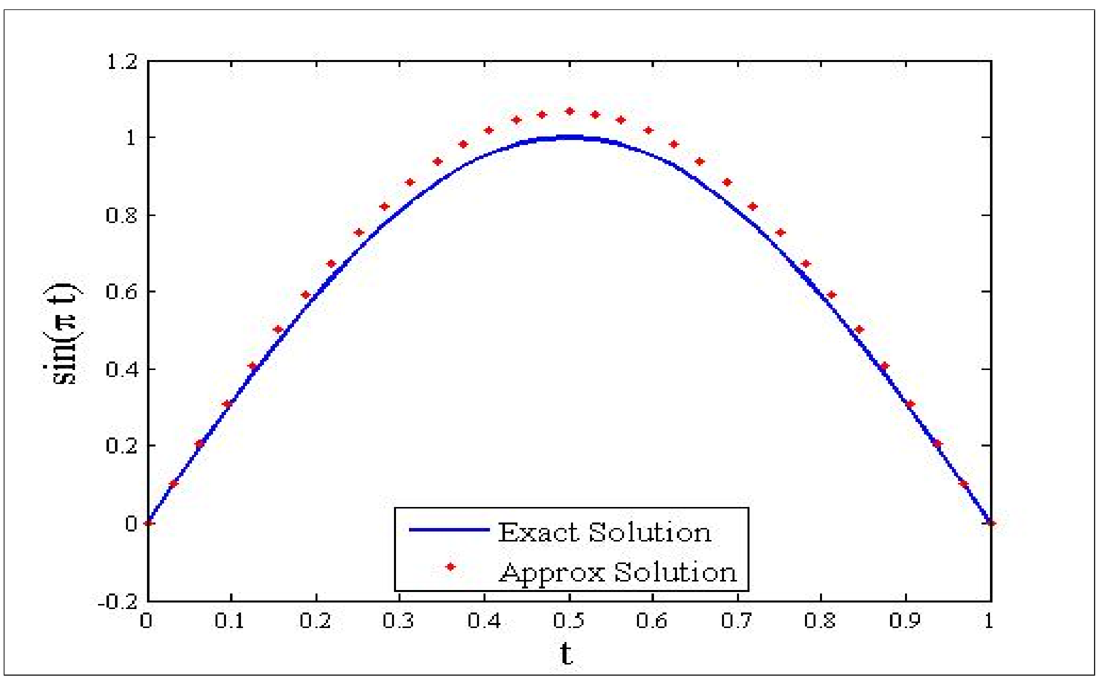

Figure 3.3: Curves of the exact and approximate solutions when n=32

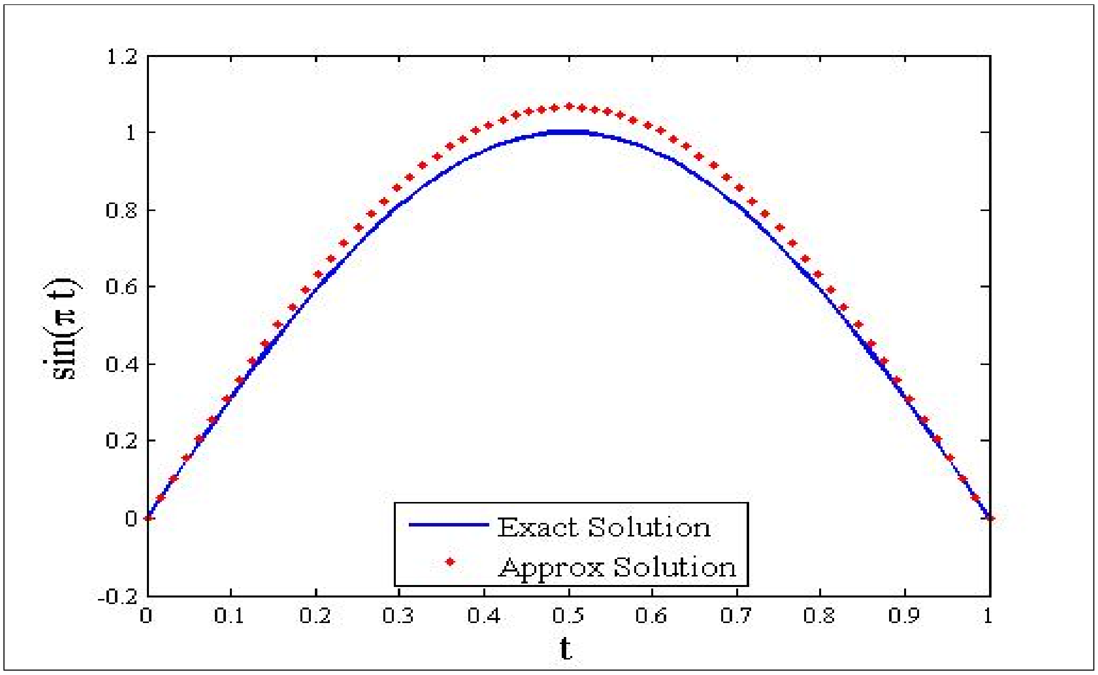

Figure 3.4: Curves of the exact and approximate solutions when n=64

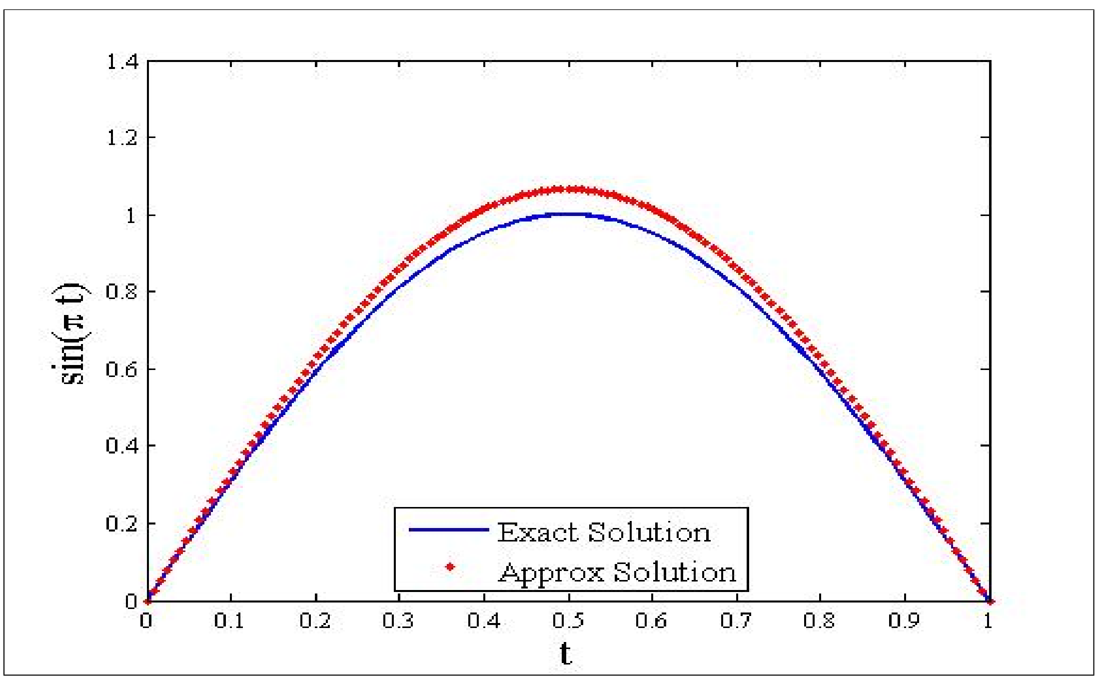

Figure 3.5: Curves of the exact and approximate solutions when n=128

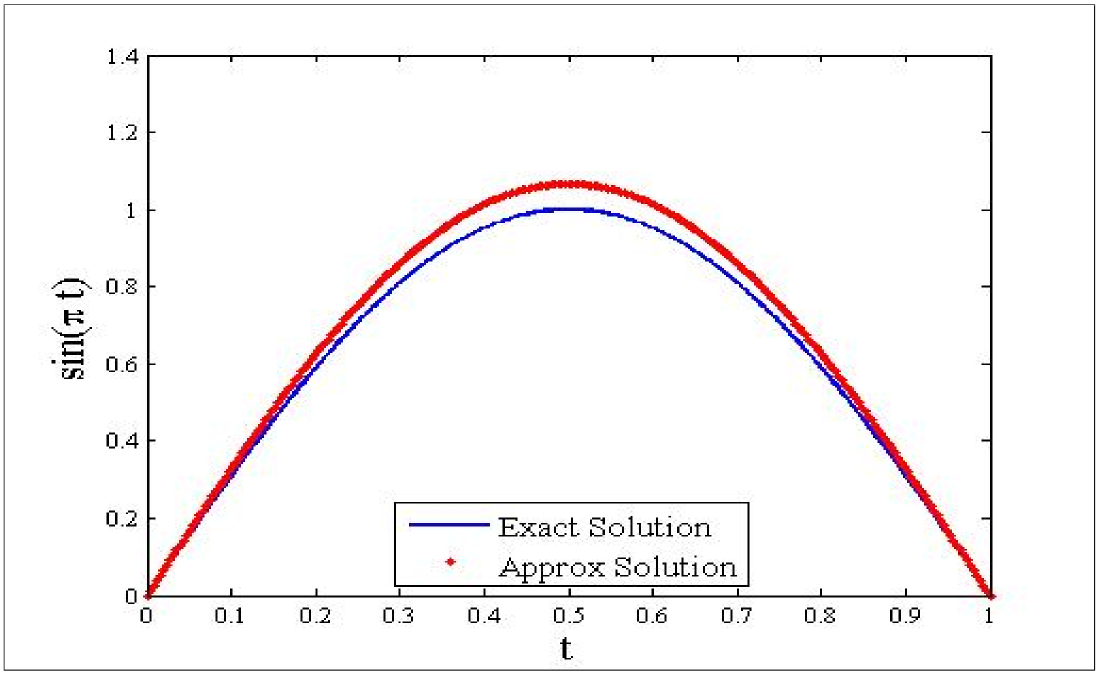

Figure 3.6: Curves of the exact and approximate solutions when n=256

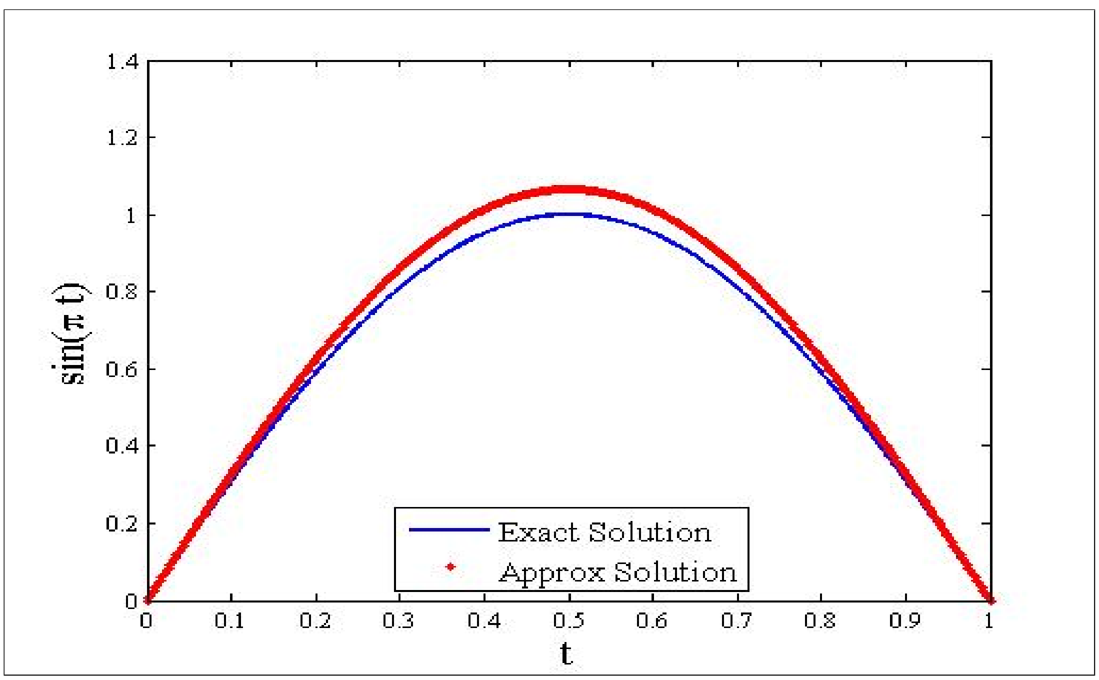

Figure 3.7: Curves of the exact and approximate solutions when n=512

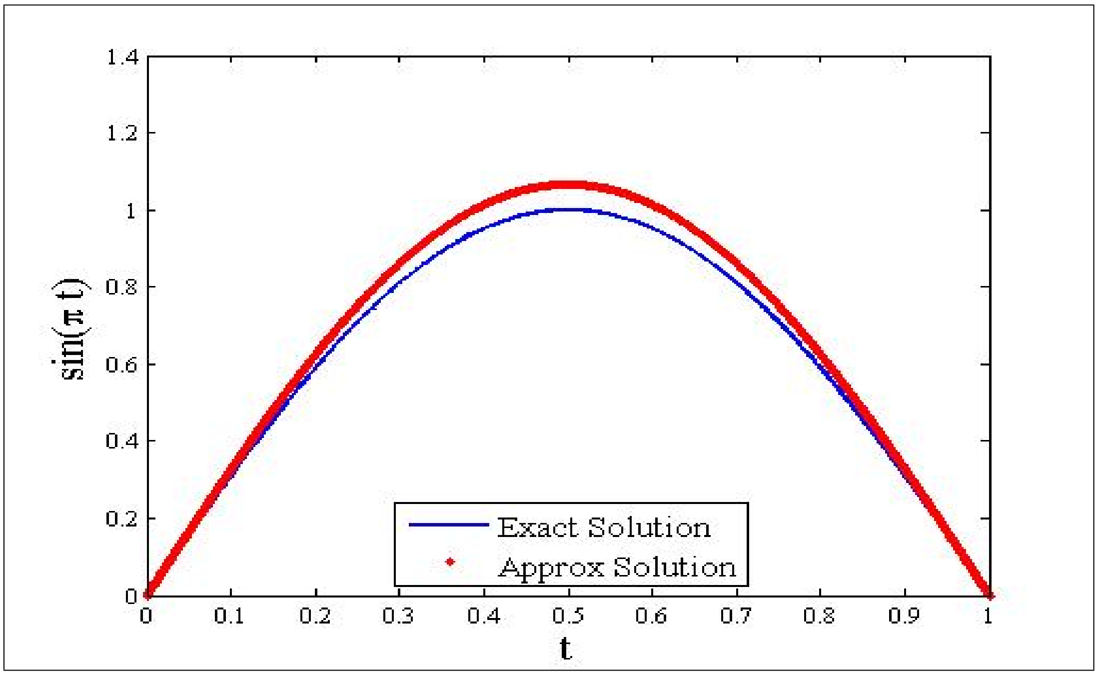

Figure 3.8: Curves of the exact and approximate solutions when n=1024

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Chapter 4

NEWTON TYPE METHODS FOR LAVRENTIEV REGULARIZATION OF NONLINEAR ILL-POSED OPERATOR EQUATIONS

A two step method, yielding cubic convergence was considered in Chapter 3, for solving nonlinear ill-posed operator equation $F(x) = f$. It is assumed that the available data is f^δ with $\|f - f^\delta\| \leq \delta$ and $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator defined on a real Hilbert space X. The method considered in this chapter converges quarticaly to the unique solution of the equation $F(x)+\alpha(x-x_0) = f^\delta$ (x_0 is the initial guess). We consider, also a finite dimensional realization of the method. An example is provided to show the efficiency of the proposed method.

4.1 INTRODUCTION

This chapter is devoted to suggest a new Newton type iterative method for apporoximating a solution of the equation (1.1.1). The method converges locally quarticaly. The proposed method for approximating the zero of (2.1.4) is defined as:

$$w_{n,\alpha}^\delta = u_{n,\alpha}^\delta - R_\alpha^{-1}(u_{n,\alpha}^\delta)[F(u_{n,\alpha}^\delta) - f^\delta + \alpha(u_{n,\alpha}^\delta - x_0)] \tag{4.1.1}$$

and

$$u_{n+1,\alpha}^\delta = w_{n,\alpha}^\delta - R_\alpha^{-1}(w_{n,\alpha}^\delta)[F(w_{n,\alpha}^\delta) - f^\delta + \alpha(w_{n,\alpha}^\delta - x_0)]. \tag{4.1.2}$$

where $R_\alpha(x)$ is as in (3.1.3) and $u^\delta_{0,\alpha} := x_0$ is the known initial guess of the solution $\hat{x}$. We will be selecting the parameter $\alpha = \alpha_i$ from some finite set defined in (2.1.3) using the adaptive method suggested by Pereverzyev and Schock (2005).

The organization of this chapter is as follows. Section 4.2 describes the convergence analysis of the method, section 4.3 deals with the error analysis carried out by choosing the regularization parameter according to the balancing principle suggested by Pereverzyev and Schock (2005). Finite dimensional realization of the method is considered in section 4.4. The error analysis in finite dimensional case is given in section 4.5. Section 4.6 gives the algorithm for implementing the proposed method, and in section 4.7 we illustrate the method through an example.

4.2 CONVERGENCE ANALYSIS

Let

$$\sigma^\delta_{n,\alpha} := \|w^\delta_{n,\alpha} - u^\delta_{n,\alpha}\|, \qquad \forall n \geq 0 \tag{4.2.3}$$

and for $0 < k_0 < \frac{2}{3}$, let $\bar{g} : (0,1) \to (0,1)$ be the function defined by

$$\bar{g}(t) = \frac{27k_0^3}{8}t^3 \qquad \forall t \in (0,1). \tag{4.2.4}$$

Note that $\sigma^\delta_{0,\alpha} = e_0$, and hence by Lemma 2.2.1 with γ_ρ as in (3.2.7) $\sigma^\delta_{0,\alpha} \leq \gamma_\rho$.

Lemma 4.2.1. *Let $w^\delta_{n,\alpha}$, $u^\delta_{n,\alpha}$ and $\sigma^\delta_{n,\alpha}$ be as in (4.1.1), (4.1.2) and (4.2.3) respectively with $\delta \in (0, \delta_0]$. And let Assumption 2.2.2 hold. Then*

(a) $\|u^\delta_{n,\alpha} - w^\delta_{n-1,\alpha}\| \leq \frac{3k_0}{2}(\sigma^\delta_{n-1,\alpha})^2$ *and*

(b) $\|u^\delta_{n,\alpha} - u^\delta_{n-1,\alpha}\| \leq \left(1 + \frac{3k_0\sigma^\delta_{n-1,\alpha}}{2}\right)\sigma^\delta_{n-1,\alpha}$.

Proof. Observe that,

$$\begin{aligned} u^\delta_{n,\alpha} - w^\delta_{n-1,\alpha} &= w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha} - R_\alpha^{-1}(w^\delta_{n-1,\alpha})[F(w^\delta_{n-1,\alpha}) - f^\delta + \alpha(w^\delta_{n-1,\alpha} - x_0)] \\ &\quad + R_\alpha^{-1}(u^\delta_{n-1,\alpha})[F(u^\delta_{n-1,\alpha}) - f^\delta + \alpha(u^\delta_{n-1,\alpha} - x_0)] \end{aligned}$$

$$
\begin{aligned}
&= w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha}\\
&\quad -R_\alpha^{-1}(w^\delta_{n-1,\alpha})[F(w^\delta_{n-1,\alpha}) - F(u^\delta_{n-1,\alpha}) + \alpha(w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha})]\\
&\quad +[R_\alpha^{-1}(u^\delta_{n-1,\alpha}) - R_\alpha^{-1}(w^\delta_{n-1,\alpha})][F(u^\delta_{n-1,\alpha}) - f^\delta + \alpha(u^\delta_{n-1,\alpha} - x_0)]\\
&= R_\alpha^{-1}(w^\delta_{n-1,\alpha})[F'(w^\delta_{n-1,\alpha})(w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha}) - (F(w^\delta_{n-1,\alpha}) - F(u^\delta_{n-1,\alpha}))]\\
&\quad +R_\alpha^{-1}(w^\delta_{n-1,\alpha})[F'(w^\delta_{n-1,\alpha}) - F'(u^\delta_{n-1,\alpha})](u^\delta_{n-1,\alpha} - w^\delta_{n-1,\alpha})\\
&:= \Gamma_1 + \Gamma_2 \qquad (4.2.5)
\end{aligned}
$$

where $\Gamma_1 = R_\alpha^{-1}(w^\delta_{n-1,\alpha})[F'(w^\delta_{n-1,\alpha})(w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha}) - (F(w^\delta_{n-1,\alpha}) - F(u^\delta_{n-1,\alpha}))]$
and $\Gamma_2 = R_\alpha^{-1}(w^\delta_{n-1,\alpha})[F'(w^\delta_{n-1,\alpha}) - F'(u^\delta_{n-1,\alpha})](u^\delta_{n-1,\alpha} - w^\delta_{n-1,\alpha})$.
Note that,

$$
\begin{aligned}
\|\Gamma_1\| &= \|R_\alpha^{-1}(w^\delta_{n-1,\alpha})\int_0^1 [F'(w^\delta_{n-1,\alpha}) - F'(u^\delta_{n-1,\alpha} + t(w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha}))]\\
&\quad \times(w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha})dt\|\\
&= \|R_\alpha^{-1}(w^\delta_{n-1,\alpha})F'(w^\delta_{n-1,\alpha}) \times\\
&\quad \int_0^1 \phi(u^\delta_{n-1,\alpha} + t(w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha}), w^\delta_{n-1,\alpha}, u^\delta_{n-1,\alpha} - w^\delta_{n-1,\alpha})dt\|\\
&\le \frac{k_0}{2}\|w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha}\|^2 \qquad (4.2.6)
\end{aligned}
$$

the last step follows from the Assumption 2.2.2 and (3.1.4). Similarly,

$$
\begin{aligned}
\|\Gamma_2\| &= \|R_\alpha^{-1}(w^\delta_{n-1,\alpha})[F'(w^\delta_{n-1,\alpha}) - F'(u^\delta_{n-1,\alpha})](u^\delta_{n-1,\alpha} - w^\delta_{n-1,\alpha})\|\\
&= \|R_\alpha^{-1}(w^\delta_{n-1,\alpha})F'(w^\delta_{n-1,\alpha})\phi(u^\delta_{n-1,\alpha}, w^\delta_{n-1,\alpha}, u^\delta_{n-1,\alpha} - w^\delta_{n-1,\alpha})\|\\
&\le k_0\|w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha}\|^2. \qquad (4.2.7)
\end{aligned}
$$

Now (a) follows from (4.2.5), (4.2.6) and (4.2.7), and (b) follows from (a) and the triangle inequality;

$\|u^\delta_{n,\alpha} - u^\delta_{n-1,\alpha}\| \le \|u^\delta_{n,\alpha} - w^\delta_{n-1,\alpha}\| + \|w^\delta_{n-1,\alpha} - u^\delta_{n-1,\alpha}\|$.

Theorem 4.2.2. *Let* $w^\delta_{n,\alpha}$, $u^\delta_{n,\alpha}$ *be as in (4.1.1) and (4.1.2) respectively with* $\delta \in (0, \delta_0]$ *and* γ_ρ, $\sigma^\delta_{n,\alpha}$ *and* $\bar{g}$ *be as in equation (3.2.7), (4.2.3) and (4.2.4) respectively. Then*

(a) $\|w^\delta_{n,\alpha} - u^\delta_{n,\alpha}\| \le \bar{g}(\sigma^\delta_{n-1,\alpha})\sigma^\delta_{n-1,\alpha}$;

(b) $\bar{g}(\sigma^\delta_{n,\alpha}) \le \bar{g}(\gamma_\rho)^{4^n}, \qquad \forall n \ge 0$;

(c) $\sigma^\delta_{n,\alpha} \leq \bar{g}(\gamma_\rho)^{(4^n-1)/3}\gamma_\rho \qquad \forall n \geq 0.$

Proof. We have,

$$\begin{aligned}
w^\delta_{n,\alpha} - u^\delta_{n,\alpha} &= u^\delta_{n,\alpha} - w^\delta_{n-1,\alpha} - R^{-1}_\alpha(u^\delta_{n,\alpha})[F(u^\delta_{n,\alpha}) - f^\delta + \alpha(u^\delta_{n,\alpha} - x_0)] \\
&\quad + R^{-1}_\alpha(w^\delta_{n-1,\alpha})[F(w^\delta_{n-1,\alpha}) - f^\delta + \alpha(w^\delta_{n-1,\alpha} - x_0)] \\
&= u^\delta_{n,\alpha} - w^\delta_{n-1,\alpha} - R^{-1}_\alpha(u^\delta_{n,\alpha})[F(u^\delta_{n,\alpha}) - F(w^\delta_{n-1,\alpha}) + \alpha(u^\delta_{n,\alpha} - w^\delta_{n-1,\alpha})] \\
&\quad + [R^{-1}_\alpha(u^\delta_{n-1,\alpha}) - R^{-1}_\alpha(u^\delta_{n,\alpha})][F(w^\delta_{n-1,\alpha}) - f^\delta + \alpha(w^\delta_{n-1,\alpha} - x_0)] \\
&= R^{-1}_\alpha(u^\delta_{n,\alpha})[F'(u^\delta_{n,\alpha})(u^\delta_{n,\alpha} - w^\delta_{n-1,\alpha}) - (F(u^\delta_{n,\alpha}) - F(w^\delta_{n-1,\alpha}))] \\
&\quad + R^{-1}_\alpha(u^\delta_{n,\alpha})[F'(u^\delta_{n,\alpha}) - F'(w^\delta_{n-1,\alpha})](w^\delta_{n-1,\alpha} - u^\delta_{n,\alpha}) \\
&:= \Gamma_3 + \Gamma_4 \qquad (4.2.8)
\end{aligned}$$

where $\Gamma_3 = R^{-1}_\alpha(u^\delta_{n,\alpha})[F'(u^\delta_{n,\alpha})(u^\delta_{n,\alpha} - w^\delta_{n-1,\alpha}) - (F(u^\delta_{n,\alpha}) - F(w^\delta_{n-1,\alpha}))]$
and $\Gamma_4 = R^{-1}_\alpha(u^\delta_{n,\alpha})[F'(u^\delta_{n,\alpha}) - F'(w^\delta_{n-1,\alpha})](w^\delta_{n-1,\alpha} - u^\delta_{n,\alpha})$.
Analogous to the proof of (4.2.6) and (4.2.7) one can prove that

$$\|\Gamma_3\| \leq \frac{k_0}{2}\|u^\delta_{n,\alpha} - w^\delta_{n-1,\alpha}\|^2 \qquad (4.2.9)$$

and

$$\|\Gamma_4\| \leq k_0\|u^\delta_{n,\alpha} - w^\delta_{n-1,\alpha}\|^2. \qquad (4.2.10)$$

Now (a) follows from the Lemma 4.2.1, (4.2.8), (4.2.9) and (4.2.10). Again, since for $\mu \in (0,1)$, $\bar{g}(\mu t) = \mu^3\bar{g}(t)$, for all $t \in (0,1)$, by (a) we get,

$$\bar{g}(\sigma^\delta_{n,\alpha}) \leq \bar{g}(\sigma_0)^{4^n} \qquad (4.2.11)$$

and

$$\begin{aligned}
\sigma^\delta_{n,\alpha} &\leq \bar{g}(\sigma^\delta_{n-1,\alpha})\sigma^\delta_{n-1,\alpha} \\
&\leq \bar{g}(\sigma_0)^{4^{n-1}}\bar{g}(\sigma^\delta_{n-2,\alpha})\sigma^\delta_{n-2,\alpha} \\
&\leq \bar{g}(\sigma_0)^{4^{n-1}}\bar{g}(\sigma_0)^{4^{n-2}}\bar{g}(\sigma^\delta_{n-3,\alpha})\sigma^\delta_{n-3,\alpha} \\
&\leq \bar{g}(\sigma_0)^{4^{n-1}+4^{n-2}+\cdots+1}\sigma_0 \\
&\leq \bar{g}(\sigma_0)^{(4^n-1)/3}\sigma_0 \qquad (4.2.12)
\end{aligned}$$

provided $\sigma^\delta_{n,\alpha} < 1$. But $\sigma^\delta_{n,\alpha} < 1$ by Lemma 2.2.1 with γ_ρ as in (3.2.7), (4.2.4) and (4.2.12). Now (b) and (c) follow from Lemma 2.2.1 with γ_ρ as in (3.2.7), (4.2.11), (4.2.12) and the relation $\bar{g}(\sigma_0) \leq \bar{g}(\gamma_\rho)$. This completes the proof of the theorem.

Theorem 4.2.3. *Suppose* $0 < \bar{g}(\gamma_\rho) < 1$, $r = \left[\frac{1}{1-\bar{g}(\gamma_\rho)} + \frac{3k_0}{2}\frac{\gamma_\rho}{1-\bar{g}(\gamma_\rho)^2}\right]\gamma_\rho$ *and let assumptions of Theorem 4.2.2 hold. Then* $u^\delta_{n,\alpha}, w^\delta_{n,\alpha} \in B_r(x_0)$ *for all* $n \geq 0$.

Proof. Note that by (b) of Lemma 4.2.1 we have,

$$\begin{aligned}\|u_{1,\alpha}^{\delta}-x_0\| &\leq [1+\frac{3k_0}{2}\sigma_0]\sigma_0 \\ &\leq [1+\frac{3k_0}{2}\gamma_\rho]\gamma_\rho \\ &< r\end{aligned} \tag{4.2.13}$$

i.e., $u_{1,\alpha}^{\delta}\in B_r(x_0)$. Again note that from (a) of Theorem 4.2.2 and (4.2.13) we get,

$$\begin{aligned}\|w_{1,\alpha}^{\delta}-x_0\| &\leq \|w_{1,\alpha}^{\delta}-u_{1,\alpha}^{\delta}\|+\|u_{1,\alpha}^{\delta}-x_0\| \\ &\leq \bar{g}(\sigma_0)\sigma_0+(1+\frac{3k_0}{2}\sigma_0)\sigma_0 \\ &\leq [1+\bar{g}(\sigma_0)+\frac{3k_0}{2}\sigma_0]\sigma_0 \\ &\leq [1+\bar{g}(\gamma_\rho)+\frac{3k_0}{2}\gamma_\rho]\gamma_\rho \\ &< r\end{aligned}$$

i.e., $w_{1,\alpha}^{\delta}\in B_r(x_0)$. Further by (b) of Lemma 4.2.1 and (4.2.13) we have,

$$\begin{aligned}\|u_{2,\alpha}^{\delta}-x_0\| &\leq \|u_{2,\alpha}^{\delta}-u_{1,\alpha}^{\delta}\|+\|u_{1,\alpha}^{\delta}-x_0\| \\ &\leq (1+\frac{3k_0}{2}\sigma_{1,\alpha}^{\delta})\sigma_{1,\alpha}^{\delta}+(1+\frac{3k_0}{2}\sigma_0)\sigma_0 \\ &\leq [1+\frac{3k_0}{2}\bar{g}(\sigma_0)\sigma_0]\bar{g}(\sigma_0)\sigma_0+(1+\frac{3k_0}{2}\sigma_0)\sigma_0 \\ &\leq [1+\bar{g}(\sigma_0)+\frac{3k_0}{2}\sigma_0(1+\bar{g}(\sigma_0)^2)]\sigma_0 \\ &\leq [1+\bar{g}(\gamma_\rho)+\frac{3k_0}{2}\gamma_\rho(1+\bar{g}(\gamma_\rho)^2)]\gamma_\rho \\ &< r\end{aligned} \tag{4.2.14}$$

and by (a) of Theorem 4.2.2 and (4.2.14) we have,

$$\begin{aligned}\|w_{2,\alpha}^{\delta}-x_0\| &\leq \|w_{2,\alpha}^{\delta}-u_{2,\alpha}^{\delta}\|+\|u_{2,\alpha}^{\delta}-x_0\| \\ &\leq \bar{g}(\sigma_{1,\alpha}^{\delta})\sigma_{1,\alpha}^{\delta}+[1+\bar{g}(\sigma_0)+\frac{3k_0}{2}\sigma_0(1+\bar{g}(\sigma_0)^2)]\sigma_0 \\ &\leq \bar{g}(\sigma_0)^5\sigma_0+[1+\bar{g}(\sigma_0)+\frac{3k_0}{2}\sigma_0(1+\bar{g}(\sigma_0)^2)]\sigma_0 \\ &\leq [1+\bar{g}(\sigma_0)+\bar{g}(\sigma_0)^5+\frac{3k_0}{2}\sigma_0(1+\bar{g}(\sigma_0)^2)]\sigma_0 \\ &\leq [1+\bar{g}(\sigma_0)+\bar{g}(\sigma_0)^2+\frac{3k_0}{2}\sigma_0(1+\bar{g}(\sigma_0)^2)]\sigma_0 \\ &\leq [1+\bar{g}(\gamma_\rho)+\bar{g}(\gamma_\rho)^2+\frac{3k_0}{2}\gamma_\rho(1+\bar{g}(\gamma_\rho)^2)]\gamma_\rho < r\end{aligned}$$

i.e., $u^\delta_{2,\alpha}$, $w^\delta_{2,\alpha} \in B_r(x_0)$. Continuing this way one can prove that $u^\delta_{n,\alpha}$, $w^\delta_{n,\alpha} \in B_r(x_0)$, $\forall n \geq 0$. This completes the proof.

The main result of this section is the following theorem.

Theorem 4.2.4. *Let $0 < \bar{g}(\gamma_\rho) < 1$, $w^\delta_{n,\alpha}$ and $u^\delta_{n,\alpha}$ be as in (4.1.1) and (4.1.2) respectively with $\delta \in (0, \delta_0]$ and assumptions of the Theorem 4.2.3 hold. Then $(u^\delta_{n,\alpha})$ is Cauchy sequence in $B_r(x_0)$ and converges to $x^\delta_\alpha \in \overline{B_r(x_0)}$. Further $F(x^\delta_\alpha) + \alpha(x^\delta_\alpha - x_0) = f^\delta$ and*

$$\|u^\delta_{n,\alpha} - x^\delta_\alpha\| \leq Ce^{-\gamma 4^n}$$

where $C = \left[\frac{1}{1-\bar{g}(\gamma_\rho)^4} + \frac{3k_0\gamma_\rho}{2}\frac{1}{1-(\bar{g}(\gamma_\rho)^2)^4}\bar{g}(\gamma_\rho)^{4^n}\right]\gamma_\rho$ and $\gamma = -\log \bar{g}(\gamma_\rho)$.

Proof. Using the relation (b) of Lemma 4.2.1 and (c) of Theorem 4.2.2, we obtain,

$$\begin{aligned}
\|u^\delta_{n+m,\alpha} - u^\delta_{n,\alpha}\| &\leq \sum_{i=0}^{m-1} \|u^\delta_{n+i+1,\alpha} - u^\delta_{n+i,\alpha}\| \\
&\leq \sum_{i=0}^{m-1} \left[1 + \frac{3k_0\sigma_0}{2}\bar{g}(\sigma_0)^{4^{n+i}}\right] \bar{g}(\sigma_0)^{4^{n+i}}\sigma_0 \\
&\leq [(1 + \bar{g}(\sigma_0)^4 + \bar{g}(\sigma_0)^{4^2} + \cdots + \bar{g}(\sigma_0)^{4^m}) + \frac{3k_0\sigma_0}{2} \\
&\quad (1 + (\bar{g}(\sigma_0)^2)^4 + (\bar{g}(\sigma_0)^2)^{4^2} + \cdots + (\bar{g}(\sigma_0)^2)^{4^m})\bar{g}(\sigma_0)^{4^n}]\bar{g}(\sigma_0)^{4^n}\sigma_0 \\
&\leq [(1 + \bar{g}(\gamma_\rho)^4 + \bar{g}(\gamma_\rho)^{4^2} + \cdots + \bar{g}(\gamma_\rho)^{4^m}) + \frac{3k_0\gamma_\rho}{2} \\
&\quad (1 + (\bar{g}(\gamma_\rho)^2)^4 + (\bar{g}(\gamma_\rho)^2)^{4^2} + \cdots + (\bar{g}(\gamma_\rho)^2)^{4^m})\bar{g}(\gamma_\rho)^{4^n}]\bar{g}(\gamma_\rho)^{4^n}\gamma_\rho \\
&\leq C\bar{g}(\gamma_\rho)^{4^n} \\
&\leq Ce^{-\gamma 4^n}.
\end{aligned}$$

Thus $u^\delta_{n,\alpha}$ is a Cauchy sequence in $B_r(x_0)$ and hence it converges, say, to $x^\delta_\alpha \in \overline{B_r(x_0)}$. Observe that,

$$\begin{aligned}
\|F(u^\delta_{n,\alpha}) - f^\delta + \alpha(u^\delta_{n,\alpha} - x_0)\| &= \|R_\alpha(u^\delta_{n,\alpha})(u^\delta_{n,\alpha} - w^\delta_{n,\alpha})\| \\
&\leq \|R_\alpha(u^\delta_{n,\alpha})\|\|u^\delta_{n,\alpha} - w^\delta_{n,\alpha}\| \\
&\leq (C_F + \alpha)\bar{g}(\sigma_0)^{4^n}\sigma_0 \\
&\leq (C_F + \alpha)\bar{g}(\gamma_\rho)^{4^n}\gamma_\rho. \qquad (4.2.15)
\end{aligned}$$

Now by letting $n \to \infty$ in (4.2.15) we obtain $F(x^\delta_\alpha) + \alpha(x^\delta_\alpha - x_0) = f^\delta$. This completes the proof.

4.3 ERROR BOUNDS UNDER SOURCE CONDITIONS

The objective of this section is to obtain an error estimate for $\|u_{n,\alpha}^{\delta} - \hat{x}\|$ under a source condition on $x_0 - \hat{x}$.

Combining the estimates in Proposition 2.3.2, Theorem 2.3.3 and Theorem 4.2.4 we obtain the following;

Theorem 4.3.1. *Let $u_{n,\alpha}^{\delta}$ be as in (4.1.2) and let assumptions in Proposition 2.3.2, Theorem 2.3.3 and Theorem 4.2.4 be satisfied. Then*

$$\|u_{n,\alpha}^{\delta} - \hat{x}\| \leq Ce^{-\gamma 4^n} + C_1 \left[\varphi(\alpha) + \frac{\delta}{\alpha}\right]$$

where $C_1 = k_0 r + 1$.

Let

$$\bar{C} := \max\{C, k_0 r)\} + 1, \tag{4.3.16}$$

and let

$$n_\delta := \min\left\{n : e^{-\gamma 4^n} \leq \frac{\delta}{\alpha}\right\}. \tag{4.3.17}$$

Theorem 4.3.2. *Let $u_{n_\delta,\alpha}^{\delta}$ be as in (4.1.2) with $n = n_\delta$ and the assumptions in Theorem 4.3.1 be satisfied. Let $\bar{C}$ be as in (4.3.16) and n_δ be as in (4.3.17). Then*

$$\|u_{n_\delta,\alpha}^{\delta} - \hat{x}\| \leq \bar{C}\left[\varphi(\alpha) + \frac{\delta}{\alpha}\right].$$

4.3.1 A priori choice of the parameter

Theorem 4.3.3. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and the assumptions in Theorem 4.3.2 hold. For $\delta > 0$, let $\alpha := \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$ and let n_δ be as in (4.3.17). Then*

$$\|u_{n_\delta,\alpha}^{\delta} - \hat{x}\| = O(\psi^{-1}(\delta)).$$

4.3.2 An adaptive choice of the parameter

Let

$$D_N(\alpha) := \{\alpha_i = \mu^i \alpha_0, i = 0, 1, \cdots, N\}$$

where $\mu > 1$.

Let

$$n_i := \min\left\{n : e^{-\gamma 4^n} \leq \frac{\delta}{\alpha_i}\right\}.$$

Then for $i = 0, 1, \cdots, N$, we have

$$\|u^\delta_{n_i,\alpha_i} - x^\delta_{\alpha_i}\| \leq C\frac{\delta}{\alpha_i}, \quad \forall i = 0, 1, \cdots N.$$

Let $u_i := u^\delta_{n_i,\alpha_i}$. In this Chapter we select $\alpha = \alpha_i$ from the set $D_N(\alpha)$ for computing u_i, for each $i = 0, 1, \cdots, N$.

Proof of the following theorem is analogous to the proof of Theorem 2.3.7.

Theorem 4.3.4. *Assume that there exists $i \in \{0, 1, 2, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}$. Let assumptions of Theorem 4.3.2 and Theorem 4.3.3 hold and let*

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}\right\} < N,$$

$$k := \max\left\{i : \|u_i - u_j\| \leq 4\bar{C}\frac{\delta}{\alpha_j}, \quad j = 0, 1, 2, \cdots, i-1\right\}.$$

Then $l \leq k$ and $\|\hat{x} - u_k\| \leq c\psi^{-1}(\delta)$ where $c = 6\bar{C}\mu$.

4.4 PROJECTION METHOD AND ITS CONVERGENCE

We consider the following sequence defined iteratively by

$$w^{h,\delta}_{n,\alpha} = u^{h,\delta}_{n,\alpha} - R^{-1}_\alpha(u^{h,\delta}_{n,\alpha})P_h[F(u^{h,\delta}_{n,\alpha}) - f^\delta + \alpha(u^{h,\delta}_{n,\alpha} - x_0)] \tag{4.4.18}$$

and

$$u^{h,\delta}_{n+1,\alpha} = w^{h,\delta}_{n,\alpha} - R^{-1}_\alpha(w^{h,\delta}_{n,\alpha})P_h[F(w^{h,\delta}_{n,\alpha}) - f^\delta + \alpha(w^{h,\delta}_{n,\alpha} - x_0)] \tag{4.4.19}$$

where $R_\alpha(x) := P_hF'(x)P_h + \alpha P_h$ and $u_{0,\alpha}^{h,\delta} := P_hx_0$, for obtaining an approximation for x_α^δ in the finite dimensional subspace $R(P_h)$ of X. Note that the iteration (4.4.18) and (4.4.19) are the finite dimensional realization of the iteration (4.1.1) and (4.1.2) in section 4.1. We will be selecting the parameter $\alpha = \alpha_i$ from some finite set defined in (2.1.3) using the adaptive method considered by Pereverzyev and Schock (2005).

Let

$$\sigma_{n,\alpha}^{h,\delta} := \|w_{n,\alpha}^{h,\delta} - u_{n,\alpha}^{h,\delta}\|, \qquad \forall n \geq 0 \tag{4.4.20}$$

and for $0 < k_0 < \frac{2}{3(1+\frac{\varepsilon_0}{\alpha_0})}$, let $\tau : (0,1) \to (0,1)$ be the function defined by

$$\tau(t) = \frac{27k_0^3}{8}(1+\frac{\varepsilon_0}{\alpha_0})^3 t^3 \qquad \forall t \in (0,1). \tag{4.4.21}$$

Lemma 4.4.1. *Let $w_{n,\alpha}^{h,\delta}$, $u_{n,\alpha}^{h,\delta}$ and $\sigma_{n,\alpha}^{h,\delta}$ be as in (4.4.18), (4.4.19) and (4.4.20) respectively with $\delta \in (0,\delta_0]$. And let Assumption 2.2.2 hold. Then*

(a) $\|u_{n,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta}\| \leq \frac{3k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})(\sigma_{n-1,\alpha}^{h,\delta})^2$ *and*

(b) $\|u_{n,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta}\| \leq (1+\frac{3k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\sigma_{n-1,\alpha}^{h,\delta})\sigma_{n-1,\alpha}^{h,\delta}$.

Proof. Observe that,

$$\begin{aligned}
u_{n,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta} &= w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta} - R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta})P_h[F(w_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(w_{n-1,\alpha}^{h,\delta} - x_0)] \\
&\quad + R_\alpha^{-1}(u_{n-1,\alpha}^{h,\delta})P_h[F(u_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(u_{n-1,\alpha}^{h,\delta} - x_0)] \\
&= w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta} \\
&\quad - R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta})P_h[F(w_{n-1,\alpha}^{h,\delta}) - F(u_{n-1,\alpha}^{h,\delta}) + \alpha(w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta})] \\
&\quad + [R_\alpha^{-1}(u_{n-1,\alpha}^{h,\delta}) - R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta})]P_h[F(u_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(u_{n-1,\alpha}^{h,\delta} - x_0)]
\end{aligned}$$

$$\begin{aligned}
&= R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta})P_h[F'(w_{n-1,\alpha}^{h,\delta})(w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta}) - (F(w_{n-1,\alpha}^{h,\delta}) - F(u_{n-1,\alpha}^{h,\delta}))] \\
&\quad + R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta})P_h(F'(w_{n-1,\alpha}^{h,\delta}) - F'(u_{n-1,\alpha}^{h,\delta}))(u_{n-1,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta}) \\
&:= \Gamma_1 + \Gamma_2
\end{aligned} \tag{4.4.22}$$

where $\Gamma_1 = R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta})P_h[F'(w_{n-1,\alpha}^{h,\delta})(w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta}) - (F(w_{n-1,\alpha}^{h,\delta}) - F(u_{n-1,\alpha}^{h,\delta}))]$ and $\Gamma_2 = R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta})P_h[F'(w_{n-1,\alpha}^{h,\delta}) - F'(u_{n-1,\alpha}^{h,\delta})](u_{n-1,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta})$.

Note that,

$$\begin{aligned}
\|\Gamma_1\| &= \|R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta})P_h\int_0^1[F'(w_{n-1,\alpha}^{h,\delta}) - F'(u_{n-1,\alpha}^{h,\delta} + t(w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta}))] \\
&\quad \times(w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta})dt\| \\
&= \|R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta})P_hF'(w_{n-1,\alpha}^{h,\delta}) \times \\
&\quad \int_0^1[\phi(u_{n-1,\alpha}^{h,\delta} + t(w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta}), w_{n-1,\alpha}^{h,\delta}, u_{n-1,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta})]dt\| \\
&\leq \frac{k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\|w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta}\|^2 \qquad (4.4.23)
\end{aligned}$$

the last step follows from the Assumption 2.2.2 and Lemma 2.4.1. Similarly,

$$\|\Gamma_2\| \leq k_0(1+\frac{\varepsilon_0}{\alpha_0})\|w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta}\|^2. \qquad (4.4.24)$$

So, (a) follows from (4.4.22), (4.4.23) and (4.4.24). And (b) follows from (a) and the triangle inequality;

$$\|u_{n,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta}\| \leq \|u_{n,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta}\| + \|w_{n-1,\alpha}^{h,\delta} - u_{n-1,\alpha}^{h,\delta}\|.$$

Theorem 4.4.2. *Let $w_{n,\alpha}^{h,\delta}$, $u_{n,\alpha}^{h,\delta}$ be as in (4.4.18) and (4.4.19) respectively with $\delta \in (0, \delta_0]$ and γ_ρ, $\sigma_{n,\alpha}^{h,\delta}$ and τ be as in equation (3.4.21), (4.4.20) and (4.4.21) respectively. Then*

(a) $\|w_{n,\alpha}^{h,\delta} - u_{n,\alpha}^{h,\delta}\| \leq \tau(\sigma_{n-1,\alpha}^{h,\delta})\sigma_{n-1,\alpha}^{h,\delta}$;

(b) $\tau(\sigma_{n,\alpha}^{h,\delta}) \leq \tau(\gamma_\rho)^{4^n}, \qquad \forall n \geq 0$;

(c) $\sigma_{n,\alpha}^{h,\delta} \leq \tau(\gamma_\rho)^{\frac{4^n-1}{3}}\gamma_\rho \qquad \forall n \geq 0.$

Proof. We have,

$$\begin{aligned}
w_{n,\alpha}^{h,\delta} - u_{n,\alpha}^{h,\delta} &= u_{n,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta} - R_\alpha^{-1}(u_{n,\alpha}^{h,\delta})P_h[F(u_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(u_{n,\alpha}^{h,\delta} - x_0)] \\
&\quad + R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta})P_h[F(w_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(w_{n-1,\alpha}^{h,\delta} - x_0)] \\
&= u_{n,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta} \\
&\quad - R_\alpha^{-1}(u_{n,\alpha}^{h,\delta})P_h[F(u_{n,\alpha}^{h,\delta}) - F(w_{n-1,\alpha}^{h,\delta}) + \alpha(u_{n,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta})] \\
&\quad + [R_\alpha^{-1}(w_{n-1,\alpha}^{h,\delta}) - R_\alpha^{-1}(u_{n,\alpha}^{h,\delta})]P_h[F(w_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(w_{n-1,\alpha}^{h,\delta} - x_0)] \\
&= R_\alpha^{-1}(u_{n,\alpha}^{h,\delta})P_h[F'(u_{n,\alpha}^{h,\delta})(u_{n,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta}) - (F(u_{n,\alpha}^{h,\delta}) - F(w_{n-1,\alpha}^{h,\delta}))] \\
&\quad + R_\alpha^{-1}(u_{n,\alpha}^{h,\delta})P_h[F'(u_{n,\alpha}^{h,\delta}) - F'(w_{n-1,\alpha}^{h,\delta})](w_{n-1,\alpha}^{h,\delta} - u_{n,\alpha}^{h,\delta}) \\
&:= \Gamma_3 + \Gamma_4 \qquad (4.4.25)
\end{aligned}$$

where $\Gamma_3 = R_\alpha^{-1}(u_{n,\alpha}^{h,\delta})P_h[F'(u_{n,\alpha}^{h,\delta})(u_{n,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta}) - (F(u_{n,\alpha}^{h,\delta}) - F(w_{n-1,\alpha}^{h,\delta}))]$ and $\Gamma_4 = R_\alpha^{-1}(u_{n,\alpha}^{h,\delta})P_h[F'(u_{n,\alpha}^{h,\delta}) - F'(w_{n-1,\alpha}^{h,\delta})](w_{n-1,\alpha}^{h,\delta} - u_{n,\alpha}^{h,\delta})$. Analogous to the proof of (4.4.23) and (4.4.24) one can prove that

$$\|\Gamma_3\| \leq \frac{k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\|u_{n,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta}\|^2 \tag{4.4.26}$$

and

$$\|\Gamma_4\| \leq k_0(1+\frac{\varepsilon_0}{\alpha_0})\|u_{n,\alpha}^{h,\delta} - w_{n-1,\alpha}^{h,\delta}\|^2. \tag{4.4.27}$$

Now (a) follows from the Lemma 4.4.1, (4.4.25), (4.4.26) and (4.4.27). Again, since for $\mu \in (0,1)$, $\tau(\mu t) = \mu^3\tau(t)$, for all $t \in (0,1)$, by (a) we get,

$$\tau(\sigma_{n,\alpha}^{h,\delta}) \leq \tau(\sigma_0)^{4^n} \tag{4.4.28}$$

and

$$\begin{aligned} \sigma_{n,\alpha}^{h,\delta} \leq \tau(\sigma_{n-1,\alpha}^{h,\delta})\sigma_{n-1,\alpha}^{h,\delta} &\leq \tau(\sigma_0)^{4^{n-1}}\tau(\sigma_{n-2,\alpha}^{h,\delta})\sigma_{n-2,\alpha}^{h,\delta} \\ &\leq \tau(\sigma_0)^{4^{n-1}}\tau(\sigma_0)^{4^{n-2}}\tau(\sigma_{n-3,\alpha}^{h,\delta})\sigma_{n-3,\alpha}^{h,\delta} \\ &\leq \tau(\sigma_0)^{4^{n-1}+4^{n-2}+\cdots+1}\sigma_0 \\ &\leq \tau(\sigma_0)^{\frac{4^n-1}{3}}\sigma_0 \end{aligned} \tag{4.4.29}$$

provided $\sigma_{n,\alpha}^{h,\delta} < 1$. But $\sigma_{n,\alpha}^{h,\delta} < 1$ by Lemma 2.4.2 with γ_ρ as in (3.4.21), (4.4.21) and (4.4.29). Now (b) and (c) follow from Lemma 2.4.2 with γ_ρ as in (3.4.21), (4.4.28), (4.4.29) and the relation $\tau(\sigma_0) \leq \tau(\gamma_\rho)$. This completes the proof of the theorem.

Theorem 4.4.3. *Suppose* $0 < \tau(\gamma_\rho) < 1$, $r = \left[\frac{1}{1-\tau(\gamma_\rho)} + \frac{3k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\frac{\gamma_\rho}{1-\tau(\gamma_\rho)^2}\right]\gamma_\rho$ *and let assumptions of Theorem 4.4.2 hold. Then* $u_{n,\alpha}^{h,\delta}, w_{n,\alpha}^{h,\delta} \in B_r(P_hx_0)$ *for all* $n \geq 0$.

Proof. Note that by (b) of Lemma 4.4.1 we have,

$$\begin{aligned} \|u_{1,\alpha}^{h,\delta} - P_hx_0\| &= \|u_{1,\alpha}^{h,\delta} - u_{0,\alpha}^{h,\delta}\| \\ &\leq [1+\frac{3k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\sigma_0]\sigma_0 \\ &\leq [1+\frac{3k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\gamma_\rho]\gamma_\rho \\ &< r \end{aligned} \tag{4.4.30}$$

i.e., $u_{1,\alpha}^{h,\delta} \in B_r(P_h x_0)$. Again note that from (a) of Theorem 4.4.2 and (4.4.30) we get

$$\begin{aligned}
\|w_{1,\alpha}^{h,\delta} - P_h x_0\| &\leq \|w_{1,\alpha}^{h,\delta} - u_{1,\alpha}^{h,\delta}\| + \|u_{1,\alpha}^{h,\delta} - P_h x_0\| \\
&\leq \tau(\sigma_0)\sigma_0 + (1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\sigma_0)\sigma_0 \\
&\leq (1 + \tau(\sigma_0) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\sigma_0)\sigma_0 \\
&\leq (1 + \tau(\gamma_\rho) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho)\gamma_\rho \\
&< r
\end{aligned}$$

i.e., $w_{1,\alpha}^{h,\delta} \in B_r(P_h x_0)$. Further by (b) of Lemma 4.4.1 and (4.4.30) we have

$$\begin{aligned}
\|u_{2,\alpha}^{h,\delta} - P_h x_0\| &\leq \|u_{2,\alpha}^{h,\delta} - u_{1,\alpha}^{h,\delta}\| + \|u_{1,\alpha}^{h,\delta} - P_h x_0\| \\
&\leq (1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\sigma_{1,\alpha}^{h,\delta})\sigma_{1,\alpha}^{h,\delta} + (1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\sigma_0)\sigma_0 \\
&\leq (1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\tau(\sigma_0)\sigma_0)\tau(\sigma_0)\sigma_0 + (1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\sigma_0)\sigma_0 \\
&\leq (1 + \tau(\sigma_0) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\sigma_0(1 + \tau(\sigma_0)^2))\sigma_0 \qquad (4.4.31) \\
&\leq (1 + \tau(\gamma_\rho) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho(1 + \tau(\gamma_\rho)^2))\gamma_\rho \\
&< r
\end{aligned}$$

and by (a) of Theorem 4.4.2 and (4.4.31) we have

$$\begin{aligned}
\|w_{2,\alpha}^{h,\delta} - P_h x_0\| &\leq \|w_{2,\alpha}^{h,\delta} - u_{2,\alpha}^{h,\delta}\| + \|u_{2,\alpha}^{h,\delta} - P_h x_0\| \\
&\leq \tau(\sigma_{1,\alpha}^{h,\delta})\sigma_{1,\alpha}^{h,\delta} + (1 + \tau(\sigma_0) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\sigma_0(1 + \tau(\sigma_0)^2))\sigma_0 \\
&\leq \tau(\sigma_0)^5\sigma_0 + (1 + \tau(\sigma_0) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\sigma_0(1 + \tau(\sigma_0)^2))\sigma_0
\end{aligned}$$

$$\begin{aligned}
&\leq (1 + \tau(\sigma_0) + \tau(\sigma_0)^5 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\sigma_0(1 + \tau(\sigma_0)^2))\sigma_0 \\
&\leq (1 + \tau(\sigma_0) + \tau(\sigma_0)^2 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\sigma_0(1 + \tau(\sigma_0)^2))\sigma_0 \\
&\leq (1 + \tau(\gamma_\rho) + \tau(\gamma_\rho)^2 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho(1 + \tau(\gamma_\rho)^2))\gamma_\rho \\
&< r
\end{aligned}$$

i.e., $u_{2,\alpha}^{h,\delta},\ w_{2,\alpha}^{h,\delta} \in B_r(P_h x_0)$. Continuing this way one can prove that $u_{n,\alpha}^{h,\delta},\ w_{n,\alpha}^{h,\delta} \in B_r(P_h x_0), \forall n \geq 0$. This completes the proof.

The main result of this section is the following theorem.

Theorem 4.4.4. *Let* $0 < \tau(\gamma_\rho) < 1$, $w_{n,\alpha}^{h,\delta}$ *and* $u_{n,\alpha}^{h,\delta}$ *be as in (4.4.18) and (4.4.19) respectively with* $\delta \in (0, \delta_0]$ *and assumptions of the Theorem 4.4.3 hold. Then* $(u_{n,\alpha}^{h,\delta})$ *is Cauchy sequence in* $B_r(P_h x_0)$ *and converges to* $x_\alpha^{h,\delta} \in \overline{B_r(P_h x_0)}$. *Further* $\|u_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \leq Ce^{-\gamma 4^n}$ *and* $P_h[F(x_\alpha^{h,\delta}) + \alpha(x_\alpha^{h,\delta} - x_0)] = P_h f^\delta$ *where* $\gamma = -\log \tau(\gamma_\rho)$ *and*

$$C = \left[\frac{1}{1-\tau(\gamma_\rho)^4} + \frac{3k_0\gamma_\rho}{2}(1+\frac{\varepsilon_0}{\alpha_0})\frac{1}{1-(\tau(\gamma_\rho)^2)^4}\tau(\gamma_\rho)^{4^n}\right]\gamma_\rho.$$

Proof. Using the relation (b) of Lemma 4.4.1 and (c) of Theorem 4.4.2, we obtain

$$\begin{aligned}
\|u_{n+m,\alpha}^{h,\delta} - u_{n,\alpha}^{h,\delta}\| &\leq \sum_{i=0}^{m-1} \|u_{n+i+1,\alpha}^{h,\delta} - u_{n+i,\alpha}^{h,\delta}\| \\
&\leq \sum_{i=0}^{m-1} \left[1 + \frac{3k_0\sigma_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})\tau(\sigma_0)^{4^{n+i}}\right]\tau(\sigma_0)^{4^{n+i}}\sigma_0 \\
&\leq [(1+\tau(\sigma_0)^4 + \tau(\sigma_0)^{4^2} + \cdots + \tau(\sigma_0)^{4^m}) + \frac{3k_0\sigma_0}{2}(1+\frac{\varepsilon_0}{\alpha_0}) \\
&\quad (1 + (\tau(\sigma_0)^2)^4 + (\tau(\sigma_0)^2)^{4^2} + \cdots + (\tau(\sigma_0)^2)^{4^m})\tau(\sigma_0)^{4^n}]\tau(\sigma_0)^{4^n}\sigma_0 \\
&\leq [(1+\tau(\gamma_\rho)^4 + \tau(\gamma_\rho)^{4^2} + \cdots + \tau(\gamma_\rho)^{4^m}) + \frac{3k_0\gamma_\rho}{2}(1+\frac{\varepsilon_0}{\alpha_0}) \\
&\quad (1 + (\tau(\gamma_\rho)^2)^4 + (\tau(\gamma_\rho)^2)^{4^2} + \cdots + (\tau(\gamma_\rho)^2)^{4^m})\tau(\gamma_\rho)^{4^n}]\tau(\gamma_\rho)^{4^n}\gamma_\rho \\
&\leq C\tau(\gamma_\rho)^{4^n} \\
&\leq Ce^{-\gamma 4^n}.
\end{aligned}$$

Thus $u_{n,\alpha}^{h,\delta}$ is a Cauchy sequence in $B_r(P_h x_0)$ and hence it converges, say, to $x_\alpha^{h,\delta} \in \overline{B_r(P_h x_0)}$. Observe that

$$\begin{aligned}
\|P_h[F(u_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(u_{n,\alpha}^{h,\delta} - x_0)]\| &= \|R_\alpha(u_{n,\alpha}^{h,\delta})(u_{n,\alpha}^{h,\delta} - w_{n,\alpha}^{h,\delta})\| \\
&\leq \|R_\alpha(u_{n,\alpha}^{h,\delta})\|\|u_{n,\alpha}^{h,\delta} - w_{n,\alpha}^{h,\delta}\| \\
&= \|(P_h F'(u_{n,\alpha}^{h,\delta})P_h + \alpha P_h)\|\sigma_{n,\alpha}^{h,\delta} \\
&\leq \|(P_h F'(u_{n,\alpha}^{h,\delta})P_h + \alpha P_h)\|\tau(\sigma_0)^{4^n}\sigma_0 \\
&\leq (C_F + \alpha)\tau(\gamma_\rho)^{4^n}\gamma_\rho. \qquad (4.4.32)
\end{aligned}$$

Now by letting $n \to \infty$ in (4.4.32) we obtain

$$P_h[F(x_\alpha^{h,\delta}) + \alpha(x_\alpha^{h,\delta} - x_0)] = P_h f^\delta.$$

This completes the proof.

4.5 ERROR BOUNDS UNDER SOURCE CONDITIONS FOR PROJECTION METHOD

The objective of this section is to obtain an error estimate for $\|u_{n,\alpha}^{h,\delta} - \hat{x}\|$ under a source condition on $x_0 - \hat{x}$. The proof of the following theorem is analogous to the proof of Theorem 3.5.1, so the proof is omitted.

Theorem 4.5.1. *Let $u_{n,\alpha}^{h,\delta}$ be as in (4.4.19), and let assumptions in Theorem 2.5.2 and Theorem 4.4.4 hold. Then*

$$\|u_{n,\alpha}^{h,\delta} - \hat{x}\| \leq Ce^{-\gamma 4^n} + \max\{1, \tilde{C}\}\left[\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}\right].$$

Let

$$n_\delta := \min\left\{n : e^{-\gamma 4^n} \leq \frac{\delta + \varepsilon_h}{\alpha}\right\} \tag{4.5.33}$$

and

$$C_0 = C + \max\{1, \tilde{C}\}. \tag{4.5.34}$$

Theorem 4.5.2. *Let $u_{n_\delta,\alpha}^{h,\delta}$ be as in (4.4.19) and the assumptions in Theorem 4.5.1 be satisfied. And let n_δ and C_0 be as in (4.5.33) and (4.5.34) respectively. Then*

$$\|u_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| \leq C_0\left[\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}\right].$$

4.5.1 A priori choice of the parameter

Theorem 4.5.3. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and the assumptions in Theorem 4.5.2 hold. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h))$ and let n_δ be as in (4.5.33). Then*

$$\|u_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

4.5.2 An adaptive choice of the parameter

Let

$$D_N(\alpha) := \{\alpha_i = \mu^i \alpha_0, i = 0, 1, \cdots, N\}$$

where $\mu > 1$, $\alpha_0 > 0$ and let

$$n_i := \min\left\{n : e^{-\gamma 4^n} \leq \frac{\delta + \varepsilon_h}{\alpha_i}\right\}.$$

Then for $i = 0, 1, \cdots, N$, we have

$$\|u^{h,\delta}_{n_i,\alpha_i} - x^{h,\delta}_{\alpha_i}\| \leq C\left[\frac{\delta + \varepsilon_h}{\alpha_i}\right], \qquad \forall i = 0, 1, \cdots N.$$

Let $u_i := u^{h,\delta}_{n_i,\alpha_i}$. We select the regularization parameter $\alpha = \alpha_i$ from the set $D_N(\alpha)$ and operate only with corresponding u_i, $\quad i = 0, 1, \cdots, N$.

Proof of the following theorem is analogous to the proof of Theorem 2.3.7.

Theorem 4.5.4. *Assume that there exists $i \in \{0, 1, 2, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta+\varepsilon_h}{\alpha_i}$. Let assumptions of Theorem 4.5.2 and Theorem 4.5.3 hold and let*

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta + \varepsilon_h}{\alpha_i}\right\} < N,$$

$$k := \max\left\{i : \|u_i - u_j\| \leq 4C_0\left[\tfrac{\delta+\varepsilon_h}{\alpha_j}\right], \quad j = 0, 1, 2, \cdots, i\right\}.$$

Then $l \leq k$ and $\|\hat{x} - u_k\| \leq c\psi^{-1}(\delta + \varepsilon_h)$ where $c = 6C_0\mu$.

4.6 IMPLEMENTATION OF ADAPTIVE CHOICE RULE

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 4.4.2 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta_0 < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \cdots, N$.

4.6.1 Algorithm

1. Set $i = 0$.
2. Choose $n_i := \min\left\{n : e^{-\gamma 4^n} \leq \frac{\delta+\varepsilon_h}{\alpha_i}\right\}$.
3. Solve $u_i := u^{h,\delta}_{n_i,\alpha_i}$ by using the iteration (4.4.18) and (4.4.19).
4. If $\|u_i - u_j\| > 4C_0\left(\frac{\delta+\varepsilon_h}{\alpha_j}\right)$, $j < i$, then take $k = i - 1$ and return u_k.
5. Else set $i = i + 1$ and go to Step 2.

4.7 NUMERICAL EXAMPLE

Once again in this section we consider the problem studied in Example 2.7.1 for illustrating the algorithm considered in section 4.6.1. We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X as in section 2.7.

Example 4.7.1. *Here also we take the kernel as in Example 2.7.1,* $f^\delta = f + \delta$, *where* $f(t) = \frac{6cos(\pi t)+cos^3(\pi t)+14t-7}{9\pi^2}$. *Then the exact solution* $\hat{x}(t) = cos(\pi t)$.

We use $x_0(t) = cos(\pi t) + \frac{3[t\pi^2 - t^2\pi^2 - sin^2(\pi t)]}{4\pi^2}$ *as our initial guess, so that the function* $x_0 - \hat{x}$ *satisfies the source condition* $x_0 - \hat{x} = \varphi(F'(\hat{x}))\frac{1}{4}$ *where* $\varphi(\lambda) = \lambda$.

We choose $\alpha_0 = (1.1)(\delta + \varepsilon_h)$, $\mu = 1.1$, $\rho = 0.11$, $\gamma_\rho = 0.7818$ *and* $g(\gamma_\rho) = 0.99$. *The results of the computation are presented in Table 4.1. The plots of the exact solution and the approximate solution obtained are given in Figures 4.1 through 4.8.*

n	k	n_k	$\delta + \varepsilon_h$	α_k	$\|u_k - \hat{x}\|$	$\frac{\|u_k - \hat{x}\|}{(\delta+\varepsilon_h)^{1/2}}$
8	2	4	0.0135	0.0180	0.3648	3.1407
16	2	4	0.0134	0.0178	0.2515	2.1751
32	2	4	0.0133	0.0178	0.1792	1.5516
64	2	4	0.0133	0.0177	0.1287	1.1141
128	2	4	0.0133	0.0177	0.0936	0.8103
256	2	4	0.0133	0.0177	0.0697	0.6033
512	2	4	0.0133	0.0177	0.0539	0.4664
1024	2	4	0.0133	0.0177	0.0439	0.3799

Table 4.1: Iterations and corresponding error estimates

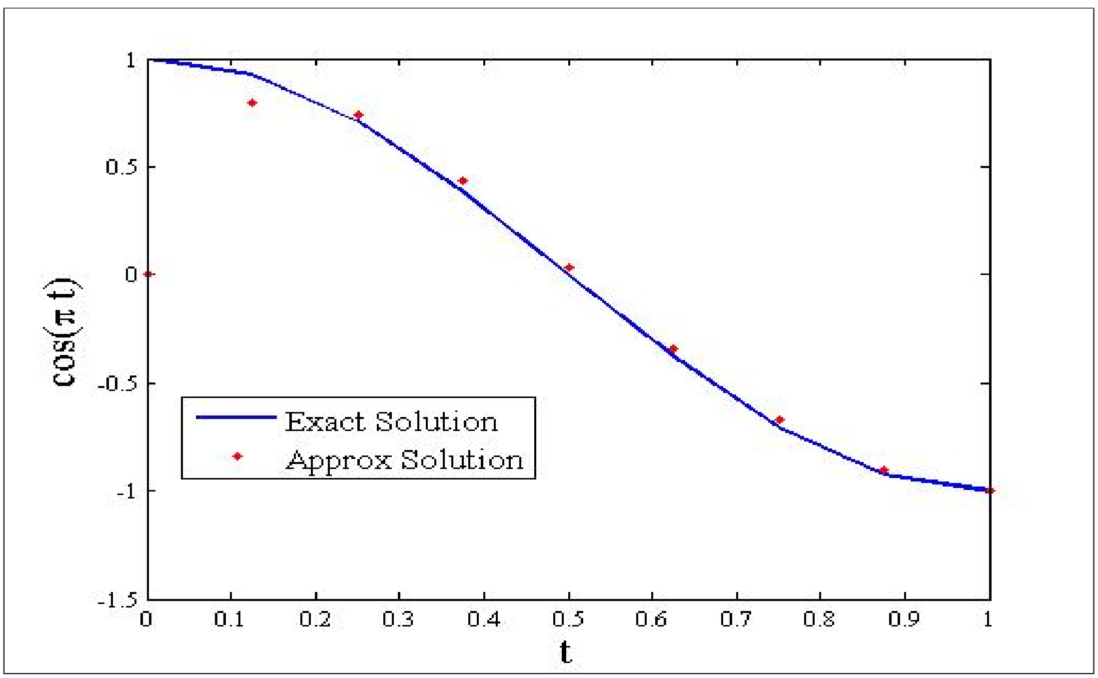

Figure 4.1: Curves of the exact and approximate solutions when n=8

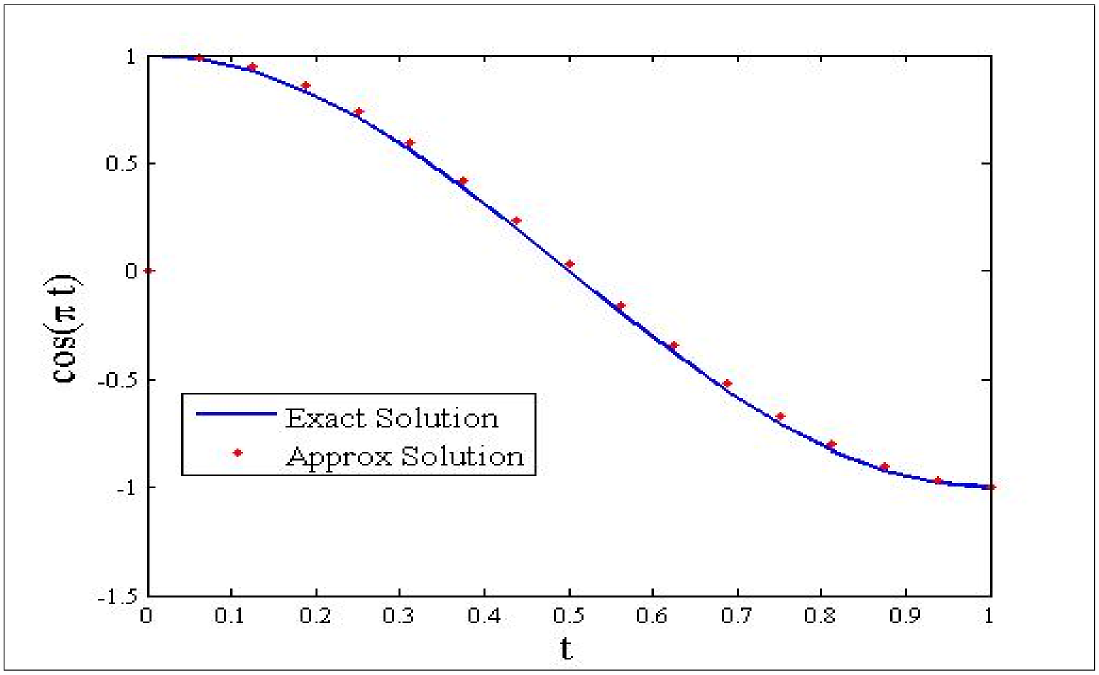

Figure 4.2: Curves of the exact and approximate solutions when n=16

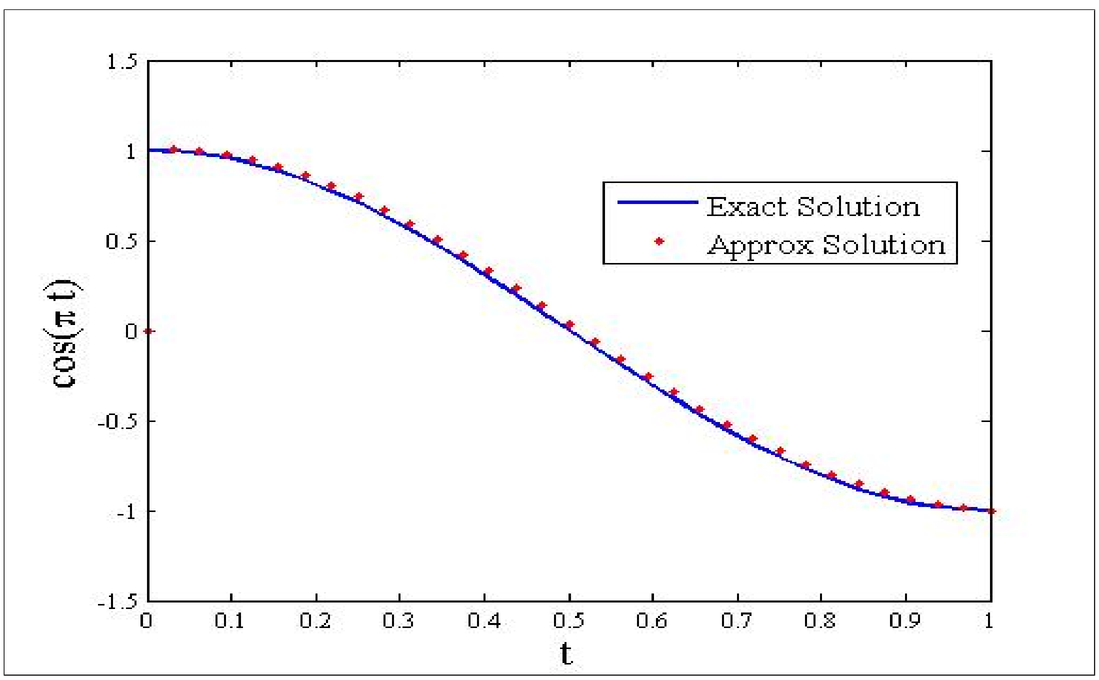

Figure 4.3: Curves of the exact and approximate solutions when n=32

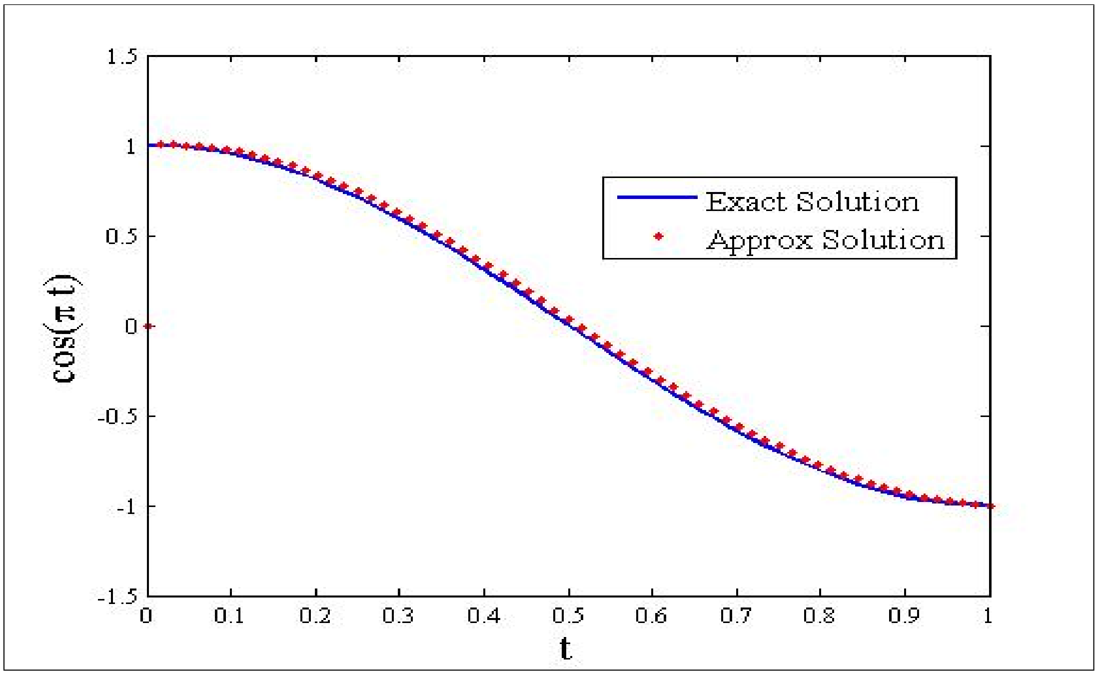

Figure 4.4: Curves of the exact and approximate solutions when n=64

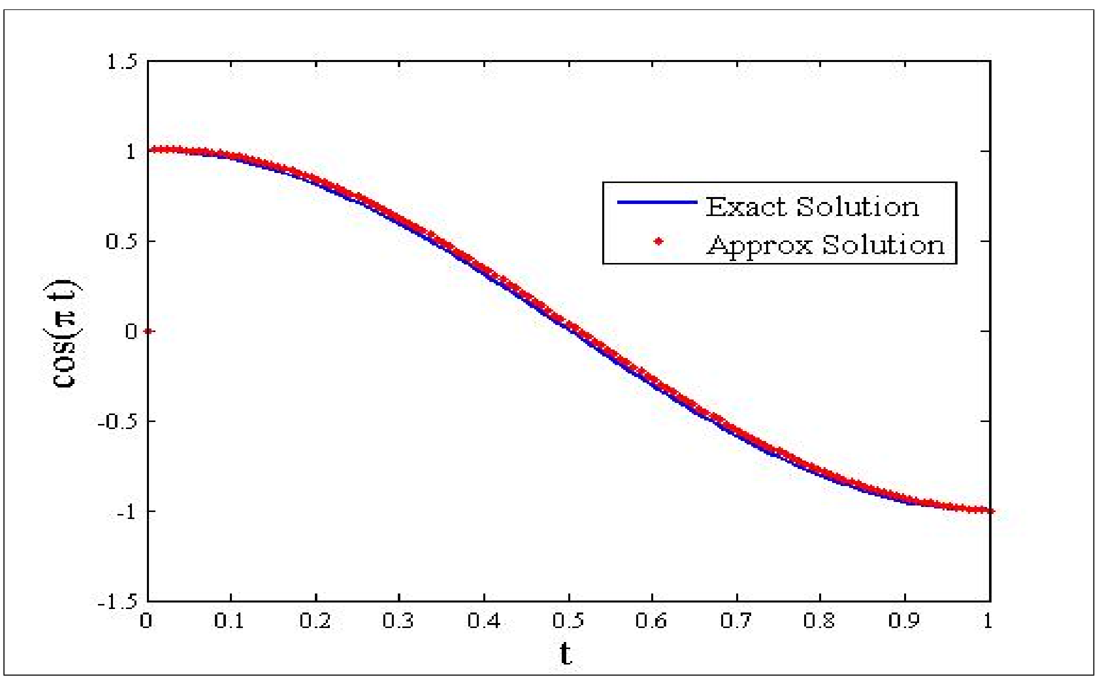

Figure 4.5: Curves of the exact and approximate solutions when n=128

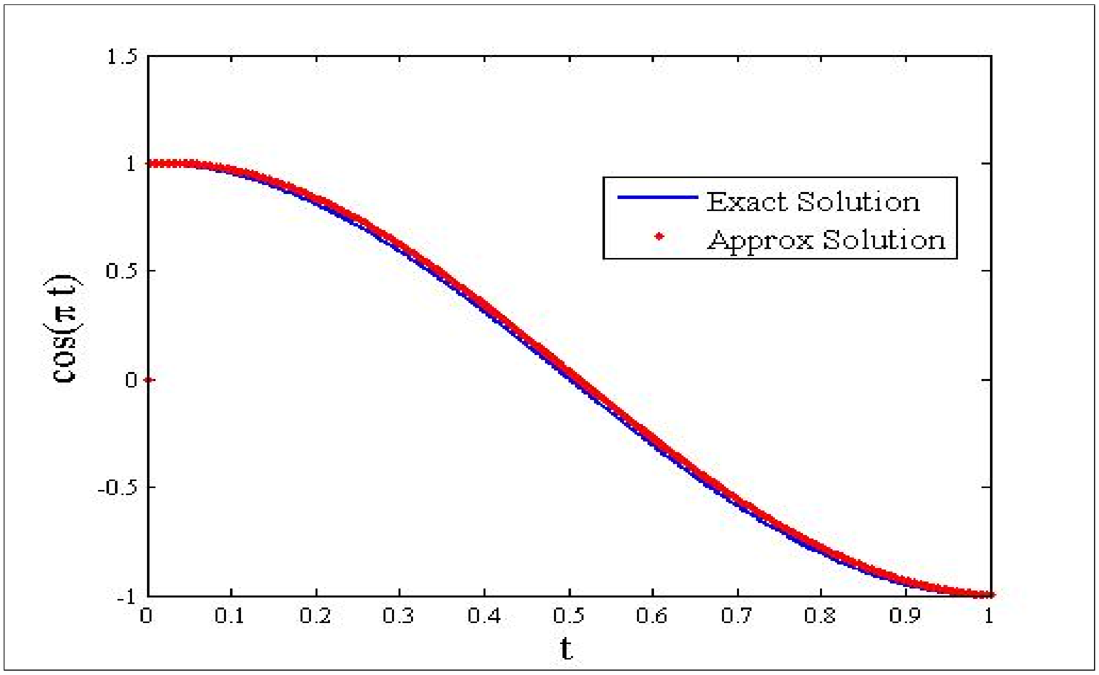

Figure 4.6: Curves of the exact and approximate solutions when n=256

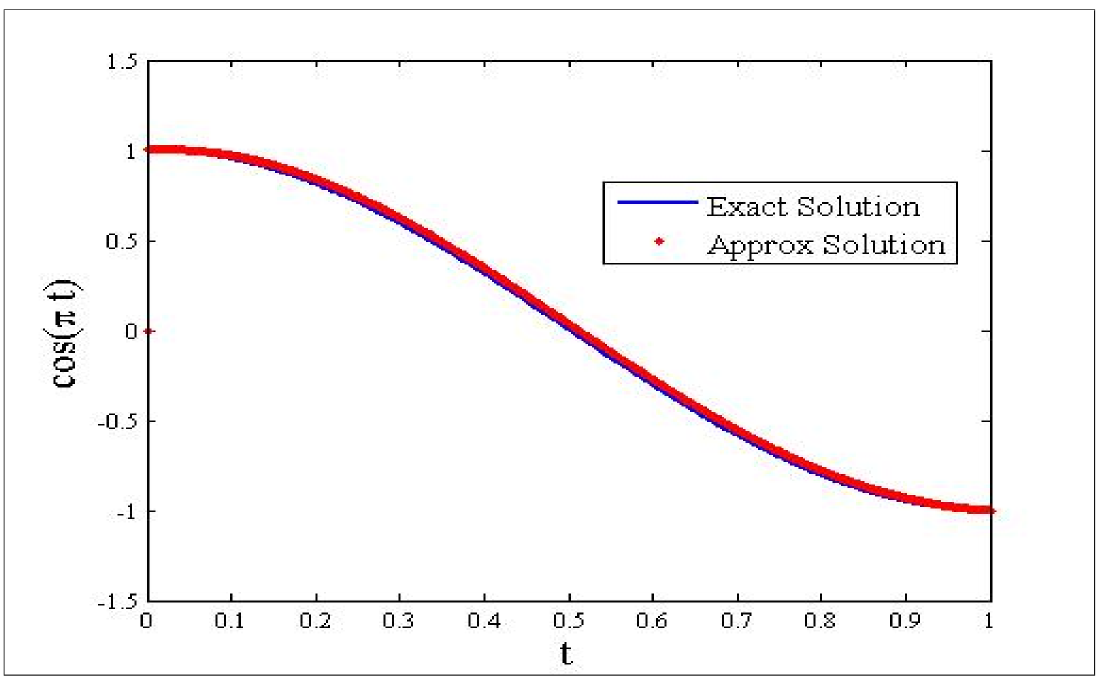

Figure 4.7: Curves of the exact and approximate solutions when n=512

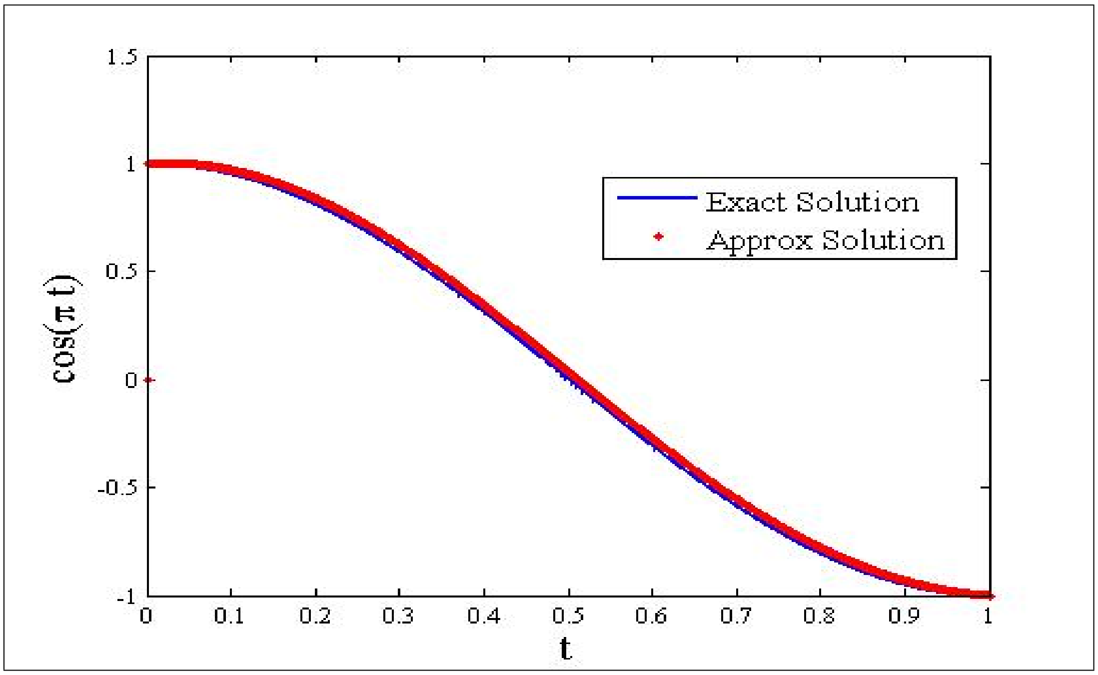

Figure 4.8: Curves of the exact and approximate solutions when n=1024

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Chapter 5

NEWTON LAVRENTIEV REGULARIZATION FOR ILL-POSED OPERATOR EQUATIONS IN HILBERT SCALES

In this chapter we present a two step method for approximately solving the ill-posed operator equation $F(x) = f$, in the setting of Hilbert scales. Here $F : D(F) \subseteq X \to X$, is a nonlinear monotone operator defined on a real Hilbert space X. Also we derive the error estimates by selecting the regularization parameter α according to the adaptive method suggested in Pereverzyev and Schock (2005). The error estimate obtained in the setting of Hilbert scales $\{X_r\}_{r\in R}$ generated by a densely defined, linear, unbounded, strictly positive self adjoint operator $L : D(L) \subset X \to X$ is of optimal order.

5.1 INTRODUCTION

This Chapter is devoted to the study of nonlinear ill-posed operator equation

$$F(x) = f, \tag{5.1.1}$$

where $F : D(F) \subset X \to X$ is a nonlinear monotone operator in the setting of Hilbert scale. Here, $D(F)$ is the domain of F and X is a real Hilbert space with inner product $\langle ., . \rangle$ and corresponding norm $\|.\|$. Throughout this Chapter we assume the existence of an x_0-MNS, $\hat{x}$ for exact data f, i.e.,

$$F(\hat{x}) = f$$

and the element x_0 is assumed to be known. Further we assume that $f^\delta \in X$ are the available noisy data with $\|f - f^\delta\| \leq \delta$.

In this Chapter we consider, the Hilbert scales variant of (2.1.1) and (2.1.2) for obtaining better convergence rates.

The Chapter is organized as: In Section 5.2, we give the preliminaries. The proposed method is given in section 5.3, the error estimates, adaptive parameter choice is given in section 5.4.

5.2 PRELIMINARIES

Let $L : D(L) \subset X \to X$ be a densely defined unbounded self adjoint strictly positive operator. We consider a Hilbert scales $\{X_r\}_{r\in R}$ (see, Natterer (1984), Tautenhahn (1996), Neubauer (2000), Egger and Neubauer (2005), Qi-Nian and Tautenhahn (2011)) induced by L, i.e., X_r is the completion of $D := \cap_{k=0}^{\infty} D(L^k)$ with respect to the Hilbert space norm

$$\|x\|_r = \|L^r x\|, \quad r \in R.$$

Throughout this Chapter we will be using the following assumptions.

Assumption 5.2.1. *There exist constants $a \geq 0$, $0 < m \leq M < \infty$ such that*

$$m\|h\|_{-a} \leq \|F'(x_0)h\| \leq M\|h\|_{-a}, h \in X.$$

Note that the above assumption is weaker than the Assumption 3(a) in Qi-Nian and Tautenhahn (2011). Let

$$A_s = L^{-\frac{s}{2}} F'(x_0) L^{-\frac{s}{2}},$$

$f(\nu) = \min\{m^\nu, M^\nu\}$ and $g(\nu) = \max\{m^\nu, M^\nu\}, \quad \nu \in R, |\nu| \leq 1$. The following proposition is important for proving the results in this chapter.

Proposition 5.2.1. *(See George and Nair (1997), Proposition 3.1) For $s \geq 0$ and $|\nu| \leq 1$,*

$$f(\frac{\nu}{2})\|x\|_{-\frac{\nu}{2}(s+a)} \leq \|A_s^{\nu/2} x\| \leq g(\frac{\nu}{2})\|x\|_{-\frac{\nu}{2}(s+a)}, \quad x \in X.$$

Using the above proposition, we prove the following lemma, which is used extensively to prove the results of this chapter.

Lemma 5.2.2. *Let Assumption 5.2.1 hold. Then for all $h \in X$,*

$$\|(F'(x_0)+\alpha L^s)^{-1}F'(x_0)h\| \leq \psi(s)\|h\|, \text{where } \psi(s) = \frac{g(\frac{s}{2(s+a)})}{f(\frac{s}{2(s+a)})}.$$

Proof. Note that,

$$\begin{aligned}
\|(F'(x_0)+\alpha L^s)^{-1}F'(x_0)h\| &= \|L^{\frac{-s}{2}}(A_s+\alpha I)^{-1}A_s L^{\frac{s}{2}}h\| \\
&\leq \frac{1}{f\left(\frac{s}{2(s+a)}\right)}\|A_s^{\frac{s}{2(s+a)}}(A_s+\alpha I)^{-1}A_s L^{\frac{s}{2}}h\| \\
&\leq \frac{1}{f\left(\frac{s}{2(s+a)}\right)}\|(A_s+\alpha I)^{-1}A_s\|\|A_s^{\frac{s}{2(s+a)}}L^{\frac{s}{2}}h\| \\
&\leq \psi(s)\|h\|.
\end{aligned}$$

The last step follows from the spectral properties of the self adjoint operator $A_s, s > 0$.

5.3 NEWTON LAVRENTIEV METHOD IN HILBERT SCALES

In this section we consider the Hilbert scales variant of the method (2.1.1) and (2.1.2). Define

$$y^\delta_{n,\alpha,s} = x^\delta_{n,\alpha,s} - (F'(x_0)+\alpha L^s)^{-1}[F(x^\delta_{n,\alpha,s}) - f^\delta + \alpha L^s(x^\delta_{n,\alpha,s} - x_0)] \tag{5.3.2}$$

and

$$x^\delta_{n+1,\alpha,s} = y^\delta_{n,\alpha,s} - (F'(x_0)+\alpha L^s)^{-1}[F(y^\delta_{n,\alpha,s}) - f^\delta + \alpha L^s(y^\delta_{n,\alpha,s} - x_0)] \tag{5.3.3}$$

where $x^\delta_{0,\alpha,s} := x_0$, is the initial approximation for the solution $\hat{x}$ of (5.1.1). We will be selecting the regularization parameter $\alpha = \alpha_i$ from some finite set defined in (2.1.3).

We assume that F possesses a uniformly bounded Fréchet derivative $F'(x)$ for all $x \in D(F)$ and $F'(x)$ satisfies the Assumption 2.2.2.

Let

$$e^\delta_{n,\alpha,s} := \|y^\delta_{n,\alpha,s} - x^\delta_{n,\alpha,s}\|, \qquad \forall n \geq 0 \tag{5.3.4}$$

and let $\delta_0 < \frac{\alpha_0^{\frac{a}{s+a}}}{4k_0\psi(s)\psi_1(s)}$ for some $\alpha_0 > 0$, where $\psi_1(s) = \frac{g\left(\frac{-s}{2(s+a)}\right)}{f\left(\frac{s}{2(s+a)}\right)}$.

Let $\|\hat{x} - x_0\| \leq \rho$, with $\rho \leq \frac{\sqrt{1+[\frac{1}{2\psi(s)^2} - \frac{2k_0\psi_1(s)\delta_0}{\psi(s)\alpha_0^{\frac{a}{s+a}}}]-1}}{k_0}$ and

$$\gamma_\rho := \frac{k_0}{2}\psi(s)\rho^2 + \psi(s)\rho + \frac{\psi_1(s)\delta_0}{\alpha_0^{\frac{a}{s+a}}}. \tag{5.3.5}$$

$$\text{Further let} \quad q = \psi(s)k_0 r, \tag{5.3.6}$$

$$\text{where } r \in \left(\frac{1-\sqrt{1-4k_0\psi(s)\gamma_\rho}}{2k_0\psi(s)}, \frac{1+\sqrt{1-4k_0\psi(s)\gamma_\rho}}{2k_0\psi(s)}\right). \tag{5.3.7}$$

Note that if q is as in (5.3.6), then $q < 1$.

Lemma 5.3.1. *Let $e^\delta_{0,\alpha,s}$ be as in (5.3.4). Then $e^\delta_{0,\alpha,s} \leq \gamma_\rho$.*

Proof. Observe that

$$\begin{aligned}
e^\delta_{0,\alpha,s} &= \|y^\delta_{0,\alpha,s} - x^\delta_{0,\alpha,s}\| \\
&= \|(F'(x_0) + \alpha L^s)^{-1}(F(x_0) - f^\delta)\| \\
&= \|(F'(x_0) + \alpha L^s)^{-1}[F(x_0) - F(\hat{x}) - F'(x_0)(x_0 - \hat{x}) \\
&\quad + F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\
&= \|(F'(x_0) + \alpha L^s)^{-1}[\int_0^1 (F'(x_0 + t(\hat{x} - x_0)) - F'(x_0))(x_0 - \hat{x})dt \\
&\quad + F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\
&= \|(F'(x_0) + \alpha L^s)^{-1}F'(x_0)[\int_0^1 \Phi(x_0 + t(\hat{x} - x_0), x_0, x_0 - \hat{x})dt + (x_0 - \hat{x})] \\
&\quad + (F'(x_0) + \alpha L^s)^{-1}(F(\hat{x}) - f^\delta)\|
\end{aligned}$$

and hence by Assumption 2.2.2 and Lemma 5.2.2 we have,

$$e^\delta_{0,\alpha,s} \leq \frac{k_0}{2}\psi(s)\|x_0 - \hat{x}\|^2 + \psi(s)\|x_0 - \hat{x}\| + \|(F'(x_0) + \alpha L^s)^{-1}(f - f^\delta)\|. \tag{5.3.8}$$

Observe that,

$$\begin{aligned}
\|(F'(x_0) + \alpha L^s)^{-1}(f - f^\delta)\| &= \|L^{\frac{-s}{2}}(A_s + \alpha I)^{-1}L^{\frac{-s}{2}}(f^\delta - f)\| \\
&\leq \frac{1}{f\left(\frac{s}{2(s+a)}\right)}\|A_s^{\frac{s}{2(s+a)}}(A_s + \alpha I)^{-1}L^{\frac{-s}{2}}(f^\delta - f)\|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{f\left(\frac{s}{2(s+a)}\right)}\|(A_s+\alpha I)^{-1}A_s^{\frac{s}{s+a}}A_s^{\frac{-s}{2(s+a)}}L^{\frac{-s}{2}}(f^\delta - f)\| \\
&\leq \frac{g\left(\frac{-s}{2(s+a)}\right)}{f\left(\frac{s}{2(s+a)}\right)}\alpha^{\frac{-a}{s+a}}\|L^{\frac{-s}{2}}(f^\delta - f)\|_{\frac{s}{2}} \\
&\leq \psi_1(s)\delta\alpha^{\frac{-a}{s+a}} \leq \psi_1(s)\delta_0\alpha_0^{\frac{-a}{s+a}}. \qquad (5.3.9)
\end{aligned}
$$

Now the result follows from (5.3.5), (5.3.8) and (5.3.9).

Theorem 5.3.2. *Let $y^\delta_{n,\alpha,s}$, $x^\delta_{n,\alpha,s}$ and $e^\delta_{n,\alpha,s}$ be as in (5.3.2), (5.3.3) and (5.3.4) respectively with $\delta \in (0,\delta_0]$ and $\alpha \in D_N(\alpha)$. Let γ_ρ, q and r be as in (5.3.5), (5.3.6) and (5.3.7) respectively. Then*

(a) $\|x^\delta_{n,\alpha,s} - y^\delta_{n-1,\alpha,s}\| \leq q\|y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s}\|$;

(b) $\|y^\delta_{n,\alpha,s} - x^\delta_{n,\alpha,s}\| \leq q^2\|y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s}\|$;

(c) $e^\delta_{n,\alpha,s} \leq q^{2n}\gamma_\rho$;

(d) $x^\delta_{n,\alpha,s}, y^\delta_{n,\alpha,s} \in B_r(x_0)$.

Proof. Observe that, if $x^\delta_{n,\alpha,s}$, $y^\delta_{n,\alpha,s} \in B_r(x_0)$, then

$$
\begin{aligned}
x^\delta_{n,\alpha,s} - y^\delta_{n-1,\alpha,s} &= y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s} - (F'(x_0)+\alpha L^s)^{-1}[F(y^\delta_{n-1,\alpha,s}) - F(x^\delta_{n-1,\alpha,s}) \\
&\quad +\alpha L^s(y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s})] \\
&= (F'(x_0)+\alpha L^s)^{-1}[F'(x_0)(y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s}) \\
&\quad -(F(y^\delta_{n-1,\alpha,s}) - F(x^\delta_{n-1,\alpha,s}))] \\
&= (F'(x_0)+\alpha L^s)^{-1}\int_0^1 [F'(x_0) - F'(x^\delta_{n-1,\alpha,s} + t(y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s}))] \\
&\quad \times(y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s})dt \\
&= (F'(x_0)+\alpha L^s)^{-1}F'(x_0) \\
&\quad \int_0^1 \Phi(x_0, x^\delta_{n-1,\alpha,s} + t(y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s}), y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s})dt,
\end{aligned}
$$

so, by Assumption 2.2.2 and Lemma 5.2.2 we have,

$$
\|x^\delta_{n,\alpha,s} - y^\delta_{n-1,\alpha,s}\| \leq \psi(s)k_0r\|y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s}\|. \qquad (5.3.10)
$$

This proves (a). Again observe that if $x^\delta_{n,\alpha,s}, y^\delta_{n,\alpha,s} \in B_r(x_0)$, by Assumption 2.2.2 and (5.3.10) we have

$$\begin{aligned}
y^\delta_{n,\alpha,s} - x^\delta_{n,\alpha,s} &= x^\delta_{n,\alpha,s} - y^\delta_{n-1,\alpha,s} - (F'(x_0)+\alpha L^s)^{-1}[F(x^\delta_{n,\alpha,s}) - f^\delta + \alpha(x^\delta_{n,\alpha,s} - x_0)] \\
&= (F'(x_0)+\alpha L^s)^{-1}[F'(x_0)(x^\delta_{n,\alpha,s} - y^\delta_{n-1,\alpha,s}) - (F(x^\delta_{n,\alpha,s}) - F(y^\delta_{n-1,\alpha,s}))] \\
&= (F'(x_0)+\alpha L^s)^{-1}\int_0^1 [F'(x_0) - F'(y^\delta_{n-1,\alpha,s} + t(x^\delta_{n,\alpha,s} - y^\delta_{n-1,\alpha,s})] \\
&\quad (x^\delta_{n,\alpha,s} - y^\delta_{n-1,\alpha,s})dt \\
&= (F'(x_0)+\alpha L^s)^{-1}F'(x_0)\int_0^1 \Phi(x_0, y^\delta_{n-1,\alpha,s} + t(x^\delta_{n,\alpha,s} - y^\delta_{n-1,\alpha,s}), \\
&\quad x^\delta_{n,\alpha,s} - y^\delta_{n-1,\alpha,s})dt
\end{aligned}$$

and hence, by Lemma 5.2.2

$$\|y^\delta_{n,\alpha,s} - x^\delta_{n,\alpha,s}\| \leq \psi(s)k_0 r\|x^\delta_{n,\alpha,s} - y^\delta_{n-1,\alpha,s}\| \leq q^2\|y^\delta_{n-1,\alpha,s} - x^\delta_{n-1,\alpha,s}\|. \quad (5.3.11)$$

This proves (b) and (c) follows from (b).

Now using induction we shall prove that $x^\delta_{n,\alpha,s}, y^\delta_{n,\alpha,s} \in B_r(x_0)$. Note that $x_0, y^\delta_{0,\alpha,s} \in B_r(x_0)$ and hence by (5.3.10)

$$\begin{aligned}
\|x^\delta_{1,\alpha,s} - x_0\| &\leq \|x^\delta_{1,\alpha,s} - y^\delta_{0,\alpha,s}\| + \|y^\delta_{0,\alpha,s} - x_0\| \\
&\leq (1+q)e^\delta_{0,\alpha,s} \\
&\leq \frac{e^\delta_{0,\alpha,s}}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r
\end{aligned}$$

i.e., $x^\delta_{1,\alpha,s} \in B_r(x_0)$, again by (5.3.11)

$$\begin{aligned}
\|y^\delta_{1,\alpha,s} - x_0\| &\leq \|y^\delta_{1,\alpha,s} - x^\delta_{1,\alpha,s}\| + \|x^\delta_{1,\alpha,s} - x_0\| \\
&\leq q^2 e^\delta_{0,\alpha,s} + (1+q)e^\delta_{0,\alpha,s} \\
&\leq \frac{e^\delta_{0,\alpha,s}}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r
\end{aligned}$$

i.e., $y^\delta_{1,\alpha,s} \in B_r(x_0)$. Suppose $x^\delta_{k,\alpha,s}, y^\delta_{k,\alpha,s} \in B_r(x_0)$ for some $k > 1$.
Then since $\|x^\delta_{k+1,\alpha,s} - x_0\| \leq \|x^\delta_{k+1,\alpha,s} - x^\delta_{k,\alpha,s}\| + \|x^\delta_{k,\alpha,s} - x^\delta_{k-1,\alpha,s}\| + \cdots + \|x^\delta_{1,\alpha,s} - x_0\|$

and by (a) and (b) we have,

$$\begin{aligned}
\|x^\delta_{k+1,\alpha,s} - x^\delta_{k,\alpha,s}\| &\leq \|x^\delta_{k+1,\alpha,s} - y^\delta_{k,\alpha,s}\| + \|y^\delta_{k,\alpha,s} - x^\delta_{k,\alpha,s}\| \\
&\leq (q+1)\|y^\delta_{k,\alpha,s} - x^\delta_{k,\alpha,s}\| \leq (1+q)q^{2k}e^\delta_{0,\alpha,s}.
\end{aligned}$$

Thus

$$\|x^\delta_{k+1,\alpha,s} - x_0\| \leq (1+q)[q^{2k} + q^{2(k-1)} + \cdots + 1]e^\delta_{0,\alpha,s} \tag{5.3.12}$$

$$\begin{aligned}
\|x^\delta_{k+1,\alpha,s} - x_0\| &\leq (1+q)\left[\frac{1-q^{2k+1}}{1-q^2}\right] e^\delta_{0,\alpha,s} \\
&\leq \frac{e^\delta_{0,\alpha,s}}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r.
\end{aligned}$$

So by induction $x^\delta_{n,\alpha,s} \in B_r(x_0)$ for all $n \geq 0$. Again by (a), (b) and (5.3.12) we have,

$$\begin{aligned}
\|y^\delta_{k+1,\alpha,s} - x_0\| &\leq \|y^\delta_{k+1,\alpha,s} - x^\delta_{k+1,\alpha,s}\| + \|x^\delta_{k+1,\alpha,s} - x_0\| \\
&\leq q^{2k+2}e^\delta_{0,\alpha,s} + (1+q)[q^{2k} + q^{2(k-1)} + \cdots + 1]e^\delta_{0,\alpha,s} \\
&\leq (1+q)\left[\frac{1-q^{2k+3}}{1-q^2}\right] e^\delta_{0,\alpha,s} \\
&\leq \frac{e^\delta_{0,\alpha,s}}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r.
\end{aligned}$$

Thus $y^\delta_{k+1,\alpha,s} \in B_r(x_0)$ and hence by induction $y^\delta_{n,\alpha,s} \in B_r(x_0)$ for all $n \geq 0$. This completes the proof.

Theorem 5.3.3. *Let $y^\delta_{n,\alpha,s}$ and $x^\delta_{n,\alpha,s}$ be as in (5.3.2) and (5.3.3) respectively with $\delta \in (0, \delta_0]$ and $\alpha \in D_N(\alpha)$ and let the assumptions of Theorem 5.3.2 hold. Then $(x^\delta_{n,\alpha,s})$ is a Cauchy sequence in $B_r(x_0)$ and converges to $x^\delta_{\alpha,s} \in \overline{B_r(x_0)}$. Further*

$$F(x^\delta_{\alpha,s}) + \alpha L^s(x^\delta_{\alpha,s} - x_0) = f^\delta \tag{5.3.13}$$

and

$$\|x^\delta_{n,\alpha,s} - x^\delta_{\alpha,s}\| \le Cq^{2n}$$

where $C = \frac{\gamma_\rho}{1-q}$.

Proof. Using the relation (b) and (c) of Theorem 5.3.2, we obtain

$$\begin{aligned}
\|x^\delta_{n+m,\alpha,s} - x^\delta_{n,\alpha,s}\| &\le \sum_{i=0}^{m-1} \|x^\delta_{n+i+1,\alpha,s} - x^\delta_{n+i,\alpha,s}\| \\
&\le \sum_{i=0}^{m-1} (1+q)e^\delta_{n+i,\alpha,s} \\
&\le \sum_{i=0}^{m-1} (1+q)q^{2(n+i)}e^\delta_{0,\alpha,s} \\
&\le \frac{q^{2n}}{1-q}e^\delta_{0,\alpha,s} \\
&\le \frac{q^{2n}}{1-q}\gamma_\rho.
\end{aligned}$$

Thus $x^\delta_{n,\alpha,s}$ is a Cauchy sequence in $B_r(x_0)$ and hence it converges, say, to $x^\delta_{\alpha,s} \in \overline{B_r(x_0)}$. Observe that

$$\begin{aligned}
\|F(x^\delta_{n,\alpha,s}) + \alpha L^s(x^\delta_{n,\alpha,s} - x_0) - f^\delta\| &= \|(F'(x_0) + \alpha L^s)(x^\delta_{n,\alpha,s} - y^\delta_{n,\alpha,s})\| \\
&\le \|(F'(x_0) + \alpha L^s)\|_{X_s \to X}\|x^\delta_{n,\alpha,s} - y^\delta_{n,\alpha,s}\| \\
&\le \|(F'(x_0) + \alpha L^s)\|_{X_s \to X} q^{2n}\gamma_\rho. \qquad (5.3.14)
\end{aligned}$$

Now by letting $n \to \infty$ in (5.3.14) we obtain $F(x^\delta_{\alpha,s}) + \alpha L^s(x^\delta_{\alpha,s} - x_0) = f^\delta$.

5.4 ERROR BOUNDS AND PARAMETER CHOICE IN HILBERT SCALES

In order to obtain error estimate in Hilbert scales an assumption of the form

$$x_0 - \hat{x} \in M_{t,E} = \{x \in X : \|x - \hat{x}\|_t \le E\} \qquad (5.4.15)$$

is used (cf. George and Nair (1997), George and Nair (2003), George and Nair (2004), Lu et al. (2010) and Jin and Tautenhahn (2011)). In this Chapter we use the following general source condition on $x_0 - \hat{x}$.

Assumption 5.4.1. *There exists a continuous, strictly monotonically increasing function $\varphi : (0, \|A_s\|] \to (0, \infty)$ satisfying;*

(i) $\lim_{\lambda \to 0} \varphi(\lambda) = 0,$

(ii) $\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq \varphi(\alpha) \qquad \forall \lambda \in (0, \|A_s\|]$ *and*

(iii) there exists $v \in X$ *with* $\|v\| \leq E_1$ *such that* $A_s^{\frac{s}{2(s+a)}} L^{\frac{s}{2}}(x_0 - \hat{x}) = \varphi(A_s)v$

Remark 5.4.1. *Note that if* $x_0 - \hat{x} \in M_{t,E}$, *then*

$$\begin{aligned}
\left\| A_s^{\frac{s-2t}{2(s+a)}} L^{\frac{s}{2}}(x_0 - \hat{x}) \right\| &\leq g\left(\frac{s-2t}{2(s+a)}\right) \|L^{\frac{s}{2}}(x_0 - \hat{x})\|_{t-\frac{s}{2}} \\
&\leq g\left(\frac{s-2t}{2(s+a)}\right) \|(x_0 - \hat{x})\|_t \\
&\leq g\left(\frac{s-2t}{2(s+a)}\right) E.
\end{aligned}$$

$$\begin{aligned}
So, \quad A_s^{\frac{s}{2(s+a)}} L^{\frac{s}{2}}(x_0 - \hat{x}) &= A_s^{\frac{t}{(s+a)}} A_s^{\frac{s-2t}{2(s+a)}} L^{\frac{s}{2}}(x_0 - \hat{x}) \\
&:= \varphi(A_s)v
\end{aligned}$$

where $\varphi(\lambda) = \lambda^{\frac{t}{s+a}}$ *and* $v = A_s^{\frac{s-2t}{2(s+a)}} L^{\frac{s}{2}}(x_0 - \hat{x})$. *In other words the 5.4.15 leads to the Assumption 5.4.1.*

Theorem 5.4.2. *Suppose* $x_{\alpha,s}^{\delta}$ *is the solution of (5.3.13) and Assumption 2.2.2 and Assumption 5.4.1 hold. Then* $\|x_{\alpha,s}^{\delta} - \hat{x}\| \leq C_s \left[\frac{\delta}{\alpha^{\frac{a}{s+a}}} + \varphi(\alpha)\right]$ *where*

$$C_s = \frac{1}{1-q} \max\left\{\psi_1(s), \frac{E_1}{f\left(\frac{s}{2(s+a)}\right)}\right\}.$$

Proof. Let $M = \int_0^1 F'(\hat{x} + t(x_{\alpha,s}^{\delta} - \hat{x}))dt$. Since $F(x_{\alpha,s}^{\delta}) - f^{\delta} + \alpha L^s(x_{\alpha,s}^{\delta} - x_0) = 0$, we have,

$$x_{\alpha,s}^{\delta} - \hat{x} = (F'(x_0) + \alpha L^s)^{-1}[(f^{\delta} - f) + \alpha L^s(x_0^{\delta} - \hat{x}) + (F'(x_0) - M)(x_{\alpha,s}^{\delta} - \hat{x})]$$

Thus by Lemma 5.2.2 and Assumption 2.2.2,

$$\begin{aligned}\|(x^\delta_{\alpha,s}-\hat{x})\| &\leq \|(F'(x_0)+\alpha L^s)^{-1}[(f^\delta-f)+\alpha L^s(x_0-\hat{x})\\ &\quad +(F'(x_0)-M)(x^\delta_{\alpha,s}-\hat{x})]\|\\ &\leq \psi_1(s)\frac{\delta}{\alpha^{\frac{a}{s+a}}}+\|(F'(x_0)+\alpha L^s)^{-1}\alpha L^s(x_0-\hat{x})\|+q\|x^\delta_{\alpha,s}-\hat{x}\|.\end{aligned}$$

So, the result follows, if we prove $\|(F'(x_0)+\alpha L^s)^{-1}\alpha L^s(x_0-\hat{x})\| \leq \frac{E_1}{f(\frac{s}{2(s+a)})}\varphi(\alpha)$. This can be seen as follows.

$$\begin{aligned}\|(F'(x_0)+\alpha L^s)^{-1}\alpha L^s(x_0-\hat{x})\| &= \|\alpha L^{\frac{-s}{2}}(A_s+\alpha I)^{-1}L^{\frac{s}{2}}(x_0-\hat{x})\|\\ &\leq \frac{1}{f\left(\frac{s}{2(s+a)}\right)}\|\alpha(A_s+\alpha I)^{-1}A_s^{\frac{s}{2(s+a)}}L^{\frac{s}{2}}(x_0-\hat{x})\|\\ &\leq \frac{1}{f\left(\frac{s}{2(s+a)}\right)}\|\alpha(A_s+\alpha I)^{-1}\varphi(A_s)v\|\\ &\leq \frac{E_1\varphi(\alpha)}{f\left(\frac{s}{2(s+a)}\right)}.\end{aligned}$$

The last step follows from the Assumption 5.4.1.

Note that the error estimate $\frac{\delta}{\alpha^{\frac{a}{s+a}}}+\varphi(\alpha)$ in Theorem (5.4.2) attains minimum for the choice $\alpha := \alpha(\delta, s, a)$ which satisfies $\varphi(\alpha) = \alpha^{-a/(s+a)}\delta$. Clearly $\alpha(\delta, s, a) = \varphi^{-1}(\psi^{-1}_{s,a}(\delta))$, where

$$\psi_{s,a}(\lambda) = \lambda[\varphi^{-1}(\lambda)]^{a/(s+a)}, \quad 0 < \lambda \leq \|A_s\|. \tag{5.4.16}$$

The following Theorem is a consequence of Theorem 5.3.3 and Theorem 5.4.2.

Theorem 5.4.3. *Let $x^\delta_{n,\alpha,s}$ be as in (5.3.3) and assumptions in Theorem 5.3.3 and Theorem 5.4.2 hold. Then*

$$\|\hat{x}-x^\delta_{n,\alpha,s}\| \leq Cq^{2n}+C_s\left[\frac{\delta}{\alpha^{\frac{a}{s+a}}}+\varphi(\alpha)\right].$$

Theorem 5.4.4. *Let $x^\delta_{n,\alpha,s}$ be as in (5.3.3) and the assumptions of Theorem 5.3.3 hold. And let $n_k := \min\left\{n : q^{2n} \leq \frac{\delta}{\alpha^{\frac{a}{s+a}}}\right\}$. Then*

$$\|\hat{x}-x^\delta_{n_k,\alpha,s}\| \leq \overline{C_s}\left[\frac{\delta}{\alpha^{\frac{a}{s+a}}}+\varphi(\alpha)\right],$$

where $\overline{C_s} = C + C_s$.

5.4.1 Adaptive scheme and stopping rule

Once again the selection of regularization parameter is done using the adaptive scheme considered in Pereverzyev and Schock (2005). We use the same strategy suitably modified for the situation for choosing the parameter α.

For convience take $x_{i,s} := x^{\delta}_{n_i,\alpha_i,s}$. Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu = \eta^{(1+s/a)}, \eta > 1$ and $\alpha_0 = \delta^{(1+s/a)}$. Let

$$l := \max\left\{i : \varphi(\alpha_i) \leq \alpha_i^{-a/(s+a)}\delta\right\} < N. \tag{5.4.17}$$

and

$$k := \max\left\{i : \|x^{\delta}_{i,s} - x^{\delta}_{j,s}\| \leq 4\overline{C_s}\alpha_j^{-a/(s+a)}\delta\right\}, j = 0, 1, 2, \cdots, i. \tag{5.4.18}$$

Now we have the following.

Theorem 5.4.5. *Let $\psi_{s,a}$, l and k be as in (5.4.16), (5.4.17) and (5.4.18) respectively. Then $l \leq k$; and*

$$\|\hat{x} - x^{\delta}_{k,s}\| \leq C_{s,\eta}\psi^{-1}_{s,a}(\delta),$$

where $C_{s,\eta} = \frac{\overline{C_s}(6\eta-2)}{(\eta-1)}$.

Proof. To see that $l \leq k$, it is enough to show that, for $i = 1, 2, \cdots, N$,

$$\varphi(\alpha_i) \leq \alpha_i^{-a/(s+a)}\delta \Longrightarrow \|x^{\delta}_{i,s} - x^{\delta}_{j,s}\| \leq 4\overline{C_s}\alpha_j^{-a/(s+a)}\delta, \qquad \forall j = 0, 1, \cdots, i.$$

For $j \leq i$, by Theorem 5.4.4

$$\begin{aligned}
\|x^{\delta}_{i,s} - x^{\delta}_{j,s}\| &\leq \|x^{\delta}_{i,s} - \hat{x}\| + \|\hat{x} - x^{\delta}_{j,s}\| \\
&\leq \overline{C_s}\left[\frac{\delta}{\alpha_i^{\frac{a}{s+a}}} + \varphi(\alpha_i)\right] + \overline{C_s}\left[\frac{\delta}{\alpha_j^{\frac{a}{s+a}}} + \varphi(\alpha_j)\right] \\
&\leq \overline{C_s}\left[2\alpha_i^{-a/(s+a)}\delta + 2\alpha_j^{-a/(s+a)}\delta\right] \\
&\leq 4\overline{C_s}\alpha_j^{-a/(s+a)}\delta.
\end{aligned}$$

This proves the relation $l \leq k$. Thus by the relation $\alpha_{l+m}^{a/(s+a)} = \eta^m\alpha_l^{a/(s+a)}$ and by using triangle inequality successively, we obtain

$$\begin{aligned}
\|\hat{x} - x_{k,s}^{\delta}\| &\leq \|\hat{x} - x_{l,s}^{\delta}\| + \sum_{i=l+1}^{k} \|x_{i,s}^{\delta} - x_{i-1,s}^{\delta}\| \\
&\leq \|\hat{x} - x_{l,s}^{\delta}\| + \sum_{m=0}^{k-l-1} 4\overline{C_s}\alpha_l^{-a/(s+a)}\eta^{-m}\delta \\
&\leq \|\hat{x} - x_{l,s}^{\delta}\| + \frac{4\eta\overline{C_s}}{\eta - 1}\alpha_l^{-a/(s+a)}\delta. \qquad (5.4.19)
\end{aligned}$$

Therefore by (5.4.19) and Theorem 5.4.4 we have

$$\begin{aligned}
\|\hat{x} - x_{k,s}^{\delta}\| &\leq \overline{C_s}\left[\frac{\delta}{\alpha_l^{\frac{a}{s+a}}} + \varphi(\alpha_l)\right] + \frac{4\eta\overline{C_s}}{\eta - 1}\alpha_l^{-a/(s+a)}\delta \\
&\leq C_{s,\eta}\psi_{s,a}^{-1}(\delta).
\end{aligned}$$

■ □ ■

Chapter 6

CONCLUDING REMARKS

In this thesis we focussed our attention exclusively on some iterative regularization methods for solving nonlinear ill-posed operator equation

$$F(x) = f, \tag{6.0.1}$$

where $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator defined on real Hilbert space X.

Throughout this thesis we assume that the available data is f^δ with $\|f - f^\delta\| \leq \delta$. The approach was to construct an iterative sequence which converges to the unique solution of $F(x) + \alpha(x - x_0) = f^\delta$. It is known that x_α^δ is an approximation for the solution $\hat{x}$, for properly chosen parameter α.

In Chapter 2, we considered a Two Step Modified Newton Lavrentiev Method (TSMNLM) which converges linearly to x_α^δ and also the finite dimensional approximation of the iterative method TSMNLM. A numerical example and the corresponding computational results are exibited to confirm the reliability and effectiveness of our method.

In Chapter 3, we presented cubically converging Two Step Newton Lavrentiev Method (CNLM) and its finite dimensional realization for finding an approximate solution for a nonlinear ill-posed operator equation (6.0.1). The CNLM converges cubically to the solution x_α^δ (x_α^δ is an approximation for the x_0-minimal norm solution of (6.0.1)) of the equation $F(x) + \alpha(x - x_0) = f^\delta$.

In Chapter 4, we have suggested and analyzed another iterative method and its finite dimensional realization for obtaining an approximate solution for nonlinear ill-posed operator equation (6.0.1), and proved that the methods converge locally quarticaly to x_α^δ. Numerical results were provided to show the efficiency of the method.

In Chapter 5, we considered the Hilbert scale variant of the two step Newton method considered in Chapter 2 for approximately solving the ill-posed operator equation (6.0.1). The derived error estimate using a general source condition is of optimal order. The sequence in this chapter converges linearly to the solution $x^\delta_{\alpha,s}$ of the equation $F(x^\delta_{\alpha,s}) + \alpha L^s(x^\delta_{\alpha,s} - x_0) = f^\delta$.

The regularization parameter α in all the chapters was selected according to the adaptive method of Pereverzyev and Schock (2005).

The methods considered in this thesis for solving nonlinear ill-posed operator equations, by no means, is exhaustive. In future works, we would like to analyze the methods in Chapter 3 and Chapter 4 in the Hilbert scale settings.

■ □ ■

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