A STUDY ON *EP* AND HYPO-*EP* OPERATORS

Thesis

Submitted in partial fulfilment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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To my family members and my teachers

DECLARATION

I hereby **declare** that the thesis entitled **"A STUDY ON *EP* AND HYPO-*EP* OPERATORS"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfilment of the requirements for the award of the degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to **certify** that the thesis entitled **"A STUDY ON *EP* AND HYPO-*EP* OPERATORS"** submitted by **VINOTH A.**, (Reg. No. 135012 MA13P02) as the record of the research work carried out by him, is *accepted as the thesis submission* in partial fulfilment of the requirements for the award of degree of **Doctor of Philosophy**.

Dr. P. Sam Johnson
Guide

Chairman - DRPC

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ABSTRACT

A complex square matrix A is said to be EP (EP stands for Equal Projections) if ranges of A and its adjoint are equal. The class of EP matrices was introduced by Schwerdtfeger (Schwerdtfeger, 1950) which contains the class of normal matrices. Later the notion of EP matrix was extended to bounded linear operators on Hilbert spaces with the additional assumption that the operators have closed ranges and then this class of operators was generalized to hypo-EP operators.

In this thesis, we characterize the hypo-EP operators with the aid of factorization of bounded linear operators. Precisely, two kinds of factorizations are involved in these characterizations. One is factorization involving direct sum of operators whereas another is similar to full rank factorization in matrix theory. Also, we prove that for a given subspace $\mathcal{M}$ of $\mathbb{C}^n$, there exists an EP matrix whose range space is $\mathcal{M}$.

The product of two hypo-EP operators is not necessarily hypo-EP and hence we derive necessary and sufficient conditions for product of two hypo-EP to be hypo-EP. Also we come up with some conditions which are necessary or sufficient for sum and restriction of hypo-EP operators to be again hypo-EP.

One of the classical results concerning normal operators is Fuglede theorem which states that if a bounded linear operator commutes with a normal operator then the bounded operator commutes with adjoint of the normal operator. We show that this celebrated result is not true for EP operators and we find some conditions so that the Fuglede-Putnam theorem is true for EP operators. Also, we evince that if we replace adjoint operation by Moore-Penrose inverse, we arrive at Fuglede-Putnam type theorems for EP operators.

We generalize quite a number of characterizations of EP operators on Hilbert spaces into Krein space settings. We extend some of the results of EP and hypo-EP bounded operators into unbounded densely defined closed operators on Hilbert spaces.

Keywords : Moore-Penrose inverse, EP operator, Hypo-EP operator.

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NOTATIONS

$\mathbb{K}$	the field of real or complex scalars
ℓ_2	the set of all square summable sequences
$C[0,1]$	the set of all continuous functions on $[0,1]$
$C^1[0,1]$	the set of all continuously differentiable functions on $[0,1]$
$L^2[0,1]$	the set of all square integrable functions
$\mathbb{C}^{m\times n}$	the set of all $m\times n$ matrices with complex entries
$\overline{\mathcal{M}}$	the closure of $\mathcal{M}$
$\mathcal{M}^\perp$	the orthogonal complement of $\mathcal{M}$
$\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n$	external direct sum or internal direct sum
$\mathcal{M}_1 \oplus^\perp \mathcal{M}_2 \oplus^\perp \cdots \oplus^\perp \mathcal{M}_n$	internal orthogonal direct sum
$c(\mathcal{M},\mathcal{N})$	cosine of the angle between two closed subspaces $\mathcal{M}$ and $\mathcal{N}$
$\mathcal{D}(A)$	domain of A
$\mathcal{N}(A)$	null space of A
$C(A)$	carrier of A
$\mathcal{R}(A)$	range space of A
A^*	adjoint of A
$\|A\|$	norm of the operator A
$A^\dagger$	the Moore-Penrose inverse of A
$A^\#$	the group inverse of A
$\gamma(A)$	the reduced minimum modulus of A
$P_{\mathcal{M}}$	orthogonal projection onto $\mathcal{M}$
$\mathcal{L}(\mathcal{H},\mathcal{K})$	the space of all linear operators from $\mathcal{H}$ into $\mathcal{K}$
$\mathcal{B}(\mathcal{H},\mathcal{K})$	the space of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$
$\mathcal{B}_c(\mathcal{H},\mathcal{K})$	the class of all operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$ with closed range
$\mathcal{C}(\mathcal{H},\mathcal{K})$	the class of all closed linear operators from $\mathcal{H}$ into $\mathcal{K}$

CHAPTER 1

PRELIMINARIES

1.1 GENERAL INTRODUCTION

Among all operators on a Hilbert space, the class of normal operators are considered to be most well understood. The theory of normal operators is so successful that much of the theory of non-normal operators is modeled after it. A natural way to extend a successful theory is to weaken some of its hypotheses obscurely and hope that the results are weakened only slightly. One weakening of normality is EP. The class of EP matrices was first introduced by Schwerdtfeger (Schwerdtfeger, 1950) as complex square matrices of rank r satisfying certain conditions concerning columns and rows. Pearl (Pearl, 1959) reformulated this condition in a simpler form: a matrix A is an EP matrix if $\mathcal{N}(A) = \mathcal{N}(A^*)$. A few years later, in 1966, Pearl (Pearl, 1966) gave an interesting characterization of EP matrix through Moore-Penrose inverse : A is an EP matrix if and only if A commutes with its Moore-Penrose inverse $A^\dagger$. Campbell and Meyer (Campbell and Meyer, 1975) extended the notion of EP matrix into a bounded linear operator with closed range defined on a Hilbert space, using the Pearl's characterization. Itoh (Itoh, 2005) introduced hypo-EP operator by weakening the Pearl's characterization as $A^\dagger A - AA^\dagger$ is a positive operator.

In the thesis, results on EP and hypo-EP operators on Hilbert spaces for bounded and unbounded cases are discussed in detail. Basic definitions and results are presented in the Chapter which are useful in the sequel.

1.2 BASIC DEFINITIONS AND RESULTS

Definition 1.2.1. *(Limaye, 2013) Let $\mathcal{X}$ be a linear space over a field of real or complex scalars $\mathbb{K}$. An **inner product** on $\mathcal{X}$ is a function $\langle \cdot, \cdot \rangle$ from $\mathcal{X} \times \mathcal{X}$ to $\mathbb{K}$ such that for all x, y, z in $\mathcal{X}$ and $\alpha \in \mathbb{K}$, we have*

(a) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$,

(b) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ *and* $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,

(c) $\langle y, x \rangle = \overline{\langle x, y \rangle}$.

A linear space with an inner product is called an **inner product space**.

Definition 1.2.2. *(Limaye, 2013) An inner product space which is complete with respect to the norm induced by the inner product is said to be a **Hilbert space**. We use the letter $\mathcal{H}$ for a Hilbert space.*

Definition 1.2.3. *(Limaye, 2013) Let $\mathcal{X}$ be an inner product space. For x and y in $\mathcal{X}$, we say that x and y are **orthogonal** if $\langle x, y \rangle = 0$. In that case, we write $x \perp y$. For a subset E of an inner product space $\mathcal{X}$,*

$$E^{\perp} = \{y \in \mathcal{X} : y \perp x \text{ for every } x \in E\}.$$

Theorem 1.2.4. *(Limaye, 2013) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a nonempty closed subspace of $\mathcal{H}$. Then $\mathcal{H} = \mathcal{M} + \mathcal{M}^{\perp}$. Moreover, $\mathcal{M}^{\perp\perp} = \mathcal{M}$.*

Definition 1.2.5. *(Groetsch, 2007) Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A map $A : \mathcal{H} \to \mathcal{K}$ is called a **linear operator** if for any $\alpha, \beta \in \mathbb{K}, x, y \in \mathcal{H}$,*

$$A(\alpha x + \beta y) = \alpha A x + \beta A y.$$

*Sometimes the operator may not be defined on the whole space $\mathcal{H}$ and it may be defined on a proper subspace of $\mathcal{H}$. In that case, we denote the **domain of operator** A (simply **domain**) by $\mathcal{D}(A)$. We denote the set of all linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{L}(\mathcal{H}, \mathcal{K})$ and $\mathcal{L}(\mathcal{H}, \mathcal{H}) = \mathcal{L}(\mathcal{H})$.*

Every $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ gives rise to two important subspaces namely, the ***null space*** *$\mathcal{N}(A)$, defined by*

$$\mathcal{N}(A) = \{x \in \mathcal{D}(A) : Ax = 0\}$$

and the ***range space*** *$\mathcal{R}(A)$ defined as*

$$\mathcal{R}(A) = \{Ax : x \in \mathcal{D}(A)\}.$$

If the quantity

$$\|A\| := \sup \left\{ \frac{\|Ax\|}{\|x\|} : x \in \mathcal{D}(A), x \neq 0 \right\} < \infty,$$

then A is called ***bounded****. If $\|A\| = \infty$, then it is called an* ***unbounded operator****. The quantity $\|A\|$ is called the* ***operator norm*** *(or, simply* ***norm****) of the operator.*

The basic difference between bounded and unbounded linear operators is the domain on which they are defined. Domains of unbounded linear operators are proper subspaces of Hilbert spaces.

Throughout the thesis, we consider only linear operators. Hence bounded operator means bounded linear operator. The set of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H}, \mathcal{K})$. A linear operator from $\mathcal{H}$ to itself is called an **operator** on $\mathcal{H}$. We denote the collection of all bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.

Theorem 1.2.6. *(Limaye, 2013) Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then there is a unique operator $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that*

$$\langle Ax, y \rangle = \langle x, By \rangle \qquad \textit{for all } x \in \mathcal{H}, y \in \mathcal{K}. \tag{1.2.1}$$

The operator B is called the ***adjoint*** *of A and it is denoted by A^*.*

In general, a bounded operator on an inner product space need not have an adjoint. The fact that the completeness is essential in the above theorem.

Theorem 1.2.7. *(Limaye, 2013) Let $\mathcal{H}$ be a Hilbert space. Consider $A, B \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{K}$. Then*

1. $(A+B)^* = A^* + B^*$, $(\alpha A)^* = \overline{\alpha} A^*$, $(AB)^* = B^* A^*$, $(A^*)^* = A$. *Further, A is invertible if and only if A^* is invertible, and in that case* $(A^*)^{-1} = (A^{-1})^*$.

2. $\|A^*\| = \|A\|$ *and* $\|A^*A\| = \|A\|^2 = \|AA^*\|$.

Definition 1.2.8. *(Limaye, 2013) An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be an* ***orthogonal projection*** *if $A^2 = A = A^*$.*

Definition 1.2.9. *(Limaye, 2013) Let $A \in \mathcal{B}(\mathcal{H})$. Then A is called*

1. *a* ***self-adjoint operator*** *if $A = A^*$.*

2. *a* ***normal operator*** *if $AA^* = A^*A$.*

3. *an* ***unitary operator*** *if $AA^* = A^*A = I$.*

4. *an* ***isometry*** *if $\|Ax\| = \|x\|$ for all $x \in \mathcal{H}$.*

Theorem 1.2.10. *(Limaye, 2013) Let $A \in \mathcal{B}(\mathcal{H})$. Then A is normal if and only if $\|Ax\| = \|A^*x\|$ for all $x \in \mathcal{H}$. In that case*

$$\|A^2\| = \|A^*A\| = \|A\|^2.$$

Theorem 1.2.11. *(Limaye, 2013) Let $A, B \in \mathcal{B}(\mathcal{H})$.*

1. *Let A and B be self-adjoint. Then $A + B$ is self-adjoint. Also AB is self-adjoint if and only if A and B commute.*

2. *Let A and B be normal operators such that A commutes with B. Then $A+B$ and AB are normal.*

Theorem 1.2.12. *(Limaye, 2013) Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. Then*

1. $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp$ *and* $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$. *Further, A is injective if and only if $\mathcal{R}(A^*)$ is dense in $\mathcal{H}$, and A^* is injective if and only if $\mathcal{R}(A)$ is dense in $\mathcal{H}$.*

2. *The closure of $\mathcal{R}(A)$ equals $\mathcal{N}(A^*)^\perp$, and the closure of $\mathcal{R}(A^*)$ equals $\mathcal{N}(A)^\perp$.*

Definition 1.2.13. *(Limaye, 2013) Let $A \in \mathcal{B}(\mathcal{H})$. A subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is said to be an* ***invariant subspace*** *for A if $A(\mathcal{M}) \subseteq \mathcal{M}$. It is called a* ***reducing subspace*** *for A if both $A(\mathcal{M}) \subseteq \mathcal{M}$ and $A(\mathcal{M}^\perp) \subseteq \mathcal{M}^\perp$. If $\mathcal{M}$ is closed, then $A(\mathcal{M}) \subseteq \mathcal{M}$ if and only if $A^*(\mathcal{M}^\perp) \subseteq \mathcal{M}^\perp$, and in that case $(A|_{\mathcal{M}})^* = PA^*|_{\mathcal{M}}$, where P is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$.*

Theorem 1.2.14. *(Limaye, 2013) Let $A \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent:*

1. *$\mathcal{R}(A)$ is closed in $\mathcal{H}$;*

2. *$\mathcal{R}(A^*)$ is closed in $\mathcal{H}$;*

3. $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$;

4. $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp$;

5. *There exists $k > 0$ such that $\|Ax\| \geq k\|x\|$ for all $x \in \mathcal{N}(A)^\perp$.*

Definition 1.2.15. *(Limaye, 2013) A self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ is said to be* ***positive*** *if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and we write $A \geq 0$. If A and B are self-adjoint operators and $A - B \geq 0$, then we write $A \geq B$ or $B \leq A$.*

Definition 1.2.16. *(Stampfli, 1962) An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be a* ***hyponormal operator*** *if $A^*A - AA^*$ is a positive operator on $\mathcal{H}$.*

Theorem 1.2.17. *(Stampfli, 1962) Let $A \in \mathcal{B}(\mathcal{H})$. Then A is hyponormal if and only if $\|Ax\| \geq \|A^*x\|$ for all $x \in \mathcal{H}$.*

Theorem 1.2.18 (Riesz representation theorem)**.** *(Limaye, 2013) Let $f \in \mathcal{B}(\mathcal{H}, \mathbb{C})$. Then there is a unique $y \in \mathcal{H}$ such that $f(x) = \langle x, y \rangle$, for all $x \in \mathcal{H}$.*

Definition 1.2.19. *(Kubrusly, 2001) Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ be linear spaces over the same field $\mathbb{K}$ (but not necessarily subspaces of the same linear space). The **external direct sum** of $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$, denoted by $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n$, is the set of all ordered n-tuples $(x_1, x_2, \dots, x_n)$ with each x_i in $\mathcal{X}_i$ where vector addition and scalar multiplication are defined as follows.*

$$(x_1, x_2, \dots, x_n) \oplus (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$

$$\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

for every $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ in $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n$ and every α in $\mathbb{K}$. The direct sum $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n$ is a linear space over $\mathbb{K}$ under vector addition and scalar multiplication defined above. The underlying set of the linear space $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n$ is the Cartesian product $\mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n$ of the underlying sets of each linear space $\mathcal{X}_i$.

Definition 1.2.20. *(Kubrusly, 2001) Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ be linear spaces over the same field $\mathbb{K}$ and consider their direct sum $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n$. Let $A_1, A_2, \dots, A_n$ be a family of bounded operators such that $A_i \in \mathcal{B}(\mathcal{X}_i)$ for every i. The **direct sum** of $A_1, A_2, \dots, A_n$, denoted by $A_1 \oplus A_2 \oplus \cdots \oplus A_n$, is the mapping from $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n$ into itself defined by*

$$(A_1 \oplus A_2 \oplus \cdots \oplus A_n)(x_1, x_2, \dots, x_n) = (A_1 x_1, A_2 x_2, \dots, A_n x_n)$$

for every $(x_1, x_2, \dots, x_n) \in \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n$ and $(A_1 \oplus A_2 \oplus \cdots \oplus A_n) \in \mathcal{B}(\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n)$.

Remark 1.2.21. *Let $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$ be closed subspaces of a linear space $\mathcal{X}$ such that $\sum_{k=1}^{n} M_k = \mathcal{X}$ and $\mathcal{M}_j \cap \sum_{\substack{k=1 \\ k \neq j}}^{n} \mathcal{M}_k = \{0\}$ when $j = 1, 2, \dots, n$. Then the linear space $\mathcal{X}$ is called the **internal direct sum** of $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$.*

If $\mathcal{M}_i \perp \mathcal{M}_j$ for $i \neq j$, then the internal direct sum is called the ***internal orthogonal direct sum*** *of $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ and it is denoted by $\mathcal{M}_1 \oplus^{\perp} \mathcal{M}_2 \oplus^{\perp} \cdots \oplus^{\perp} \mathcal{M}_n$. The use of the word external or internal is optional when referring to linear space direct sums. Normally, the context does make it clear which type of direct sum being considered.*

1.3 MOORE-PENROSE INVERSES

Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces over $\mathbb{C}$. Consider the problem of solving a linear equation of the type

$$Ax = b \tag{1.3.2}$$

where $b \in \mathcal{K}$ and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. If the operator A has an inverse then equation (1.3.2) has the unique solution $x = A^{-1}b$. But in general such a linear equation may have no solution or may have more than one solution. Even if the equation has no solution in the traditional meaning, it is still possible to assign what is in a sense a "best possible" solution to the problem. Such a solution is assured by the generalized inverse of A.

Definition 1.3.1. *(Groetsch, 1977) Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have closed range. The mapping $A^{\dagger} : \mathcal{K} \to \mathcal{H}$ defined by $A^{\dagger}b = u$, where u is the least squares solution of minimal norm of the equation $Ax = b$, is called the* ***generalized inverse*** *of A.*

Generalized inverses of matrices and linear operators may be defined in many different ways. The above definition is called variational definition of generalized inverse. Moore was the first to give an explicit definition of the generalized inverse of an arbitrary matrix.

Definition 1.3.2. *(Groetsch, 1977) Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have closed range. Then $A^{\dagger}$ is the unique linear operator in $\mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying*

$$AA^{\dagger} = P_{\mathcal{R}(A)} \text{ and } A^{\dagger}A = P_{\mathcal{R}(A^{\dagger})}$$

where $P_{\mathcal{M}}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$.

This definition was given by Moore in the paper published in Bulletin of the American Mathematical Society in 1920 and its significance was not realized much.

Penrose was unaware of the work of Moore when he published his paper. Penrose defined generalized inverse with four conditions which is equivalent to Moore's definition with two conditions. The following is the Penrose's definition of generalized inverse.

Definition 1.3.3. *(Groetsch, 1977) Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have closed range. Then $A^\dagger$ is the unique operator in $\mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying*

1. $AA^\dagger A = A$,
2. $A^\dagger AA^\dagger = A^\dagger$,
3. $AA^\dagger = (AA^\dagger)^*$,
4. $A^\dagger A = (A^\dagger A)^*$.

Moore-Penrose inverse of a closed range operator between Hilbert spaces exists and it is unique. Moreover, all these three definitions of generalized inverse of an operator A are equivalent and we call $A^\dagger$ as the **Moore-Penrose inverse** of A.

Theorem 1.3.4. *(Groetsch, 1977) Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have closed range. Then the following statements are true.*

1. $A^\dagger \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.
2. $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{R}(A^\dagger A)$.
3. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$.

1.4 UNBOUNDED OPERATORS

Definition 1.4.1. *(Rudin, 1991) An operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ with domain $\mathcal{D}(A)$ is said to be* ***densely defined*** *if $\overline{\mathcal{D}(A)} = \mathcal{H}$.*

Definition 1.4.2. *(Rudin, 1991) Let* $A, B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$*. If* $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ *and*

$$Ax = Bx \quad \textit{for all } x \in \mathcal{D}(A),$$

then A *is called a* ***restriction*** *of* B *(or,* B *is called an* ***extension*** *of* A*) and is denoted by* $A \subseteq B$ *(or, by* $B \supseteq A$*). Note that if* $A \subseteq B$ *and* $B \subseteq A$*, then* $A = B$*.*

Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{K}$ be a bounded operator. Then

$$\|Ax - Ay\| \leq \|A\| \, \|x - y\| \quad \text{for all } x, y \in \mathcal{D}(A).$$

The operator A defined on the subspace $\mathcal{D}(A)$ of $\mathcal{H}$ can be extended continuously to the closure of $\mathcal{D}(A)$ and then it can be extended further to the whole space $\mathcal{H}$ by defining 0 on $\mathcal{D}(A)^{\perp}$. Thus, without loss of generality, we assume that a bounded operator is an everywhere defined operator. This type of operators arise in boundary value problems and domains of unbounded operators are proper subspaces of Hilbert spaces. Thus specification of a domain is an essential part of the definition of an unbounded operator.

Example 1.4.3. *Consider the differential map* A *defined on* $C^1[0,1]$*, a subspace consisting of all differentiable functions whose derivatives are continuous on* $[0,1]$*. Then the operator* $A : C^1[0,1] \subseteq C[0,1] \to C[0,1]$ *(with the sup norm* $\|.\|_\infty$*) is a densely defined unbounded operator.*

Example 1.4.4. *Let* $\mathcal{H} := \ell^2$ *and*

$$\mathcal{D}(A) = \Big\{ (x_1, x_2, \dots) \in \mathcal{H} : (x_1, 2x_2, 3x_3, \dots) \in \mathcal{H} \Big\}.$$

Define

$$A(x_1, x_2, x_3, \dots) = (x_1, 2x_2, 3x_3 \dots) \quad \textit{for all } (x_1, x_2, \dots) \in \mathcal{D}(A).$$

If $\{e_n : n \in \mathbb{N}\}$*, where* $e_n(m) = \delta_{nm}$*, the* ***Kronecker delta*** *function, then* $Ae_n = ne_n$*. Hence the operator* A *is unbounded.*

We have seen that if A is a bounded operator and if the relation (1.2.1) holds for all $x \in \mathcal{H}, y \in \mathcal{K}$, then B would be the uniquely defined bounded operator, called the "adjoint of A." However, in the unbounded case, the relation (1.2.1) by itself does not define B uniquely. It is possible although not obvious that of all the operators satisfying (1.2.1) there will be one with a domain which is maximal (in the sense of set inclusion). If $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}$, the maximal operator, A^* say, provides the required generalization of the adjoint of A.

Definition 1.4.5. *(Rudin, 1991) Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be a densely defined operator. Then there exists a unique operator A^* such that*

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \quad \textit{for all } x \in \mathcal{D}(A) \textit{ and } \quad y \in \mathcal{D}(A^*).$$

*This operator is known as the **Hilbert adjoint** or simply the **adjoint** of A. In this case*

$$\mathcal{D}(A^*) := \Big\{ y \in \mathcal{K} : x \mapsto \langle Ax, y\rangle \quad \textit{for all } x \in \mathcal{D}(A), \quad \textit{is continuous} \Big\}.$$

Equivalently,

$$\mathcal{D}(A^*) : = \Big\{ y \in \mathcal{K} : \textit{for some } y^* \in \mathcal{H},\ \langle Ax, y\rangle = \langle x, y^*\rangle \quad \textit{for all } x \in \mathcal{D}(A) \Big\}$$

and in this case,

$$A^*y = y^* \quad \textit{for all } y \in \mathcal{D}(A^*).$$

Definition 1.4.6. *(Dunford and Schwartz, 1988) Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. If A is one-to-one, then the **inverse** of A is the linear operator $A^{-1}\colon \mathcal{R}(A) \to \mathcal{H}$ defined by $A^{-1}(Ax) = x$ for all $x \in \mathcal{D}(A)$. It can be seen that $AA^{-1}y = y$ for all $y \in \mathcal{R}(A)$.*

*An unbounded operator $A : \mathcal{D}(A) \to \mathcal{H}$ on $\mathcal{H}$ is said to be **invertible** if there exists an everywhere defined bounded operator B such that $BA \subseteq AB = I$.*

Definition 1.4.7. *(Riesz and Sz.-Nagy, 1955) Let $A, B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $C \in \mathcal{L}(\mathcal{K}, \mathcal{I})$ and $\alpha \in \mathbb{C}\backslash\{0\}$. Then*

(a) $A + B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and

$$(A + B)x = Ax + Bx \quad \textit{for all} \quad x \in \mathcal{D}(A + B)$$

(b) $CA \in \mathcal{L}(\mathcal{H}, \mathcal{I})$ *with domain* $\mathcal{D}(CA) := \{x \in \mathcal{D}(A) : Ax \in \mathcal{D}(C)\}$ *and*

$$(CA)x = C(Ax) \quad \text{for all} \quad x \in \mathcal{D}(CA).$$

(c) $\alpha A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *with domain* $\mathcal{D}(\alpha A) = \mathcal{D}(A)$ *and*

$$(\alpha A)x = \alpha Ax \quad \text{for all} \quad x \in \mathcal{D}(A).$$

Proposition 1.4.8. *(Riesz and Sz.-Nagy, 1955)*

1. *If* $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *is densely defined, then* $(\alpha A)^* = \bar{\alpha} A^*$, *for any scalar* α.

2. *If* $A, B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *are densely defined such that* $A + B$ *is densely defined, then* $(A + B)^* \supseteq A^* + B^*$ (*equality holds if* A *is everywhere defined*).

3. *If* A, B *are densely defined such that* $\mathcal{D}(AB)$ *is dense, then* $(AB)^* \supseteq B^*A^*$ (*equality holds if* A *is everywhere defined*).

4. *If* A *is one-to-one and* $\mathcal{R}(A)$ *is dense in* $\mathcal{K}$, *then* $(A^*)^{-1} = (A^{-1})^*$.

5. *If* A *is densely defined such that* $A \subseteq B$, *then* $B^* \subseteq A^*$.

CHAPTER 2

CHARACTERIZATIONS OF HYPO-EP OPERATORS

2.1 INTRODUCTION

A square matrix A over the complex field is said to be an EP matrix if ranges of A and A^* are equal. The EP matrix was defined by Schwerdtfeger (Schwerdtfeger, 1950). But it did not get any greater attention until Pearl (Pearl, 1966) characterized it through Moore-Penrose inverse which is shown below.

Theorem 2.1.1. *(Pearl, 1966) Let $A \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*

(i) A is an EP matrix ;

(ii) $AA^\dagger = A^\dagger A$;

(iii) $A^\dagger$ can be expressed as a polynomial in A with scalar coefficients.

Because of Pearl's characterizations, there are several characterizations of EP matrices available in literature. The following characterizations describe the structure of EP matrices.

Theorem 2.1.2. *(Katz and Pearl, 1966) Let A be a complex square matrix of order n with rank r. Then the following are equivalent:*

(i) A is an EP matrix ;

(ii) A can be represented as

$$A = P \begin{bmatrix} D & DX^* \\ XD & XDX^* \end{bmatrix} P^*,$$

where P is a permutation matrix, D is a non-singular matrix of order r and X is an $n - r \times r$ matrix ;

(iii) There exist a non-singular matrix Q of order n and a non-singular matrix D of order r such that $QAQ^* = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$;

(iv) There is a non-singular matrix Q of order n such that $A^* = QA$ *;*

(v) There is a matrix Q of order n such that $A^* = QA$ *;*

(vi) There exist a unitary matrix U and a non-singular matrix D of order r such that $UAU^* = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$;

(vii) A is the matrix of linear transformation T acting on $\mathbb{C}^n$*, and there are mutually orthogonal subspaces* V_1 *and* V_2 *of* $\mathbb{C}^n$ *such that* V_1 *has dimension r,*

$$T(V_1) = V_1 \text{ and } T(V_2) = 0.$$

Theorem 2.1.3. *(Baksalary and Trenkler, 2008; Cheng and Tian, 2003) Let* $A \in \mathbb{C}^{n\times n}$*. Then the following are equivalent:*

(i) A is an EP matrix ;

(ii) A commutes with $AA^\dagger$ *;*

(iii) $A^\dagger$ *commutes with* $AA^\dagger$ *;*

(iv) $r(A) = r(A^2)$ *and* $A^\dagger A$ *commutes with* $AA^\dagger$ *;*

(v) $r(A) = r(A^2)$ *and* A^*A *commutes with* $AA^\dagger$ *;*

(vi) $(AA^\dagger)^2 = A^2(A^\dagger)^2$ *;*

(vii) $AAA^\dagger + (AAA^\dagger)^* = A + A^*$.

The notion of EP operator was introduced by Campbell and Meyer (Campbell and Meyer, 1975) in 1975. Brock (Brock, 1990) gave few more characterizations of EP operators.

Definition 2.1.4. *(Campbell and Meyer, 1975) An operator $A \in \mathcal{B}(\mathcal{H})$ is called an EP* ***operator*** *if A has closed range and $\mathcal{R}(A) = \mathcal{R}(A^*)$.*

Theorem 2.1.5. *(Brock, 1990) Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent:*

(i) A is an EP operator ;

(ii) $AA^\dagger = A^\dagger A$;

(iii) $\mathcal{N}(A)^\perp = \mathcal{R}(A)$;

(iv) $\mathcal{N}(A) = \mathcal{N}(A^*)$;

(v) $A^* = PA$, where P is a bijective bounded operator on $\mathcal{H}$.

Note that the set of all EP operators contains the set of all normal operators with closed range. We denote the set of all operators on $\mathcal{H}$ with closed range by $\mathcal{B}_c(\mathcal{H})$.

Example 2.1.6. *Let $A : \ell_2 \to \ell_2$ be defined by*

$$A(x_1, x_2, x_3, x_4, x_5, \ldots) = (x_1 + x_2, 2x_1 + x_2 + x_3, -x_1 - x_3, x_4, x_5, \ldots).$$

Then

$$A^*(x_1, x_2, x_3, x_4, x_5, \ldots) = (x_1 + 2x_2 - x_3, x_1 + x_2, x_2 - x_3, x_4, \ldots)$$

and

$$\mathcal{N}(A) = \mathcal{N}(A^*) = \{(x_1, -x_1, -x_1, 0, 0, \ldots) : x_1 \in \mathbb{C}\}.$$

But $AA^* \neq A^*A$. Hence A is an EP operator but not normal.

Some results for EP matrices are not true for EP operators. For example, Theorem 2.1.3 (vi,vii) will not be true in general for EP operators.

Example 2.1.7. *(Djordjević, 2007) Consider the real Hilbert space ℓ_2 and let A be a bounded operator with closed range defined as*

$$A(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

Then

$$A^*(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

and $A^\dagger = A^*$. *In this case* $AA^\dagger = I$ *and*

$$A^\dagger A(x_1, x_2, \ldots) = (0, x_2, x_3, \ldots).$$

Hence A *is not an* EP *operator. But still it is true that*

$$AA^\dagger = A^2(A^\dagger)^2 \text{ and } AAA^\dagger + (AAA^\dagger)^* = A + A^*.$$

Definition 2.1.8. *(Djordjević and Koliha, 2007) The* ***ascent*** *and* ***descent*** *of* $A \in \mathcal{B}(\mathcal{H})$ *are defined by*

$$asc\ A = \inf \left\{p : \mathcal{N}(A^p) = \mathcal{N}(A^{p+1})\right\},$$

$$dsc\ A = \inf \left\{p : \mathcal{R}(A^p) = \mathcal{R}(A^{p+1})\right\}.$$

If they are finite, they are equal and their common value is called the ***index*** *of* A *and it is denoted by* $ind(A)$.

In Example 2.1.7, $asc(A) = \infty$, if we include the additional condition that $asc(A) < \infty$, then the Theorem 2.1.3 can be extended to operators.

Theorem 2.1.9. *(Djordjević, 2007) Let* $A \in \mathcal{B}_c(\mathcal{H})$. *Then the following statements are equivalent:*

(i) A *is* EP ;

(ii) $AA^\dagger = A^2(A^\dagger)^2$ *and* $asc(A) < \infty$;

(iii) $AAA^\dagger + (AAA^\dagger)^* = A + A^*$ *and* $asc(A) < \infty$.

Definition 2.1.10. *Let* $A \in \mathcal{B}(\mathcal{H})$. *The* ***group inverse*** *of* A *is the unique operator* $A^\# \in \mathcal{B}(\mathcal{H})$ *such that*

1. $AA^\# = A^\# A$,

2. $AA^\# A = A$,

3. $A^\# AA^\# = A^\#$.

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be **group invertible** if and only if $ind(A) \leq 1$.

Theorem 2.1.11. *(Djordjević and Koliha, 2007) Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following statements are equivalent:*

(i) A is EP ;

(ii) $ind(A) \leq 1$ and A^*A commutes with $AA^\dagger$;

(iii) $ind(A) \leq 1$ and A commutes with $A^\dagger A^\#$;

(iv) $ind(A) \leq 1$ and $A^\#$ commutes with $AA^\dagger$.

2.2 CONSTRUCTION OF EP MATRICES

Let $\mathcal{H}$ be a complex Hilbert space. Given an EP operator A on $\mathcal{H}$, we get a closed subspace $\mathcal{R}(A)$ which is the same as $\mathcal{R}(A^*)$. On the other hand, one may ask whether every closed subspace $\mathcal{M}$ of $\mathcal{H}$ is the range of some EP operator (not necessarily normal) on $\mathcal{H}$. The answer is affirmative in a finite dimensional Hilbert space $\mathcal{H}$. We give a procedure to construct such EP matrices and this construction has been used in the sequel to provide suitable examples of EP matrices.

Theorem 2.2.1. *If $\mathcal{W}$ is a subspace of $\mathbb{C}^n$, then there exists an EP matrix A of order n such that $\mathcal{R}(A) = \mathcal{W}$.*

Proof. If $\mathcal{W}$ is a trivial subspace of $\mathbb{C}^n$, then it holds trivially. Without loss of generality, let $\mathcal{W}$ be a subspace of $\mathbb{C}^n$ with of dimension $n-1$. Then $\mathcal{W}$ can be expressed as

$$\Big\{(x_1, x_2, \ldots, x_{i-1}, \sum_{k=1}^{n-1} a_k x_k, x_i, \ldots, x_{n-1}) \ : \ x_k \in \mathbb{C}, k = 1, 2, \ldots, n-1\Big\}.$$

Let

$$\left\{ v_j = \big(x_{j1}, x_{j2}, \ldots, x_{j(i-1)}, \sum_{k=1}^{n-1} a_k x_{jk}, x_{ji}, \ldots, x_{j(n-1)}\big), j = 1, 2, \ldots, n-1 \right\}$$

be a basis for $\mathcal{W}$ which can be regarded as column vectors.

Take

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_{i-1} & v' & v_i & \cdots & v_{n-1} \end{bmatrix}$$

where

$$v' = \left(\sum_{k=1}^{n-1} \overline{a_k} x_{k1}, \sum_{k=1}^{n-1} \overline{a_k} x_{k2}, \ldots, \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} a_j \overline{a_k} x_{kj}, \ldots, \sum_{k=1}^{n-1} \overline{a_k} x_{k(n-1)} \right).$$

Since the columns of A contain a basis of $\mathcal{W}$, $\mathcal{R}(A) = \mathcal{W}$. Now we need to show that A is EP. But the selection of v' ensures that each row of A is in $\mathcal{W}$. Hence $\mathcal{R}(A^*) = \mathcal{W}$. Therefore the result is true when dimension of $\mathcal{W}$ is $n-1$.

For the sake of completeness we also prove the result when the dimension of $\mathcal{W}$ is $n-2$. Thus one can construct EP matrices for a given subspace $\mathcal{W}$ having any dimension. Suppose that $\mathcal{W}$ is of dimension $n-2$. Then $\mathcal{W}$ can be expressed as

$$\Big\{ (x_1, x_2, \ldots, x_{i-1}, \sum_{k=1}^{n-2} a_k x_k, x_i, \ldots, x_{\ell-1}, \sum_{k=1}^{n-2} b_k x_k, x_\ell, \ldots, x_{n-2}) \ : \ x_k \in \mathbb{C}, \quad k = 1, 2, \ldots, n-2 \Big\}.$$

Let

$$\Big\{ v_j = \big(x_{j1}, \ldots, x_{j(i-1)}, \sum_{k=1}^{n-2} a_k x_{jk}, x_{ji}, \ldots, x_{j(\ell-1)}, \sum_{k=1}^{n-2} b_k x_{jk}, x_{j\ell}, \ldots x_{j(n-2)} \big), \quad j = 1, 2, \ldots, n-2 \Big\}$$

be a basis for $\mathcal{W}$ which can be regarded as column vectors. Take

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_{i-1} & v' & v_i & \cdots & v_{\ell-1} & v'' & v_\ell & \cdots & v_{n-2} \end{bmatrix}$$

where

$$v' = \left(\sum_{k=1}^{n-2} \overline{a_k} x_{k1}, \sum_{k=1}^{n-2} \overline{a_k} x_{k2}, \ldots, \sum_{j=1}^{n-2} \sum_{k=1}^{n-2} a_j \overline{a_k} x_{kj}, \ldots, \sum_{j=1}^{n-2} \sum_{k=1}^{n-2} b_j \overline{a_k} x_{kj}, \ldots, \sum_{k=1}^{n-2} \overline{a_k} x_{k(n-2)} \right)$$

and

$$v'' = \left(\sum_{k=1}^{n-2} \overline{b_k} x_{k1}, \sum_{k=1}^{n-2} \overline{b_k} x_{k2}, \ldots, \sum_{j=1}^{n-2} \sum_{k=1}^{n-2} a_j \overline{b_k} x_{kj}, \ldots, \sum_{j=1}^{n-2} \sum_{k=1}^{n-2} b_j \overline{b_k} x_{kj}, \ldots, \sum_{k=1}^{n-2} \overline{b_k} x_{k(n-2)} \right).$$

As in the first case, $\mathcal{R}(A) = \mathcal{R}(A^*) = \mathcal{W}$. $\square$

Remark 2.2.2. *If A is a complex EP matrix of rank 1, then it must be normal: Let $A \in \mathbb{C}^{n\times n}$. Then A can be expressed as $A = uv^*$ for some $u, v \in \mathbb{C}^{n\times 1}$ and $A^* = vu^*$. Since A is EP, one has $R(A) = R(A^*)$ and so $sp\{u\} = sp\{v\}$, where sp denotes the linear span. Thus $v = \alpha u$ for some complex scalar α. It now follows that $AA^* = A^*A$.*

Remark 2.2.3. *If A is a real EP matrix of rank 1, then it must be a symmetric matrix. Indeed, as in Remark 2.2.2, $A = \alpha uu^T$ for some $\alpha \in \mathbb{R}$ and u^T denotes the transpose of u. This proves that A is symmetric.*

Example 2.2.4. *Let $\mathcal{W} = \{(x_1, x_1 + x_2, x_2) \ : \ x_1, x_2 \in \mathbb{C}\}$ be a subspace of $\mathbb{C}^3$ with basis*

$$v_1 = (1, 1 + i, i) \text{ and } v_2 = (1, 0, -1).$$

By the proof of the Theorem 2.2.1, we have

$$v' = (2, 1 + i, i - 1).$$

Then

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1+i & 1+i & 0 \\ i & i-1 & -1 \end{bmatrix}.$$

Here A is an EP matrix (non-normal) with $\mathcal{R}(A) = \mathcal{W}$.

Conjecture 2.2.5. *Let $\mathcal{W}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then there exists an EP (non-normal) operator A on $\mathcal{H}$ such that $\mathcal{R}(A) = \mathcal{W}$.*

2.3 CHARACTERIZATIONS OF HYPO - *EP* OPERATORS

All characterizations of EP matrices and EP operators available in literature are algebraic in nature as the definitions of EP matrices and EP operators involve Moore-Penrose inverses.

Definition 2.3.1. *(Itoh, 2005) An operator $A \in \mathcal{B}(\mathcal{H})$ is called* ***hypo-****EP* ***operator*** *if A has closed range and $A^\dagger A - AA^\dagger \geq 0$.*

We first derive an interesting characterization of hypo-EP operator which does not involve Moore-Penrose inverse. Consequences of this characterization and few more characterizations of hypo-EP operators through factorizations are given in the Chapter.

We now start with some known characterizations of hypo-EP operators.

Theorem 2.3.2. *(Itoh, 2005) Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent:*

1. *A is hypo-EP ;*
2. *$\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$;*
3. *$\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$;*
4. *$A = A^*C$, for some $C \in \mathcal{B}(\mathcal{H})$.*

Example 2.3.3. *Let $A : \ell_2 \to \ell_2$ be defined by*

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots).$$

Then $A^*(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$. Here $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$ and $\mathcal{R}(A)$ is closed. Hence A is a hypo-EP operator.

Remark 2.3.4. *The class of all hypo-EP operators contains the class of all EP operators and hyponormal operators with closed ranges. Hence it contains all normal with closed ranges and invertible operators. In the case of finite dimensional, EP and hypo-EP are same.*

Theorem 2.3.5. *(Douglas, 1966)[Douglas' Theorem] Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}), B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$. Then the following are equivalent:*

1. *$A = BC$, for some $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$;*
2. *$\|A^*x\| \leq k\|B^*x\|$, for some $k > 0$ and for all $x \in \mathcal{H}$;*
3. *$\mathcal{R}(A) \subseteq \mathcal{R}(B)$.*

Theorem 2.3.6. *Let $A \in \mathcal{B}_c(\mathcal{H})$. Then A is hypo-EP if and only if for each $x \in \mathcal{H}$, there exists $k > 0$ such that*

$$|\langle Ax, y\rangle| \leq k\|Ay\|, \text{ for all } y \in \mathcal{H}. \tag{2.3.1}$$

Proof. Suppose A is hypo-EP. If $x \in \mathcal{N}(A)$, then the result is trivial. Let $x \in \mathcal{H}$ such that $Ax \neq 0$. Then $Ax \in \mathcal{R}(A) \subseteq \mathcal{R}(A^*)$. Therefore there exists a non-zero $z \in \mathcal{H}$ such that $A^*z = Ax$. Then for all $y \in \mathcal{H}$,

$$|\langle Ax, y\rangle| = |\langle A^*z, y\rangle| = |\langle z, Ay\rangle| \leq \|z\|\|Ay\|.$$

Taking $k = \|z\|$, we get

$$|\langle Ax, y\rangle| \leq k\|Ay\|,$$

for all $y \in \mathcal{H}$.

Conversely, assume that for each $x \in \mathcal{H}$, there exists $k > 0$ such that

$$|\langle Ax, y\rangle| \leq k\|Ay\|$$

for all $y \in \mathcal{H}$. Let $x \in \mathcal{H}$ be fixed. Then for all $y \in \mathcal{H}$,

$$k\|Ay\| \geq |\langle x, A^*y\rangle| = |f_x(A^*y)|$$

setting $f_x(A^*y) = \langle A^*y, x\rangle$. Hence

$$|(Af_x^*)^*y| \leq k\|(A^*)^*y\|$$

for some $k > 0$, for all $y \in \mathcal{H}$. By Douglas' theorem,

$$Af_x^* = A^*D,$$

for some $D \in \mathcal{B}(\mathbb{C}, \mathcal{H})$. Taking adjoint on both sides gives

$$f_xA^* = g_xA$$

where $g_x = D^* \in \mathcal{B}(\mathcal{H}, \mathbb{C})$. By Riesz representation theorem, there exists $x' \in \mathcal{H}$ such that

$$g_x(Az) = \langle Az, x'\rangle$$

for all $z \in \mathcal{H}$. Hence for $z \in \mathcal{H}$,

$$f_x A^* z = g_x A z$$

implies that

$$\langle A^* z, x \rangle = \langle Az, x' \rangle.$$

Therefore for each $x \in \mathcal{H}$ there exists $x' \in \mathcal{H}$ such that $Ax = A^* x'$. Thus $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$. $\square$

The following example given by Barnes shows that $A \in \mathcal{B}_c(\mathcal{H})$ but $A^2 \notin \mathcal{B}_c(\mathcal{H})$.

Example 2.3.7. *(Barnes, 2007) Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with closed subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $\mathcal{K}_1 + \mathcal{K}_2$ is not closed. Let P_1 and P_2 be orthogonal projections with*

$$\mathcal{R}(P_1) = \mathcal{K}_1 \text{ and } \mathcal{R}(P_2) = \mathcal{K}_2^{\perp}.$$

Since $\mathcal{K}_1$ and $\mathcal{K}_2^{\perp}$ are both separble Hilbert spaces which are not finite dimensional, there is an isometry S from $\mathcal{K}_2^{\perp}$ onto $\mathcal{K}_1$. Take

$$A = P_1 S P_2 \in \mathcal{B}(\mathcal{H}).$$

Clearly $\mathcal{R}(A) = \mathcal{K}_1$. Let $x \in \mathcal{N}(A)$. Then

$$0 = Ax = P_1(SP_2 x)$$

and hence $SP_2 x \in \mathcal{K}_1^{\perp}$. But $SP_2 x \in \mathcal{K}_1$. Hence $SP_2 x = 0$. Since S is an isometry, $P_2 x = 0$. Therefore $\mathcal{N}(A) \subseteq \mathcal{N}(P_2)$. Clearly $\mathcal{N}(P_2) \subseteq \mathcal{N}(A)$. Hence

$$\mathcal{N}(A) = \mathcal{N}(P_2) = \mathcal{K}_2.$$

Thus A has closed range and

$$\mathcal{R}(A) + \mathcal{N}(A) = \mathcal{K}_1 + \mathcal{K}_2.$$

Now we claim that $A(\mathcal{R}(A) + \mathcal{N}(A)) = \mathcal{R}(A^2)$. Let $Ax \in \mathcal{R}(A^2)$. Then there exists $y \in \mathcal{H}$ such that $Ax = A^2 y$. Hence $x - Ay \in \mathcal{N}(A)$. Now

$$x = Ay + (x - Ay) \in \mathcal{R}(A) + \mathcal{N}(A).$$

Therefore

$$\mathcal{R}(A^2) \subseteq A(\mathcal{R}(A) + \mathcal{N}(A)).$$

The other inclusion relation is obvious. From this we have

$$A^{-1}\{\mathcal{R}(A^2)\} = \mathcal{R}(A) + \mathcal{N}(A).$$

Since $\mathcal{R}(A) + \mathcal{N}(A)$ is not closed, $\mathcal{R}(A^2)$ is not closed.

We have seen an example of a closed range operator A such that A^2 does not have closed range. But we now prove that if A is hypo-EP, then A^2 has closed range always. Moreover any natural power of A has closed range. Thus it is redundant that $\mathcal{R}(A^n)$ is closed for any $n \in \mathbb{N}$ when A is hypo-EP.

Theorem 2.3.8. *If A is hypo-EP, then A^n has closed range for any $n \in \mathbb{N}$.*

Proof. Suppose that A is hypo-EP. Then for any $m, n \in \mathbb{N}$ with $m \leq n$,

$$A^m[\mathcal{N}(A^n)^\perp] \subseteq A^m[\mathcal{N}(A)^\perp] \subseteq \mathcal{R}(A) \subseteq \mathcal{R}(A^*) = \mathcal{N}(A)^\perp. \tag{2.3.2}$$

As A has closed range, there exists $k > 0$ such that

$$\|Ax\| \geq k\|x\|,$$

for all $x \in \mathcal{N}(A)^\perp$. Let $x \in \mathcal{N}(A^n)^\perp$. Then by (2.3.2),

$$\|A^n x\| = \|A(A^{n-1}x)\| \geq k\|A^{n-1}x\| \geq \cdots \geq k^n\|x\|.$$

Thus A^n has closed range, for any $n \in \mathbb{N}$. □

If we start with any $A \in \mathcal{B}(\mathcal{H})$, the null spaces of A^n are growing in nature along with increasing values of n. But interestingly, all null spaces are same when A is hypo-EP. However, range spaces of A^n may not be the same for any $n \in \mathbb{N}$. For instance, the right shift operator A on ℓ_2 is hypo-EP, but $\mathcal{R}(A) \neq \mathcal{R}(A^n)$ for any $n > 1$.

Theorem 2.3.9. *If A is hypo-EP, then $\mathcal{N}(A^n) = \mathcal{N}(A)$, for each $n \in \mathbb{N}$. Moreover, if A is nilpotent, then $A = 0$.*

Proof. It is enough to prove that $\mathcal{N}(A^n) = \mathcal{N}(A^{n+1})$ for each $n \in \mathbb{N}$. Let $z \in \mathcal{H}$ be fixed. If we apply Theorem 2.3.6 to an element $x = A^{n-1}z$, there exists $k > 0$ such that

$$|\langle A(A^{n-1}z), y\rangle| \leq k\|Ay\|, \text{ for all } y \in \mathcal{H}.$$

In particular taking $y = A^n z$, we get

$$|\langle A^n z, A^n z\rangle| \leq k\|A^{n+1}z\|.$$

If $z \in \mathcal{N}(A^{n+1})$, then $z \in \mathcal{N}(A^n)$. Hence $\mathcal{N}(A^n) = \mathcal{N}(A^{n+1})$ for each $n \in \mathbb{N}$. Thus $\mathcal{N}(A^n) = \mathcal{N}(A)$, for each $n \in \mathbb{N}$. □

Remark 2.3.10. *The condition $\mathcal{N}(A) = \mathcal{N}(A^n)$, for each $n \in \mathbb{N}$ is necessary for A to be hypo-EP. It is not a sufficient condition for A to be hypo-EP. For example, let $A \in \mathcal{B}(\ell_2)$ be defined by*

$$A(x_1, x_2, x_3, x_4, \ldots) = (x_1 + x_2, 0, x_3, x_4, \ldots).$$

Here A is not hypo-EP, but $\mathcal{N}(A^n) = \mathcal{N}(A)$ for each $n \in \mathbb{N}$.

Theorem 2.3.11. *If A is hypo-EP, then A^n is hypo-EP, for any $n \in \mathbb{N}$.*

Proof. Suppose that A is hypo-EP. Then for any $n \in \mathbb{N}$,

$$\mathcal{N}(A^n) = \mathcal{N}(A) \subseteq \mathcal{N}(A^*) \subseteq \mathcal{N}(AA^{(n-1)*}),$$

so $\mathcal{N}(AA^{(n-1)*})^\perp \subseteq \mathcal{N}(A^n)^\perp$. Since $\mathcal{R}(A^{n*})$ is closed and $\mathcal{R}(A^{(n-1)}A^*) \subseteq \overline{\mathcal{R}(A^{(n-1)}A^*)}$,

$$\mathcal{R}(A^{(n-1)}A^*) \subseteq \mathcal{R}(A^{n*}).$$

Then by Douglas' theorem

$$\|AA^{(n-1)*}x\| \leq \ell\|A^n x\|, \text{ for some } \ell > 0, \text{ for all } x \in \mathcal{H} \text{ and } n \in \mathbb{N}.$$

By Theorem 2.3.6, for each $x \in \mathcal{H}$, there exists $k > 0$ such that

$$|\langle A^n x, y\rangle| = |\langle Ax, A^{(n-1)*}y\rangle| \leq k\|AA^{(n-1)*}y\| \leq k\ell\|A^n y\|, \text{ for all } y \in \mathcal{H} \text{ and } n \in \mathbb{N}.$$

Thus for any natural number n, A^n is hypo-EP. □

Remark 2.3.12. *Theorems 2.3.8, 2.3.9 and 2.3.11 have been observed in the paper (Patel and Shekhawat, 2016), but our characterization given in Theorem 2.3.6 was used to prove the results.*

Pearl (Pearl, 1959) showed that a matrix A is EP if and only if A can be expressed as $U(B \oplus 0)U^*$ with U unitary and B an invertible matrix. Drivalliaris (Drivaliaris et al., 2008) extended the results to EP operators on Hilbert spaces. Here we also extend the results to hypo-EP operators on Hilbert spaces. We extend Pearl's characterizations of matrices to hypo-EP operators through factorizations.

Lemma 2.3.13. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{B}_c(\mathcal{H})$ and $B \in \mathcal{B}_c(\mathcal{K})$. Then $A \oplus B$ is hypo-EP if and only if A and B are hypo-EP.*

Proof. Suppose that $A \oplus B$ is hypo-EP and $x \in \mathcal{N}(A)$. Then

$$(x, 0) \in \mathcal{N}(A \oplus B) \subseteq \mathcal{N}(A^* \oplus B^*)$$

and $x \in \mathcal{N}(A^*)$. Hence A is hypo-EP. Similarly B is also hypo-EP. Conversely, suppose that A, B are hypo-EP and $(x, y) \in \mathcal{N}(A \oplus B)$, then $Ax = 0$ and $By = 0$. This implies $A^*x = 0$ and $B^*y = 0$. Hence

$$(x, y) \in \mathcal{N}(A^* \oplus B^*).$$

Therefore $A \oplus B$ is hypo-EP. □

Lemma 2.3.14. *Let $A \in \mathcal{B}_c(\mathcal{H}), B \in \mathcal{B}_c(\mathcal{K})$ and $U \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be injective such that $A = UBU^*$. Then A is hypo-EP if and only if B is hypo-EP.*

Proof. Suppose that B is hypo-EP and $x \in \mathcal{N}(A)$. Then $UBU^*x = 0$. Since U is injective, $BU^*x = 0$ implies that $B^*U^*x = 0$ (B is hypo-EP), which in turn implies that $UB^*U^*x = 0$, equivalently $x \in \mathcal{N}(A^*)$. Hence A is hypo-EP.

Conversely, suppose that A is hypo-EP and $x \in \mathcal{N}(B)$. Therefore $Bx = 0$. Since U is injective, U^* is surjective. Hence for $x \in \mathcal{K}$ there exists $y \in \mathcal{H}$ such that $U^*y = x$. Therefore $BU^*y = 0$ implies that $UBU^*y = Ay = 0$. Since A is hypo-EP, $A^*y = UB^*U^*y = 0$. Using injectivity of U and $U^*y = x$, we get $x \in \mathcal{N}(B^*)$. Hence B is hypo-EP. □

Theorem 2.3.15. *Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent:*

1. *A is hypo-EP ;*

2. *There exist Hilbert spaces $\mathcal{K}_1$ and $\mathcal{L}_1$, $U_1 \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{L}_1, \mathcal{H})$ unitary and $B_1 \in \mathcal{B}(\mathcal{K}_1)$ injective such that $A = U_1(B_1 \oplus 0)U_1^*$;*

3. *There exist Hilbert spaces $\mathcal{K}_2$ and $\mathcal{L}_2$, $U_2 \in \mathcal{B}(\mathcal{K}_2 \oplus \mathcal{L}_2, \mathcal{H})$ isomorphism and $B_2 \in \mathcal{B}(\mathcal{K}_2)$ injective such that $A = U_2(B_2 \oplus 0)U_2^*$;*

4. *There exist Hilbert spaces $\mathcal{K}_3$ and $\mathcal{L}_3$, $U_3 \in \mathcal{B}(\mathcal{K}_3 \oplus \mathcal{L}_3, \mathcal{H})$ injective and $B_3 \in \mathcal{B}(\mathcal{K}_3)$ injective such that $A = U_3(B_3 \oplus 0)U_3^*$.*

Proof. It is enough to prove $(1 \Rightarrow 2)$ and $(4 \Rightarrow 1)$. All other implications follow trivially. Let $\mathcal{K}_1 = \mathcal{R}(A^*)$ and $\mathcal{L}_1 = \mathcal{N}(A)$. Define $U_1 : \mathcal{K}_1 \oplus \mathcal{L}_1 \to \mathcal{H}$ by

$$U_1(y, z) = y + z$$

for $y \in \mathcal{R}(A^*), z \in \mathcal{N}(A)$. Then

$$U_1^* x = (P_{\mathcal{R}(A^*)}x, P_{\mathcal{N}(A)}x),$$

for all $x \in \mathcal{H}$ and U_1 is unitary. Take

$$B_1 = A|_{\mathcal{R}(A^*)} : \mathcal{R}(A^*) \to \mathcal{R}(A^*)$$

which is injective. Since $AP_{\mathcal{R}(A^*)} = A$,

$$A = U_1(B_1 \oplus 0)U_1^*.$$

Hence the implication $(1 \Rightarrow 2)$ is proved. Lemma 2.3.13 and Lemma 2.3.14 give $(4 \Rightarrow 1)$. $\square$

Theorem 2.3.16. *Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent:*

1. *A is hypo-EP ;*

2. *There exist Hilbert spaces $\mathcal{K}_1$ and $\mathcal{L}_1$, $V_1 \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{L}_1, \mathcal{H})$ injective, $W_1 \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{L}_1, \mathcal{H}), S_1 \in \mathcal{B}(\mathcal{H}, \mathcal{K}_1 \oplus \mathcal{L}_1), B_1 \in \mathcal{B}(\mathcal{K}_1)$ injective and $C_1 \in \mathcal{B}(\mathcal{K}_1)$ such that $A = V_1(B_1 \oplus 0)S_1$ and $A^* = W_1(C_1 \oplus 0)S_1$.*

Proof. The implication ($1 \Rightarrow 2$) follows from Theorem 2.3.15. Now assume (2), then from $A = V_1(B_1 \oplus 0)S_1$ and injectivity of V_1 and B_1, we get

$$\mathcal{N}(A) = S_1^{-1}(\{0\} \oplus \mathcal{L}_1).$$

From $A^* = W_1(C_1 \oplus 0)S_1$, we get

$$S_1^{-1}(\{0\} \oplus \mathcal{L}_1) \subseteq \mathcal{N}(A^*).$$

Therefore $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$. Hence A is hypo-EP. □

Theorem 2.3.17. *Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent:*

1. *A is hypo-EP ;*
2. *There exist Hilbert spaces $\mathcal{K}_1$ and $\mathcal{L}_1$, $U_1 \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{L}_1, \mathcal{H})$ isomorphism, $B_1 \in \mathcal{B}(\mathcal{K}_1)$ injective and $C_1 \in \mathcal{B}(\mathcal{K}_1)$ such that*

$$A = U_1(B_1 \oplus 0)U_1^{-1} \text{ and } A^* = U_1(C_1 \oplus 0)U_1^{-1}.$$

Proof. The implication ($1 \Rightarrow 2$) follows from Theorem 2.3.15.
The proof of ($2 \Rightarrow 1$) follows from Theorem 2.3.16. □

Next we are going to prove another characterization through the factorization of the form $A = BC$ which involves the Moore-Penrose inverse of an operator.

Let $A \in \mathcal{B}_c(\mathcal{H})$. Then $A = A|_{\mathcal{R}(A^*)}P_{\mathcal{R}(A^*)}$, where $A|_{\mathcal{R}(A^*)}$ is the restriction of the operator A to $\mathcal{R}(A^*)$ and $P_{\mathcal{R}(A^*)}$ is the projection onto $\mathcal{R}(A^*)$. Here $B = A|_{\mathcal{R}(A^*)}$ and $C = P_{\mathcal{R}(A^*)}$ in the factorization $A = BC$. Also, B is an injective operator with closed range and C is a surjective operator. The factorization of the form $A = BC$ is not unique because of the following reason.

Suppose that $U \in \mathcal{B}(\mathcal{K}, \mathcal{R}(A^*))$ is an isomorphism, $BU \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is injective with closed range and $U^{-1}C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is surjective. Thus

$$A = (BU)(U^{-1}C)$$

is also a factorization of the same type. Therefore the factorization $A = BC$ is not unique. Thus if $A \in \mathcal{B}_c(\mathcal{H})$, then there exists a Hilbert space $\mathcal{K}$ such that $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ injective and $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ surjective with $A = BC$. Moreover, $\mathcal{R}(A) = \mathcal{R}(B), \mathcal{R}(A^*) = \mathcal{R}(C^*), B^\dagger B = I_{\mathcal{K}}$, $CC^\dagger = I_{\mathcal{H}}$ and $A^\dagger = C^\dagger B^\dagger$.

Theorem 2.3.18. *(Bouldin, 1982) Let $A, B \in \mathcal{B}_c(\mathcal{H})$ such that $AB \in \mathcal{B}_c(\mathcal{H})$. Then $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.*

Theorem 2.3.19. *Let $A \in \mathcal{B}_c(\mathcal{H})$ and $A = BC$ be a factorization of A, for some $B, C \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent:*

1. *A is hypo-EP ;*
2. *$C^\dagger C \geq BB^\dagger$;*
3. *$\mathcal{R}(B) \subseteq \mathcal{R}(C^*)$;*
4. *$B = C^\dagger CB$;*
5. *$B^\dagger = B^\dagger C^\dagger C$;*
6. *$AA^* = BCC^*B^*C^*(C^*)^\dagger$;*
7. *$A^*A = C^*B^*C^\dagger CBC$.*

Proof. Since $A^\dagger = C^\dagger B^\dagger$, $CC^\dagger = I$ and $B^\dagger B = I$, A is hypo-EP if and only if $A^\dagger A \geq AA^\dagger$ if and only if $C^\dagger C \geq BB^\dagger$. Hence (1) and (2) are equivalent. The equivalence of (1) and (3) are trivial from the relations $\mathcal{R}(A) = \mathcal{R}(B)$ and $\mathcal{R}(A^*) = \mathcal{R}(C^*)$. Let $R(B) \subseteq R(C^*)$. Since $R(C^*) = R(C^\dagger C)$ and $C^\dagger C$ acts like identity on its range, it follows that $C^\dagger CB = B$. Assume $B = C^\dagger CB$. Since the conditions for Theorem 2.3.18 are satisfied for $C^\dagger C$ and B, taking the Moore-Penrose inverse on both sides gives (5). Now we prove (5) $\Rightarrow$ (3). Suppose that $B^\dagger = B^\dagger C^\dagger C$, then

$$\mathcal{N}(C) \subseteq \mathcal{N}(B^\dagger C^\dagger C) = \mathcal{N}(B^\dagger).$$

Since $\mathcal{N}(B^\dagger) = \mathcal{N}(B^*)$, we have

$$\mathcal{N}(C) \subseteq \mathcal{N}(B^*).$$

Hence

$$\mathcal{R}(B) \subseteq \mathcal{R}(C^*).$$

Suppose that $B = C^\dagger CB$, then (6) and (7) follow directly. Suppose that

$$AA^* = BCC^*B^*C^*(C^*)^\dagger,$$

then

$$\mathcal{N}(A) = \mathcal{N}(C) = \mathcal{N}(C^*)^\dagger \subseteq \mathcal{N}(AA^*).$$

Since $\mathcal{N}(AA^*) = \mathcal{N}(A^*)$, we have $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$. Hence A is hypo-EP.
Finally if $A^*A = C^*B^*C^\dagger CBC$, then $A^*A = A^*P_{\mathcal{R}(A^*)}A$. This implies

$$\|Ax\|^2 = \|P_{\mathcal{R}(A^*)}Ax\|^2.$$

Therefore $Ax = P_{\mathcal{R}(A^*)}Ax$ and hence $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$. Thus A is hypo-EP. $\square$

CHAPTER 3

SUM, PRODUCT AND RESTRICTION OF HYPO-EP OPERATORS

3.1 INTRODUCTION

The sum of two self-adjoint operators on a Hilbert space is again a self-adjoint operator. But the similar result will not hold for normal operators (hence, for hypo-EP operators). Suppose that $A, B \in \mathcal{B}(\mathcal{H})$ are normal operators. Then $A + B$ is normal if A commutes with B^* (Mortad, 2012). In the Chapter, we first discuss necessary and sufficient conditions for sum of two hypo-EP operators to be again a hypo-EP operator. The work of Meenakshi (Meenakshi, 1983) on sum of EP matrices motivated us to analyze the sum of hypo-EP operators.

We next consider a problem of finding conditions (necessary or sufficient or both) such that the product of hypo-EP operators is again a hypo-EP operator. The problem on the product of EP matrices was open around twenty five years. Later a necessary and sufficient condition was given by Hartwig in (Hartwig and Katz, 1997) and the product of EP operators was studied by Djordjevic in (Djordjević, 2001). Nevertheless, Patel in (Patel and Shekhawat, 2016) discussed the product of hypo-EP operators, we give results for the product to be hypo-EP

or EP, if either A or B, is hypo-EP or EP. To prove most of the results regarding product of EP and hypo-EP operators, we use the tool of "angle between a pair of closed subspaces of a Hilbert space". The restriction of hypo-EP operators is discussed in the final section of the Chapter.

3.2 SUM OF HYPO-EP OPERATORS

In general, the sum two hypo-EP operators is not necessarily hypo-EP which is illustrated in the following example.

Example 3.2.1. *Let $A, B \in \mathcal{B}(\ell_2)$ be defined by*

$$A(x_1, x_2, x_3, x_4, \dots) = (x_1 + x_2, x_2, x_3, x_4, \dots)$$

and $B = -I$. Then $A + B(x_1, x_2, x_3, x_4, \dots) = (x_2, 0, 0, \dots)$. Here A and B are hypo-EP, but $A + B$ is not hypo-EP.

Meenakshi (Meenakshi, 1983) discussed results on sum of EP matrices. The next theorem gives a sufficient condition for the sum of hypo-EP operators to be a hypo-EP operator.

Theorem 3.2.2. *Let A, B be hypo-EP operators such that $A+B$ has closed range. If*

$$\|Ax\| \leq k\|(A + B)x\|, \text{ for some } k > 0 \text{ and for all } x \in \mathcal{H}, \tag{3.2.1}$$

then $A + B$ is hypo-EP.

Proof. From (3.2.1), for all $x \in \mathcal{H}$, we have

$$\begin{aligned} \|Bx\| &\leq \|(A + B)x\| + \|Ax\| \\ &\leq \|(A + B)x\| + k\|(A + B)x\| \\ &\leq (k + 1)\|(A + B)x\|. \end{aligned}$$

Since A and B are hypo-EP, for each $x \in \mathcal{H}$ there exist $k_1, k_2 > 0$ such that

$$|\langle Ax, y\rangle| \leq k_1\|Ay\| \quad \text{and} \quad |\langle Bx, y\rangle| \leq k_2\|By\| \quad \text{for all } y \in \mathcal{H}.$$

Now, we have

$$
\begin{aligned}
|\langle (A+B)x, y\rangle| &\leq |\langle Ax, y\rangle| + |\langle Bx, y\rangle| \\
&\leq k_1\|Ay\| + k_2\|By\| \\
&\leq k_1 k\|(A+B)y\| + k_2(k+1)\|(A+B)y\|.
\end{aligned}
$$

Thus

$$|\langle (A+B)x, y\rangle| \leq [k_1 k + k_2(k+1)]\, \|(A+B)y\|.$$

Hence $A+B$ is hypo-EP. □

Corollary 3.2.3. *Let A, B be hypo-EP operators such that $A+B$ has closed range. If $A^*B + B^*A = 0$, then $A+B$ is hypo-EP.*

Proof. The assumption $A^*B + B^*A = 0$ gives

$$(A+B)^*(A+B) = A^*A + B^*B.$$

Then

$$\|(A+B)x\|^2 = \langle (A+B)x, (A+B)x\rangle = \langle (A^*A + B^*B)x, x\rangle \geq \|Ax\|^2.$$

From Theorem 3.2.2, $A+B$ is hypo-EP. □

Remark 3.2.4. *In the above theorem, the condition (3.2.1) is equivalent to*

$$\mathcal{N}(A+B) \subseteq \mathcal{N}(A).$$

But the condition (3.2.1) is not necessary for the sum of A and B to be hypo-EP. For example, let $A, B \in \mathcal{B}(\ell_2)$ be defined by

$$A(x_1, x_2, x_3, x_4, \ldots) = (x_1, -x_2, x_3, x_4, \ldots)$$

and $B = I$. Then A, B and $A+B$ are hypo-EP. But $\mathcal{N}(A+B) \nsubseteq \mathcal{N}(A)$.

Suppose A and B are hypo-EP. Then by Douglas' theorem $A^* = D_A A$ and $B^* = D_B B$ for some operators $D_A, D_B \in \mathcal{B}(\mathcal{H})$. The next theorem shows that the condition (3.2.1) is both necessary and sufficient condition for the sum to be hypo-EP under the assumption that $D_A - D_B$ is invertible.

Theorem 3.2.5. *Let $A, B \in \mathcal{B}_c(\mathcal{H})$ be hypo-EP operators such that $A + B$ has closed range and $D_A - D_B$ be invertible where D_A, D_B as defined above. Then $A + B$ is hypo-EP if and only if*

$$\|Ax\| \leq k\|(A + B)x\|$$

for some $k > 0$ and for all $x \in \mathcal{H}$.

Proof. Assume $A + B$ is hypo-EP. Then

$$A^* + B^* = (A + B)^* = E(A + B)$$

for some $E \in \mathcal{B}(\mathcal{H})$. Hence

$$D_A A + D_B B = E(A + B)$$

which implies that

$$(D_A - E)A = (E - D_B)B.$$

Taking $K = D_A - E, L = E - D_B$, we have $KA = LB$ and $(K + L)A = L(A + B)$. Then

$$A = (K + L)^{-1}L(A + B),$$

since $K + L = D_A - D_B$ is invertible. Hence

$$\|Ax\| \leq k\|(A + B)x\|$$

for all $x \in \mathcal{H}$, where $k = \|(K + L)^{-1}L\|$. The converse follows from Theorem 3.2.2. □

Remark 3.2.6. *The following example ensures that there are operators $A, B \in \mathcal{B}_c(\mathcal{H})$ such that $D_A - D_B$ is invertible: Let $A, B \in \mathcal{B}_c(\ell_2)$ be EP operators defined by $A(x_1, x_2, x_3, x_4, \ldots) = (x_1 + x_2, x_1, x_3, x_4, \ldots)$ and $B(x_1, x_2, x_3, x_4, \ldots) = (ix_1, ix_2, ix_3, ix_4, \ldots)$. Here $D_A = I$, $D_B = -I$ and $D_A - D_B = 2I$.*

3.3 PRODUCT OF HYPO-EP OPERATORS

Every hypo-EP operator is necessarily an operator with closed range. There is an example in (Barnes, 2007) for a bounded operator A in $\mathcal{B}_c(\mathcal{H})$ such that $A^2 \notin \mathcal{B}_c(\mathcal{H})$. But it has been observed that if A is hypo-EP, then A^2 has closed range always. Moreover, any natural power of A has closed range.

We derive few results on product of operators with closed ranges to analyze closed rangeness of "product of hypo-EP operators". We use the notion of angle between a pair of closed subspaces in a Hilbert space and give some of the basic results.

Definition 3.3.1. *(Deutsch, 1995) Let $\mathcal{M}$ and $\mathcal{N}$ be closed subspaces of a Hilbert space $\mathcal{H}$. The angle between $\mathcal{M}$ and $\mathcal{N}$ is the angle $\alpha(\mathcal{M},\mathcal{N})$ in $[0,\pi/2]$ whose cosine is defined by*

$$\begin{aligned} c(\mathcal{M},\mathcal{N}) \;=\; \sup\Big\{ &|\langle x,y\rangle| : x \in \mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^{\perp}, \|x\| \leq 1, \\ &y \in \mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^{\perp}, \|y\| \leq 1\Big\}. \end{aligned}$$

We list some consequences of the definition of angle and a result pertaining to the product of operators with closed range.

Theorem 3.3.2. *(Deutsch, 1995) Let $\mathcal{M}$ and $\mathcal{N}$ be closed subspaces of a Hilbert space $\mathcal{H}$. Then*

1. $0 \leq c(\mathcal{M},\mathcal{N}) \leq 1$.
2. $c(\mathcal{M},\mathcal{N}) = c(\mathcal{N},\mathcal{M})$.
3. $|\langle x,y\rangle| \leq c(\mathcal{M},\mathcal{N})\|x\|\|y\|$, *for all $x \in \mathcal{M}$ and $y \in \mathcal{N}$, and at least one of x or y is in $(\mathcal{M}\cap\mathcal{N})^{\perp}$.*
4. $c(\mathcal{M},\mathcal{N}) = 0$ *if and only if the orthogonal projection onto $\mathcal{M}$ commutes with the orthogonal projection onto $\mathcal{N}$.*
5. $c(\mathcal{M},\mathcal{N}) = c(\mathcal{M}^{\perp},\mathcal{N}^{\perp})$.

Theorem 3.3.3. *(Deutsch, 1995) Let A and B be bounded operators on $\mathcal{H}$ with closed ranges. Then the following statements are equivalent:*

1. *AB has closed range ;*

2. *$c(\mathcal{R}(B), \mathcal{N}(A)) < 1$;*

3. *$\mathcal{R}(B) + \mathcal{N}(A)$ is closed.*

The following example illustrates the fact that there are operators A and B in $\mathcal{B}_c(\mathcal{H})$ such that $AB \in \mathcal{B}_c(\mathcal{H})$ but $BA \notin \mathcal{B}_c(\mathcal{H})$. We shall prove that when A and B are EP operators, the closed rangeness of AB implies the closed rangeness of BA and vice-versa.

Example 3.3.4. *(Sam Johnson and Ganesa Moorthy, 2006) Let A and B be operators on ℓ_2 defined by*

$$A(x_1, x_2, x_3, \ldots) = (x_1, 0, x_2, 0, \ldots)$$

and

$$B(x_1, x_2, x_3, \ldots) = \Big(\frac{x_1}{1} + x_2, \frac{x_3}{3} + x_4, \frac{x_5}{5} + x_6, \ldots\Big).$$

One can verify that both A and B are bounded operators and are having closed ranges. Also, $\mathcal{R}(AB)$ is closed but $\mathcal{R}(BA)$ is not closed.

Theorem 3.3.5. *Let A and B be EP operators on $\mathcal{H}$. Then $\mathcal{R}(AB)$ is closed if and only if $\mathcal{R}(BA)$ is closed.*

Proof. Suppose that $\mathcal{R}(AB)$ is closed. Then by Theorem 3.3.3,

$$c(\mathcal{R}(B), \mathcal{N}(A)) < 1.$$

Now using Theorem 3.3.2, we get

$$c(\mathcal{R}(B)^\perp, \mathcal{N}(A)^\perp) = c(\mathcal{R}(A^*), \mathcal{N}(B^*)) < 1.$$

Since $\mathcal{R}(A) = \mathcal{R}(A^*)$ and $\mathcal{N}(B) = \mathcal{N}(B^*)$,

$$c(\mathcal{R}(A), \mathcal{N}(B)) = c(\mathcal{R}(A^*), \mathcal{N}(B^*)).$$

Therefore $c(\mathcal{R}(A), \mathcal{N}(B)) < 1$. Hence $\mathcal{R}(BA)$ is closed.

Conversely, suppose $\mathcal{R}(BA)$ is closed. Then

$$c(\mathcal{R}(A), \mathcal{N}(B)) < 1.$$

Now again using Theorem 3.3.2, we get

$$c(\mathcal{R}(A)^{\perp}, \mathcal{N}(A)^{\perp}) = c(\mathcal{R}(B^*), \mathcal{N}(A^*)) < 1.$$

Since $\mathcal{R}(A) = \mathcal{R}(A^*)$ and $\mathcal{N}(B) = \mathcal{N}(B^*)$,

$$c(\mathcal{R}(B), \mathcal{N}(A)) = c(\mathcal{R}(B^*), \mathcal{N}(A^*)).$$

Hence $c(\mathcal{R}(B), \mathcal{N}(A)) < 1$ which implies that $\mathcal{R}(AB)$ is closed. $\square$

We now discuss results for the product to be hypo-EP if either A or B is hypo-EP. We first give an example to show that product AB is not necessarily a hypo-EP operator even though A and B are hypo-EP.

Example 3.3.6. *Let A and B be operators on ℓ_2 defined by*

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$$

and

$$B(x_1, x_2, x_3, \ldots) = (0, x_2, 0, x_4, \ldots).$$

Both A and B are hypo-EP operators. Since

$$\mathcal{R}(AB) = \left\{(0, 0, x_1, 0, x_2, 0, \ldots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\right\}$$

and

$$\mathcal{R}((AB)^*) = \left\{(0, x_1, 0, x_2, 0, \ldots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\right\},$$

AB is not a hypo-EP operator.

Theorem 3.3.7. *Let A be a hypo-EP operator and P be the orthogonal projection onto $\mathcal{R}(A)$. Then AP is a hypo-EP operator.*

Proof. Since A has closed range, there is a $k > 0$ such that

$$\|Ax\| \geq k\|x\|$$

for all $x \in \mathcal{N}(A)^{\perp}$. Now let us take $x \in \mathcal{N}(AP)^{\perp}$, then

$$x \in \mathcal{N}(P)^{\perp} = \mathcal{R}(P) = \mathcal{R}(A) \subseteq \mathcal{R}(A^*) = \mathcal{N}(A)^{\perp}$$

and $Px = x$. Hence for $x \in \mathcal{N}(AP)^{\perp}$, we have

$$\|APx\| = \|Ax\| \geq k\|x\|.$$

Thus $\mathcal{R}(AP)$ is closed. Now

$$\mathcal{R}(AP) \subseteq \mathcal{R}(A) = P(\mathcal{R}(A)) \subseteq P(\mathcal{R}(A^*)) = \mathcal{R}(PA^*)$$

which implies that AP is hypo-EP. □

Corollary 3.3.8. *Let A be an EP operator and P be the orthogonal projection onto $\mathcal{R}(A)$. Then AP is an EP operator.*

Proof. From the proof of the Theorem 3.3.7, we can say $\mathcal{R}(AP)$ is closed. Since P is the orthogonal projection onto $\mathcal{R}(A)$,

$$\mathcal{R}(AP) = \mathcal{R}(A) = P(\mathcal{R}(A)) = P(\mathcal{R}(A^*)) = \mathcal{R}(PA^*).$$

Hence AP is EP. □

Theorem 3.3.9. *Let A be a hypo-EP operator and $B \in \mathcal{B}_c(\mathcal{H})$. If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$, then AB is hypo-EP.*

Proof. Since $\mathcal{R}(B)$ and $\mathcal{N}(A)$ are closed subspaces of $\mathcal{H}$, the angle between $\mathcal{R}(B)$ and $\mathcal{N}(A)$ is the angle $\alpha \in [0, \pi/2]$ whose cosine is defined by

$$\begin{aligned} c(\mathcal{R}(B), \mathcal{N}(A)) &= \sup\Big\{|\langle x, y\rangle| : x \in \mathcal{R}(B) \cap (\mathcal{R}(B) \cap \mathcal{N}(A))^{\perp}, \|x\| \leq 1, \\ &\qquad y \in \mathcal{N}(A) \cap (\mathcal{R}(B) \cap \mathcal{N}(A))^{\perp}, \|y\| \leq 1\Big\}. \end{aligned} \tag{3.3.2}$$

Since A is hypo-EP,

$$\mathcal{R}(B) \subseteq \mathcal{R}(A) \subseteq \mathcal{R}(A^*) = \mathcal{N}(A)^{\perp}$$

and hence $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$, so (3.3.2) becomes

$$\begin{aligned} c(\mathcal{R}(B), \mathcal{N}(A)) &= \sup\Big\{|\langle x, y\rangle| : x \in \mathcal{R}(B), \|x\| \leq 1, y \in \mathcal{N}(A), \|y\| \leq 1\Big\} \\ &\leq \sup\Big\{|\langle x, y\rangle| : x \in \mathcal{N}(A)^{\perp}, \|x\| \leq 1, y \in \mathcal{N}(A), \|y\| \leq 1\Big\} \\ &= 0. \end{aligned}$$

Hence AB has closed range. Since A is hypo-EP, $\mathcal{N}(B) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(A^*)$ and hence $\mathcal{R}(A) \subseteq \mathcal{R}(B^*)$.

Now $\mathcal{R}(AB) = A(\mathcal{R}(B)) \subseteq A(\mathcal{R}(A)) \subseteq A(\mathcal{R}(A^*)) = \mathcal{R}(AA^*) = \mathcal{R}(A) \subseteq \mathcal{R}(B^*) = \mathcal{R}(B^*B) = B^*(\mathcal{R}(B)) \subseteq B^*(\mathcal{R}(A)) \subseteq B^*(\mathcal{R}(A^*)) = \mathcal{R}(B^*A^*)$. Hence AB is hypo-EP. $\square$

Corollary 3.3.10. *Let A be a hypo-EP operator on $\mathcal{H}$. Then A^n is hypo-EP for any integer $n \geq 1$.*

Proof. The conditions in Theorem 3.3.9 are trivial when $A = B$. Hence A^2 is hypo-EP. Continuing this process, we get A^n is hypo-EP for any integer $n \geq 1$. $\square$

Remark 3.3.11. *When A and B are EP matrices, the conditions $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ imply that A and B have the same range and null spaces, that is, $\mathcal{R}(A) = \mathcal{R}(B)$ and $\mathcal{N}(A) = \mathcal{N}(B)$. The following examples illustrate that there are hypo-EP operators A and B on an infinite dimensional Hilbert space such that the inclusion relation either in $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ or in $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ is proper.*

Example 3.3.12. *Let A and B be operators on ℓ_2 defined by*

$$A(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

and

$$B(x_1, x_2, \ldots) = (0, x_1, 0, x_2, \ldots).$$

Here both A and B are hypo-EP operators. Also $\mathcal{R}(B) \subsetneq \mathcal{R}(A)$ and $\mathcal{N}(A) = \mathcal{N}(B) = \{0\}$.

Example 3.3.13. *Let A and B be operators on ℓ_2 defined by*

$$A(x_1, x_2, \ldots) = (x_1, 0, x_3, 0, \ldots)$$

and

$$B(x_1, x_2, \ldots) = (x_1, 0, x_2, 0, \ldots).$$

Even though both A and B are hypo-EP operators with $\mathcal{R}(A) = \mathcal{R}(B)$ but $\mathcal{N}(B) \subsetneq \mathcal{N}(A)$.

Remark 3.3.14. *If one of the sufficient conditions in Theorem 3.3.9 is not true, then the product of hypo-EP operator and an operator with closed range need not be a hypo-EP operator. The operators A and B given in Example 3.3.6 are hypo-EP operators and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ but AB is not hypo-EP. Note that $\mathcal{N}(B) \nsubseteq \mathcal{N}(A)$.*

Theorem 3.3.15. *Let A and B be EP operators on $\mathcal{H}$ such that $AB \in \mathcal{B}_c(\mathcal{H})$. Then AB is EP if and only if $\mathcal{R}(AB^*) = \mathcal{R}(B^*A)$.*

Proof. Suppose that A and B are EP operators. Then the following equality relations are true.

$$\mathcal{R}(AB) = A(\mathcal{R}(B)) = A(\mathcal{R}(B^*)) = \mathcal{R}(AB^*)$$

and

$$\mathcal{R}(B^*A) = B^*(\mathcal{R}(A)) = B^*(\mathcal{R}(A^*)) = \mathcal{R}(B^*A^*).$$

Hence AB is EP if and only if $\mathcal{R}(AB^*) = \mathcal{R}(B^*A)$. $\square$

Corollary 3.3.16. *Let A and B be EP operators on $\mathcal{H}$ such that $AB \in \mathcal{B}_c(\mathcal{H})$. Then AB is hypo-EP if and only if $A(\mathcal{R}(B^*)) \subseteq B^*(\mathcal{R}(A))$.*

Corollary 3.3.17. *Let A and B be hypo-EP operators on $\mathcal{H}$ such that $AB \in \mathcal{B}_c(\mathcal{H})$. If*

$$A(\mathcal{R}(B^*)) \subseteq B^*(\mathcal{R}(A)), \tag{3.3.3}$$

then AB is hypo-EP.

Proposition 3.3.18. *Let $A \in \mathcal{B}(\mathcal{H})$ be hypo-EP and $B \in \mathcal{B}(\mathcal{H})$ such that $AB \in \mathcal{B}_c(\mathcal{H})$. If there exists $k > 0$ such that*

$$\|Ax\| \leq k\|ABx\| \text{ for all } x \in \mathcal{H}, \tag{3.3.4}$$

then AB is hypo-EP.

Proof. Let $x \in \mathcal{H}$. Since A is hypo-EP, for $ABx \in \mathcal{R}(A)$ there exists $z \in \mathcal{H}$ such that $ABx = A^*z$. Hence for each $y \in \mathcal{H}$,

$$|\langle ABx, y\rangle| = |\langle A^*z, y\rangle| = |\langle z, Ay\rangle| \leq \|z\|\|Ay\| \leq k\|z\|\|ABy\|. \tag{3.3.5}$$

Take $\ell = k\|z\|$. Therefore for each $x \in \mathcal{H}$, there exists $\ell > 0$ such that

$$|\langle ABx, y\rangle| \leq \ell\|ABy\|$$

for all $y \in \mathcal{H}$. Hence by Theorem 2.3.6, AB is hypo-EP. $\square$

Remark 3.3.19. *The condition (3.3.4) is equivalent to $\mathcal{N}(AB) \subseteq \mathcal{N}(A)$. Also this condition is not necessary for AB to be hypo-EP. For example, let $A, B \in \mathcal{B}(\ell_2)$ be defined by $A = I$ and*

$$B(x_1, x_2, x_3, x_4, \ldots) = (x_1 + x_2, x_1 + x_2, x_3, x_4, \ldots).$$

Here $\mathcal{N}(AB) \nsubseteq \mathcal{N}(A)$. But A, B and AB are all hypo-EP.

Proposition 3.3.20. *Let $A \in \mathcal{B}_c(\mathcal{H})$ and B be hypo-EP operator. If $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and A is injective, then AB is hypo-EP.*

Proof. Since B is hypo-EP, by Theorem 2.3.6, for each $x \in \mathcal{H}$, there is $k_1 > 0$ such that

$$|\langle Bx, y\rangle| \leq k_1\|By\|$$

for all $y \in \mathcal{H}$. Let $x \in \mathcal{H}$. Since $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $ABx \in \mathcal{R}(A)$, there exists $x' \in \mathcal{H}$ such that $ABx = Bx'$. Hence for each $y \in \mathcal{H}$,

$$|\langle ABx, y\rangle| = |\langle Bx', y\rangle| \leq k_1\|By\|. \tag{3.3.6}$$

Since A is injective and $\mathcal{R}(A)$ is closed, there exists $k_2 > 0$ such that

$$\|ABy\| \geq k_2\|By\|$$

for all $y \in \mathcal{H}$. Therefore

$$|\langle ABx, y\rangle| \leq k_1\frac{1}{k_2}\|ABy\|$$

for all $y \in \mathcal{H}$. Hence AB is hypo-EP. $\square$

3.4 RESTRICTION OF HYPO-EP OPERATORS

In this section we discuss restriction of hypo-EP operators. The restriction of $A \in \mathcal{B}(\mathcal{H})$ to an invariant subspace $\mathcal{M}$ is denoted by $A|_{\mathcal{M}}$. The adjoint of $A|_{\mathcal{M}}$ is denoted by $(A|_{\mathcal{M}})^*$ and defined by $(A|_{\mathcal{M}})^* = PA^*|_{\mathcal{M}}$ where P is the orthogonal projection onto $\mathcal{M}$. The restriction operator $A|_{\mathcal{M}}$ coincides with the following properties as in the operator $A \in \mathcal{B}(\mathcal{H})$. The proof of the following proposition is obvious from the definition.

Proposition 3.4.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ be an invariant subspace for both A and B. Then*

1. $(A|_{\mathcal{M}})^{**} = A|_{\mathcal{M}}$.
2. $(AB|_{\mathcal{M}})^* = (B|_{\mathcal{M}})^*(A|_{\mathcal{M}})^*$.

From the definition of hypo-EP operator, for any $A \in \mathcal{B}(\mathcal{H})$, we say $A|_{\mathcal{M}}$ is hypo-EP if $\mathcal{R}(A|_{\mathcal{M}})$ is closed and $\mathcal{R}(A|_{\mathcal{M}}) \subseteq \mathcal{R}((A|_{\mathcal{M}})^*)$.

Theorem 3.4.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ be an invariant subspace for A such that $A|_{\mathcal{M}}$ has closed range. Then $A|_{\mathcal{M}}$ is hypo-EP if and only if for each $x \in \mathcal{M}$ there exists $k > 0$ such that*

$$|\langle A|_{\mathcal{M}}x, y\rangle| \leq k\|A|_{\mathcal{M}}y\|, \text{ for all } y \in \mathcal{M}.$$

Proof. We get the proof by applying Theorem 2.3.6 and Proposition 3.4.1. □

Corollary 3.4.3. *Let A be a hypo-EP operator and $\mathcal{M}$ be an invariant subspace for A such that $A|_{\mathcal{M}}$ has closed range. Then $A|_{\mathcal{M}}$ is hypo-EP.*

Remark 3.4.4. *There are sufficient conditions available in literature that range of $A|_{\mathcal{M}}$ is closed when $A \in \mathcal{B}_c(\mathcal{H})$. In (Barnes, 2007) Barnes gave a sufficient condition that "$\mathcal{R}(A|_{\mathcal{M}}) = \mathcal{R}(A) \cap \mathcal{M}$" to have $\mathcal{R}(A|_{\mathcal{M}})$ is closed. The following example tells that the condition is not necessary.*

Example 3.4.5. *Let A be the right shift operator on ℓ_2 and $\mathcal{M} = \mathcal{R}(A)$. Then $A|_{\mathcal{M}}$ is hypo-EP operator, but $\mathcal{R}(A|_{\mathcal{M}}) \neq \mathcal{R}(A) \cap \mathcal{M}$.*

Theorem 3.4.6. *Let $A \in \mathcal{B}_c(\mathcal{H})$ and $\mathcal{R}(A)$ be a reducing subspace for A. If $A|_{\mathcal{R}(A)}$ is hypo-EP, then A is hypo-EP.*

Proof. Let $y \in \mathcal{H}$. Then y can be expressed as $y = y_1 + y_2$ such that $y_1 \in \mathcal{R}(A)$ and $y_2 \in \mathcal{R}(A)^{\perp}$. For all $y \in \mathcal{H}$,

$$|\langle Ax, y\rangle| = |\langle Ax, y_1\rangle + \langle Ax, y_2\rangle|$$

where $y_1 \in \mathcal{R}(A), y_2 \in \mathcal{R}(A)^{\perp}$. As $Ax \in \mathcal{R}(A)$ and $y_2 \in \mathcal{R}(A)^{\perp}$, we get $|\langle Ax, y\rangle| = |\langle Ax, y_1\rangle|$.

Since $A|_{\mathcal{R}(A)}$ is hypo-EP, there exists $k > 0$ such that

$$|\langle Ax, y\rangle| = |\langle Ax, y_1\rangle| \leq k\|Ay_1\|.$$

Since $\mathcal{R}(A)$ is a reducing subspace for A, we have

$$\|Ay\|^2 = \|Ay_1\|^2 + \|Ay_2\|^2.$$

Hence

$$|\langle Ax, y\rangle| \leq k\|Ay_1\| \leq k\|Ay\|.$$

Thus A is hypo-EP. $\square$

CHAPTER 4

FUGLEDE-PUTNAM TYPE THEOREMS FOR *EP* OPERATORS

4.1 INTRODUCTION

The Fuglede-Putnam theorem (first proved by B. Fuglede (Fuglede, 1950) and then by C. R. Putnam (Putnam, 1951) in a more general version) plays a major role in the theory of bounded (and unbounded) operators. Many authors have worked on it since the papers of Fuglede and Putnam got published (Duggal, 2001; Gong, 1987; Gupta and Patel, 1988; Mecheri, 2004). There are various generalizations of the Fuglede-Putnam theorem to non-normal operators, for instance, hyponormal, subnormal, etc. This Chapter deals with the study of Fuglede-Putnam type theorems for EP operators.

We show that the Fuglede theorem (Fuglede, 1950) is not true in general for EP operators and we prove that the commutativity relation in Fuglede theorem is true for EP operators if the adjoint operation is replaced by Moore-Penrose inverse. Moreover, several versions of Fuglede-Putnam type theorems are given for EP operators. In the last section of the Chapter, we prove some interesting results using Fuglede-Putnam type theorems for EP operators on Hilbert spaces.

4.2 FUGLEDE-PUTNAM TYPE THEOREMS

The well-known Fuglede theorem for a bounded operator is stated as follows.

Theorem 4.2.1. *(Fuglede, 1950). Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator and $T \in \mathcal{B}(\mathcal{H})$. If $TN = NT$, then $TN^* = N^*T$.*

The following example illustrates that Fuglede theorem does not hold good for EP operators. The theorem cannot be extended to the set of EP operators on a Hilbert space $\mathcal{H}$. Howsoever, every normal operator with closed range is EP.

Example 4.2.2. *Consider the EP operator A on ℓ_2 defined by*

$$A(x_1, x_2, x_3, x_4, \ldots) = (x_1 - x_2, x_1 + x_3, 2x_1 - x_2 + x_3, x_4, \ldots)$$

and $T \in \mathcal{B}(\ell_2)$ defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (x_2, -x_1 + x_2 - x_3, -2x_1 + x_2, x_4, \ldots).$$

Here $TA = AT$ but $TA^* \neq A^*T$.

We have seen in the above example that Fuglede theorem is not true in general for EP operators. The following theorem is a Fuglede type theorem which says that if an EP operator commutes with a bounded operator, then the EP operator commutes with the Moore-Penrose inverse of the bounded operator. Our result just replaces the "adjoint" operation by the "Moore-Penrose inverse" in the Fuglede theorem stated in Theorem 4.2.1, however proofs are totally different.

Theorem 4.2.3. *Let A be an EP operator on $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$. If $TA = AT$, then $TA^\dagger = A^\dagger T$.*

Proof. As A is an EP operator, we have $AA^\dagger = A^\dagger A$. From the assumption $TA = AT$, we have

$TA^\dagger = TA^\dagger AA^\dagger = TA(A^\dagger)^2 = AT(A^\dagger)^2 = AA^\dagger AT(A^\dagger)^2 = A^\dagger ATA(A^\dagger)^2 = A^\dagger ATA^\dagger = A^\dagger TAA^\dagger = A^\dagger AA^\dagger TA^\dagger A = (A^\dagger)^2 ATA^\dagger A = (A^\dagger)^2 TAA^\dagger A = (A^\dagger)^2 TA = (A^\dagger)^2 AT = A^\dagger T.$ $\square$

Example 4.2.4. *The assumption that A is an EP operator cannot be dropped in Theorem 4.2.3. For instance, $T = A$ is a bounded operator on ℓ_2 defined by*

$$A(x_1, x_2, x_3, \ldots) = (x_1 + x_2, 2x_1 + 2x_2, x_3, \ldots).$$

Then

$$A^\dagger(x_1, x_2, x_3, \ldots) = \Big(\frac{1}{10}(x_1 + 2x_2), \frac{1}{10}(x_1 + 2x_2), x_3, \ldots\Big).$$

Note that A is not an EP operator and $TA = AT$ but $TA^\dagger \neq A^\dagger T$.

Under some conditions, we prove that Fuglede theorem is true for EP operators and we give examples which embellish that those conditions are necessary.

Theorem 4.2.5. *Let A be an EP operator on $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$. If $TA = AT$ and $TA^*A = A^*AT$, then $TA^* = A^*T$.*

Proof. Suppose $A \in \mathcal{B}(\mathcal{H})$ is an EP operator with $TA = AT$ and $TA^*A = A^*AT$. Then by Theorem 4.2.3, we have $TA^* = T(AA^\dagger A)^* = TA^*(AA^\dagger)^* = TA^*AA^\dagger = A^*ATA^\dagger = A^*AA^\dagger T = (AA^\dagger A)^*T = A^*T$. □

Example 4.2.6. *The condition $TA^*A = A^*AT$ is essential in Theorem 4.2.5. Consider the EP operator A on ℓ_2 defined by*

$$A(x_1, x_2, x_3, \ldots) = (x_1 + x_3, 0, x_3, \ldots)$$

and $T \in \mathcal{B}(\ell_2)$ defined by

$$T(x_1, x_2, x_3, \ldots) = (x_1 + 2x_3, -x_2, x_3, \ldots).$$

Then $TA = AT$ and $TA^*A \neq A^*AT$. But $TA^* \neq A^*T$.

Theorem 4.2.7. *Let A be an EP operator on $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$. If $TA = AT$ and $TA^\dagger A^* = A^\dagger A^*T$, then $TA^* = A^*T$.*

Proof. As $A \in \mathcal{B}(\mathcal{H})$ is an EP operator, we have $AA^\dagger = A^\dagger A$. From the given facts $TA = AT$ and $TA^\dagger A^* = A^\dagger A^*T$, we have $TA^* = T(AA^\dagger A)^* = TA^\dagger AA^* = TAA^\dagger A^* = ATA^\dagger A^* = AA^\dagger A^*T = (AA^\dagger A)^*T = A^*T$. □

Example 4.2.8. *The condition $TA^\dagger A^* = A^\dagger A^* T$ cannot be dropped in Theorem 4.2.7. Let T and A be as in Example 4.2.2. But $TA^\dagger A^* \neq A^\dagger A^* T$ and $TA^* \neq A^* T$.*

Remark 4.2.9. *In general, the operator equations $TA^*A = A^*AT$ and $TA^\dagger A^* = A^\dagger A^* T$ are not equivalent. For example, let us consider the EP operator A on ℓ_2 defined by*

$$A(x_1, x_2, x_3, x_4, \ldots) = (x_1 + x_3, 0, x_3, x_4, \ldots)$$

and the bounded operator T on ℓ_2 defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (2x_1 + x_3, x_2, x_3, x_4, \ldots).$$

Then $A^\dagger(x_1, x_2, x_3, x_4, \ldots) = (x_1 - x_3, 0, x_3, x_4, \ldots)$. Here $TA^\dagger A^* = A^\dagger A^* T$, but $TA^*A \neq A^*AT$.

Fuglede theorem was generalized for two normal operators by Putnam, which is well-known as Fuglede-Putnam theorem and is stated as follows.

Theorem 4.2.10. *(Putnam, 1951) Let N, M be bounded normal operators on $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$. If $TN = MT$, then $TN^* = M^*T$.*

Fuglede-Putnam theorem is not true in general if we replace bounded normal operators by EP operators, as shown in the following example.

Example 4.2.11. *Consider the EP operators A and B on ℓ_2 defined by*

$$A(x_1, x_2, x_3, \ldots) = (x_1 + x_3, 0, x_3, \ldots),$$

$$B(x_1, x_2, x_3, x_4, \ldots) = (x_1 + x_2, x_2, , 0, x_4, \ldots)$$

and $T \in \mathcal{B}(\ell_2)$ defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (x_1 - x_3, x_3, 2x_2, x_4, \ldots).$$

Then $TA = BT$. But $TA^* \neq B^*T$.

Theorem 4.2.12. *Let A, B be EP operators on $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$. If $TA = BT$ and $TA^*A = B^*BT$, then $TA^* = B^*T$.*

Proof. Suppose that $A, B \in \mathcal{B}(\mathcal{H})$ are EP operators with $TA = BT$ and $TA^*A = B^*BT$. Then we have $TA^* = T(AA^\dagger A)^* = TA^*AA^\dagger = B^*BTA^\dagger = B^*BB^\dagger T = (BB^\dagger B)^*T = B^*T$. $\square$

Example 4.2.13. *The condition $TA^*A = B^*BT$ in Theorem 4.2.12 is essential. Let A, B be EP operators and T be the bounded operator as in Example 4.2.11. Here $TA^*A \neq B^*BT$ and $TA = BT$ but $TA^* \neq B^*T$.*

Theorem 4.2.14. *Let A, B be EP operators on $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$. If $TA = BT$ and $TA^\dagger A^* = B^\dagger B^*T$, then $TA^* = B^*T$.*

Proof. As A and B are EP operators with $TA = BT$ and $TA^\dagger A^* = B^\dagger B^*T$, we have $TA^* = T(AA^\dagger A)^* = TA^\dagger AA^* = TAA^\dagger A^* = BTA^\dagger A^* = BB^\dagger B^*T = (BB^\dagger B)^*T = B^*T$. $\square$

Example 4.2.15. *The condition $TA^\dagger A^* = B^\dagger B^*T$ in Theorem 4.2.14 is essential. Let A, B be EP operators and T be the operator as in Example 4.2.11. Here $TA^\dagger A^* \neq B^\dagger B^*T$ and $TA = BT$ but $TA^* \neq B^*T$.*

The following Fuglede-Putnam type theorem for EP operators is a generalization of Theorem 4.2.3 involving two EP operators.

Theorem 4.2.16. *Let A, B be EP operators on $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$. If $TA = BT$, then $TA^\dagger = B^\dagger T$.*

Proof. As A and B are EP operators, we have $AA^\dagger = A^\dagger A$ and $BB^\dagger = B^\dagger B$. From the given fact $TA = BT$, we have $TA^\dagger = TA^\dagger AA^\dagger = TA(A^\dagger)^2 = BT(A^\dagger)^2 = BB^\dagger BT(A^\dagger)^2 = B^\dagger BTA(A^\dagger)^2 = B^\dagger BTA^\dagger = B^\dagger TAA^\dagger = B^\dagger BB^\dagger TA^\dagger A = (B^\dagger)^2BTA^\dagger A = (B^\dagger)^2TAA^\dagger A = (B^\dagger)^2TA = (B^\dagger)^2BT = B^\dagger T$. $\square$

Example 4.2.17. *In the Theorem 4.2.16, if one of the operators, A or B fails to be EP, then the theorem is not valid. Consider the EP operator A on ℓ_2 defined by*

$$A(x_1, x_2, x_3, \ldots) = (x_1 + x_3, 0, x_3, \ldots)$$

and the non-EP operator B on ℓ_2 defined by

$$B(x_1, x_2, x_3, x_4, \ldots) = (x_1 + x_2, 0, 0, x_4, \ldots).$$

Let $T \in \mathcal{B}(\ell_2)$ be defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (x_2 + 2x_3, -x_2, -x_2, x_4, \ldots).$$

Then $TA = BT$. But $TA^\dagger \neq B^\dagger T$.

Theorem 4.2.18. *Let A, B be EP operators on $\mathcal{H}$. If $T, S \in \mathcal{B}(\mathcal{H})$ with $TA = BS$ and $TA^2 = B^2S$, then $TA^\dagger = B^\dagger S$.*

Proof. Suppose that $T, S, A, B \in \mathcal{B}(\mathcal{H})$ with $TA = BS$ and $TA^2 = B^2S$, where A and B are EP operators. Then $TA^\dagger = T(A^\dagger AA^\dagger) = TAA^\dagger A^\dagger = BSA^\dagger A^\dagger = BB^\dagger BSA^\dagger A^\dagger = B^\dagger B^2SA^\dagger A^\dagger = B^\dagger TA^2A^\dagger A^\dagger = B^\dagger TAA^\dagger = B^\dagger B^\dagger BTAA^\dagger = B^\dagger B^\dagger B^2SA^\dagger = B^\dagger B^\dagger TA^2A^\dagger = B^\dagger B^\dagger TA = B^\dagger B^\dagger BS = B^\dagger S$. □

Example 4.2.19. *The assumptions that A and B are EP operators in Theorem 4.2.18 cannot be dropped. For instance, let $T, S, A, B \in \mathcal{B}(\ell_2)$ be defined by*

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_1, x_3, \ldots),$$

$$S = I,$$

$$A(x_1, x_2, x_3, \ldots) = (x_1 + x_2, -x_1 - x_2, x_3, \ldots)$$

and

$$B(x_1, x_2, x_3, \ldots) = (-x_1 - x_2, x_1 + x_2, x_3, \ldots).$$

Here both A, B are not EP operators with $TA = B = BS$. But $TA^\dagger \neq B^\dagger S$.

Example 4.2.20. *The condition $TA^2 = B^2S$ in Theorem 4.2.18 is essential. For instance, let $A, B \in \mathcal{B}(\ell_2)$ be EP operators defined by*

$$A(x_1, x_2, x_3, x_4, \ldots) = (x_1 - x_2, x_1 + x_3, 2x_1 - x_2 + x_3, x_4, \ldots),$$
$$B(x_1, x_2, x_3, \ldots) = (x_1 + x_2, x_2, x_3, \ldots)$$

and let $T, S \in \mathcal{B}(\ell_2)$ be defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (x_1 + 2x_2 - x_3, -x_1 - x_2 + x_3, 2x_1 + 2x_2 - 2x_3, x_4, \ldots),$$
$$S(x_1, x_2, x_3, x_4, \ldots) = (x_1 + x_3, 0, x_1 + x_2, x_4, \ldots).$$

Then $TA = BS$ and $TA^2 \neq B^2S$. But $TA^\dagger \neq B^\dagger S$.

Theorem 4.2.21. *Let A be an EP operator on $\mathcal{H}$ and $T, S \in \mathcal{B}(\mathcal{H})$. If $TA = AS$ and $SA = AT$, then $TA^\dagger = A^\dagger S$ and $SA^\dagger = A^\dagger T$.*

Proof. From given hypotheses, $(T+S)A = A(T+S)$. By Theorem 4.2.3,

$$\begin{aligned} (T+S)A^\dagger &= A^\dagger(T+S) \\ TA^\dagger + SA^\dagger &= A^\dagger T + A^\dagger S \\ TA^\dagger - A^\dagger S &= A^\dagger T - SA^\dagger. \end{aligned} \tag{4.2.1}$$

Again using given hypotheses, $(T-S)A = -A(T-S)$. By Theorem 4.2.16,

$$\begin{aligned} (T-S)A^\dagger &= -A^\dagger(T-S) \\ TA^\dagger - SA^\dagger &= -A^\dagger T + A^\dagger S \\ TA^\dagger - A^\dagger S &= -A^\dagger T + SA^\dagger. \end{aligned} \tag{4.2.2}$$

Adding (4.2.1) and (4.2.2), we have $TA^\dagger = A^\dagger S$. Similarly subtracting (4.2.2) from (4.2.1), we have $SA^\dagger = A^\dagger T$. □

Theorem 4.2.22. *Let A, B be EP operators on $\mathcal{H}$ and $T, S \in \mathcal{B}(\mathcal{H})$. If $TA = BS$ and $SA = BT$, then $TA^\dagger = B^\dagger S$ and $SA^\dagger = B^\dagger T$.*

Proof. From given hypotheses, $(T+S)A = B(T+S)$. By Theorem 4.2.16,

$$\begin{aligned} (T+S)A^\dagger &= B^\dagger(T+S) \\ TA^\dagger + SA^\dagger &= B^\dagger T + B^\dagger S \\ TA^\dagger - B^\dagger S &= B^\dagger T - SA^\dagger. \end{aligned} \tag{4.2.3}$$

Again using given hypotheses, $(T-S)A = -B(T-S)$. By Theorem 4.2.16,

$$\begin{aligned} (T-S)A^\dagger &= -B^\dagger(T-S) \\ TA^\dagger - SA^\dagger &= -B^\dagger T + B^\dagger S \\ TA^\dagger - B^\dagger S &= -B^\dagger T + SA^\dagger. \end{aligned} \tag{4.2.4}$$

Adding (4.2.3) and (4.2.4), we have $TA^\dagger = B^\dagger S$. Similarly subtracting (4.2.4) from (4.2.3), we have $SA^\dagger = B^\dagger T$.

□

4.3 CONSEQUENCES OF FUGLEDE - PUTNAM TYPE THEOREMS FOR EP OPERATORS

The product of EP operators is not an EP operator in general, which is illustrated in the following example.

Example 4.3.1. *Let $A, B \in \mathcal{B}(\ell_2)$ be defined by*

$$A(x_1, x_2, x_3, \ldots) = (x_1 + x_2, x_1 + x_2, x_3, \ldots)$$

and

$$B(x_1, x_2, x_3, \ldots) = (0, x_2, x_3, \ldots).$$

Here A and B are EP operators, but the product AB is not an EP operator.

Djordjević has given a necessary and sufficient condition for product of two EP operators to be an EP operator again.

Theorem 4.3.2. *(Djordjević, 2001) Let A, B be EP operators on $\mathcal{H}$. Then the following statements are equivalent :*

1. *AB is an EP operator ;*
2. *$\mathcal{R}(AB) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(AB) = \mathcal{N}(A) + \mathcal{N}(B)$.*

Example 3.3.4 shows that there are operators A and B on $\mathcal{H}$ in which $\mathcal{R}(AB)$ is closed but $\mathcal{R}(BA)$ is not closed. Also we proved in Theorem 3.3.5 that if A and B are EP operators, then $\mathcal{R}(AB)$ is closed if and only if $\mathcal{R}(BA)$ is closed.

Example 4.3.3. *Consider the EP operators $A, B \in \mathcal{B}(\ell_2)$ defined by*

$$A(x_1, x_2, x_3, \ldots) = (x_1 + x_2, x_2, x_3, \ldots)$$

and

$$B(x_1, x_2, x_3, \ldots) = (x_1, 0, x_3, \ldots).$$

Here AB is an EP operator, but BA is not EP.

Theorem 4.3.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $(AB)^\dagger = B^\dagger A^\dagger$. Then AB and BA are EP if and only if $A^\dagger AB = BAA^\dagger$ and $ABB^\dagger = B^\dagger BA$.*

Proof. Suppose AB and BA are EP. Then $(AB)^\dagger$ and $(BA)^\dagger$ are also EP. Hence we have $A^\dagger(AB)^\dagger = A^\dagger B^\dagger A^\dagger = (BA)^\dagger A^\dagger$. Therefore by Theorem 4.2.16, we have $A^\dagger AB = BAA^\dagger$. In a similar way we have $(AB)^\dagger B^\dagger = B^\dagger A^\dagger B^\dagger = B^\dagger(BA)^\dagger$. Now we use Theorem 4.2.16, we get $ABB^\dagger = B^\dagger BA$.
Conversely, suppose we have

$$A^\dagger AB = BAA^\dagger \tag{4.3.5}$$

$$ABB^\dagger = B^\dagger BA. \tag{4.3.6}$$

From the equation (4.3.5), we get $B^\dagger A^\dagger AB = B^\dagger BAA^\dagger$ and from the equation (4.3.6), we get $ABB^\dagger A^\dagger = B^\dagger BAA^\dagger$. Since the right side of these two equations are same, we have $B^\dagger A^\dagger AB = ABB^\dagger A^\dagger$. Hence $(AB)^\dagger AB = AB(AB)^\dagger$. Therefore AB is EP. Similarly from the equation (4.3.5), we get $A^\dagger ABB^\dagger = BAA^\dagger B^\dagger$ and from the equation (4.3.6), we get $A^\dagger ABB^\dagger = A^\dagger B^\dagger BA$. Therefore $BAA^\dagger B^\dagger = A^\dagger B^\dagger BA$. Hence $BA(BA)^\dagger = (BA)^\dagger BA$. Thus BA is EP. □

Corollary 4.3.5. *Let $A = UP \in \mathbb{C}^{n\times n}$ be a polar decomposition of A where $U \in \mathbb{C}^{n\times n}$ is unitary and $P \in \mathbb{C}^{n\times n}$ is positive semi-definite Hermitian and let $B \in \mathbb{C}^{n\times n}$ with $(AB)^\dagger = B^\dagger A^\dagger$. If BU is EP and $PBU = BUP$, then AB and BA are EP.*

Proof. Suppose BU is EP and $PBU = BUP$, then $BAA^\dagger = B(UP)(UP)^\dagger = BUPP^\dagger U^* = PBUP^\dagger U^* = PP^\dagger BUU^* = PP^\dagger B = P^\dagger PB = P^\dagger U^*UPB = (UP)^\dagger UPB = A^\dagger AB$. Since BU is EP and $PBU = BUP$, we have $P(BU)^\dagger = (BU)^\dagger P$. Therefore $ABB^\dagger = UPBUU^*B^\dagger = UPBU(BU)^\dagger = UBUP(BU)^\dagger = UBU(BU)^\dagger P = U(BU)^\dagger BUP = UU^*B^\dagger BUP = B^\dagger BA$. Thus by Theorem 4.3.4, AB and BA are EP. □

CHAPTER 5

EP OPERATORS ON KREIN SPACES

5.1 INTRODUCTION

An indefinite inner product space is a real (complex) vector space together with a symmetric (Hermitian) bilinear form prescribed on it so that the corresponding quadratic form assumes both positive and negative values. As complete inner product space is called Hilbert space, complete indefinite inner product space is called Krein space with respect to the induced metric. Positive definite inner product spaces are well known objects. Negative definite inner product spaces do not possess any new properties and semi-definite inner product spaces can be reduced to definite ones. Many results in classical inner product spaces will not follow in indefinite inner product settings. In this Chapter, we extend results of EP operators on Hilbert spaces to Krein space settings. Any indefinite inner product space $\mathcal{K}$ can have the following subsets:

- $\beta^+ = \{x \in \mathcal{K} : [x, x] > 0\}$ is called the "**positive cone**,"
- $\beta^- = \{x \in \mathcal{K} : [x, x] < 0\}$ is called the "**negative cone**,"
- $\beta^0 = \{x \in \mathcal{K} : [x, x] = 0\}$ is called the "**neutral cone**."

Here β^+, β^- and β^0 may not be subspaces of $\mathcal{K}$, but all definite subspaces are subsets of β^+ or β^-. Through the definite subspaces, we now define Krein space.

Definition 5.1.1. *If the inner product space $\mathcal{K}$ admits a fundamental decomposition of the form $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$; $\mathcal{K}^+ \subset \beta^+ \cup \{0\}, \mathcal{K}^- \subset \beta^- \cup \{0\}$, where the subspaces $\mathcal{K}^+, \mathcal{K}^-$ are complete with respect to the norm $\|x\| = \big|[x,x]\big|^{1/2}$, then we say that $\mathcal{K}$ is a **Krein space**.*

We can also make any Hilbert space $\mathcal{H}$ into a Krein space by suitablely changing the inner product with a help of self-adjoint bounded operator on $\mathcal{H}$. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and let J be a self-adjoint bounded operator on $\mathcal{H}$ with $J^2 = I$. Then the inner product $\langle \cdot, \cdot \rangle$ defined in $\mathcal{H}$ can be made into sesquilinear form $[\cdot, \cdot]$ as follows:

$$[x, y] = \langle Jx, y \rangle, \quad \text{for } x, y \in \mathcal{H}.$$

Unless $J = I$ or $J = -I$, this quadratic form $[x, x]$ is indefinite which means that for some $x, y \in \mathcal{H}$, we have $[x, x] < 0$ and $[y, y] > 0$. The space $\mathcal{H}$ with the sesquilinear form $[\cdot, \cdot]$ generated by J as defined above is called a **Krein space**.

Definition 5.1.2. *A **Krein space** is an indefinite inner product space $(\mathcal{K}, [\cdot, \cdot])$ such that there exists an automorphism J of $\mathcal{K}$ which squares to the identity and*

$$\langle x, y \rangle = [Jx, y]$$

*defines a positive definite inner product and $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ is a Hilbert space. The operator J is called a **fundamental symmetry**.*

The study of EP matrices on finite dimensional Krein spaces was done by Jayaraman (Jayaraman, 2012). In this Chapter we are giving characterizations of EP operators on the Krein space settings.

If a bounded operator on a Hilbert space has closed range, then the unique Moore-Penrose inverse exists which is bounded and having closed range. But in the case of Krein space, closed range is not sufficient for existence of Moore-Penrose inverse. One of the main reasons for this to happen is that closed subspace of a Krein space is not necessarily a Krein space.

Example 5.1.3. *(Bognár, 1974) Consider* $\mathcal{K} = \{(x_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |x_i|^2 < \infty, x_i \in \mathbb{C}\}$ *with the inner product*

$$[(x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty}] = \sum_{i=1}^{\infty} (-1)^i x_i \overline{y_i}.$$

Let $L = \{(x_i)_{i=1}^{\infty} : x_{2i} = \frac{2i}{2i-1} x_{2i-1}, i = 1, 2, 3, \ldots\}$*. Here* L *is a closed subspace of* $\mathcal{K}$*, but* L *is not complete with respect to the given inner product.*

For any given Krein spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, we denote the set of all bounded operators from $\mathcal{K}_1$ to $\mathcal{K}_2$ by $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ and $\mathcal{B}(\mathcal{K}, \mathcal{K}) = \mathcal{B}(\mathcal{K})$.

Definition 5.1.4. *(Mary, 2008) A subspace* $\mathcal{L}$ *of a Krein space* $\mathcal{K}$ *is said to be* ***regular*** *if* $\mathcal{L} \oplus \mathcal{L}^{\perp} = \mathcal{K}$*.*

Definition 5.1.5. *(Mary, 2008) An operator* $A \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ *is* ***regular*** *if both* $\mathcal{R}(A)$ *and* $\mathcal{N}(A)$ *are regular.*

Theorem 5.1.6. *(Mary, 2008) Let* $A \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ *be regular. Then there exists a unique regular operator* $B \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$ *such that*

1. $ABA = A$,
2. $BAB = B$,
3. $(AB)^* = AB$,
4. $(BA)^* = BA$.

The operator B is called the **Moore-Penrose inverse** of A and it is denoted by $A^{\dagger}$. Xavier Mary (Mary, 2008) has given necessary and sufficient conditions for existence of Moore-Penrose inverse in Krein spaces.

Theorem 5.1.7. *(Mary, 2008) Let* $A \in \mathcal{B}(\mathcal{K})$ *be regular. Then the following are equivalent:*

1. $AA^{\dagger} = A^{\dagger}A$;
2. $\mathcal{N}(A)^{\perp} = \mathcal{R}(A)$;
3. $\mathcal{N}(A) = \mathcal{N}(A^*)$;
4. $\mathcal{R}(A) = \mathcal{R}(A^*)$.

5.2 FACTORIZATION OF EP OPERATORS ON KREIN SPACES

Definition 5.2.1. *An operator $A \in \mathcal{B}(\mathcal{K})$ is called an EP **operator** if A is regular and $AA^\dagger = A^\dagger A$.*

In this section, we see some characterizations of EP operators on Krein spaces through factorization.

Lemma 5.2.2. *Let $A_1 \in \mathcal{B}(\mathcal{K}_1)$ and $A_2 \in \mathcal{B}(\mathcal{K}_2)$ be regular. Then $A_1 \oplus A_2$ is EP if and only if A_1 and A_2 are EP.*

Proof. Suppose $A_1 \oplus A_2$ is EP and $x \in \mathcal{N}(A_1)$. Then $(x, 0) \in \mathcal{N}(A_1 \oplus A_2) = \mathcal{N}(A_1^* \oplus A_2^*)$ and $x \in \mathcal{N}(A_1^*)$. On the other hand if $x \in \mathcal{N}(A_1^*)$, then we have $x \in \mathcal{N}(A_1)$. Hence A_1 is EP. Similarly A_2 is also EP.

Conversely, suppose A_1, A_2 are EP and $(x, y) \in \mathcal{N}(A_1 \oplus A_2)$, then $A_1x = 0$ and $A_2y = 0$. This implies $x \in \mathcal{N}(A_1) = \mathcal{N}(A_1^*)$ and $y \in \mathcal{N}(A_2) = \mathcal{N}(A_2^*)$. Hence $(x, y) \in \mathcal{N}(A_1^* \oplus A_2^*)$. Therefore $A_1 \oplus A_2$ is EP. □

Lemma 5.2.3. *Let $A_1 \in \mathcal{B}(\mathcal{K}_1)$ and $A_2 \in \mathcal{B}(\mathcal{K}_2)$ be regular and $U \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$ be injective such that $A_1 = UA_2U^*$. Then A_1 is EP if and only if A_2 is EP.*

Proof. Suppose A_2 is EP and $x \in \mathcal{N}(A_1)$. Then $UA_2U^*x = 0$. Since U is injective, $A_2U^*x = 0$ implies that $U^*x \in \mathcal{N}(A_2) = \mathcal{N}(A_2^*)$, which in turn implies that $UA_2^*U^*x = 0$, equivalently $x \in \mathcal{N}(A_1^*)$. The other implication follows in a similar way. Hence A_1 is EP.

Conversely, suppose A_1 is EP and $x \in \mathcal{N}(A_2)$. Therefore $A_2x = 0$. Since U is injective, U^* is surjective. Hence for $x \in \mathcal{K}_2$ there exists $y \in \mathcal{K}_1$ such that $U^*y = x$. Therefore $A_2U^*y = 0$ implies that $UA_2U^*y = A_1y = 0$. Since A_1 is EP, $A_1^*y = UA_2^*U^*y = 0$. Using injectivity of U and $U^*y = x$, we get $x \in \mathcal{N}(A_2^*)$. The other implication follows in a similar way. Hence A_2 is EP. □

Remark 5.2.4. *The Lemma 5.2.3 is not true if U is not injective. Consider the Krein space $\mathcal{K}$ in Example 5.1.3. Let A_1, U, A_2 be operators on $\mathcal{K}$ defined by*

$$A_1(x_1, x_2, \ldots,) = (x_1, 0, x_3, 0, \ldots),$$

$$U(x_1, x_2, \ldots) = (x_1 - x_2, x_3 - x_4, \ldots),$$

and

$$A_2(x_1, x_2, \ldots) = (x_2, 0, x_4, \ldots)$$

respectively. Then we have $U^*(x_1, x_2, \ldots,) = (x_1, x_1, x_2, x_2, \ldots)$. *Here* $A_1 = UA_2U^*$ *and* A_1 *is EP, but* U *is not injective and* A_2 *is not EP.*

Theorem 5.2.5. *Let* $A \in \mathcal{B}(\mathcal{K})$ *be regular. Then the following are equivalent:*

1. *A is EP ;*
2. *There exist Krein spaces* $\mathcal{H}_1, \mathcal{L}_1, U_1 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{L}_1, \mathcal{K})$ *unitary and* $B_1 \in \mathcal{B}(\mathcal{H}_1)$ *isomorphism such that* $A = U_1(B_1 \oplus 0)U_1^*$ *;*
3. *There exist Krein spaces* $\mathcal{H}_2, \mathcal{L}_2, U_2 \in \mathcal{B}(\mathcal{H}_2 \oplus \mathcal{L}_2, \mathcal{K})$ *isomorphism and* $B_2 \in \mathcal{B}(\mathcal{H}_2)$ *isomorphism such that* $A = U_2(B_2 \oplus 0)U_2^*$ *;*
4. *There exist Krein spaces* $\mathcal{H}_3, \mathcal{L}_3, U_3 \in \mathcal{B}(\mathcal{H}_3 \oplus \mathcal{L}_3, \mathcal{K})$ *injective and* $B_3 \in \mathcal{B}(\mathcal{H}_3)$ *isomorphism such that* $A = U_3(B_3 \oplus 0)U_3^*$.

Proof. Assume that A is EP. Let $\mathcal{H}_1 = \mathcal{R}(A), \mathcal{L}_1 = \mathcal{N}(A)$. Since $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are regular, they are Krein spaces. Then the map $U_1 : \mathcal{H}_1 \oplus \mathcal{L}_1 \to \mathcal{K}$ is defined by

$$U_1(x, y) = x + y$$

for all $x \in \mathcal{R}(A), y \in \mathcal{N}(A)$.

To say U_1 is unitary we have to show U_1 is surjective and $[U_1(y_1, z_1), U_1(y_2, z_2)] = [(y_1, z_1), (y_2, z_2)]$. This can be done since $\mathcal{R}(A) \oplus^{\perp} \mathcal{N}(A) = \mathcal{K}$. In fact we can explicitly say

$$U_1^* k = (P_{\mathcal{R}(A)}k, P_{\mathcal{N}(A)}k),\ k \in \mathcal{K}.$$

Then $B_1 = A\ |_{\mathcal{R}(A)}: \mathcal{R}(A) \to \mathcal{R}(A)$ is isomorphism, since $\mathcal{R}(A^*) = \mathcal{R}(A)$. Hence $A = U_1(B_1 \oplus 0)U_1^*$. This proves $(1 \Rightarrow 2)$.

The implications $(2 \Rightarrow 3)$ and $(3 \Rightarrow 4)$ are obvious. $(4 \Rightarrow 1)$ follows from Lemmas 5.2.2 and 5.2.3. □

Remark 5.2.6. *Theorem 5.2.5 gives a key idea to construct Moore-Penrose inverse of an EP operator. If $A = U_1(B_1 \oplus 0)U_1^*$, then $A^\dagger = U_1(B_1^{-1} \oplus 0)U_1^*$. Also if we do not assume U_3 is injective, then A is not necessarily EP.*

In Theorem 5.2.5 if we assume B_1 is injective with closed range, then A is not necessarily EP. The next characterization is given through simultaneous factorization of A and A^* of the form $A = U(B \oplus 0)U^*$ and $A^* = U(C \oplus 0)U^*$ with U, B and C injective.

Theorem 5.2.7. *Let $A \in \mathcal{B}(\mathcal{K})$ be regular. Then the following are equivalent:*

1. *A is EP ;*

2. (a) *There exist Krein spaces $\mathcal{H}_1$ and $\mathcal{L}_1$, $V_1 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{L}_1, \mathcal{K})$ injective, $W_1 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{L}_1, \mathcal{K}), S_1 \in \mathcal{B}(\mathcal{K}, \mathcal{H}_1 \oplus \mathcal{L}_1), B_1 \in \mathcal{B}(\mathcal{H}_1)$ injective and $C_1 \in \mathcal{B}(\mathcal{H}_1)$ such that $A = V_1(B_1 \oplus 0)S_1$ and $A^* = W_1(C_1 \oplus 0)S_1$.*

   (b) *There exist Krein spaces $\mathcal{H}_2$ and $\mathcal{L}_2$, $V_2 \in \mathcal{B}(\mathcal{H}_2 \oplus \mathcal{L}_2, \mathcal{K})$, $W_2 \in \mathcal{B}(\mathcal{H}_2 \oplus \mathcal{L}_2, \mathcal{K})$ injective, $S_2 \in \mathcal{B}(\mathcal{K}, \mathcal{H}_2 \oplus \mathcal{L}_2), B_2 \in \mathcal{B}(\mathcal{H}_2)$ and $C_2 \in \mathcal{B}(\mathcal{H}_2)$ injective such that $A = V_2(B_2 \oplus 0)S_2$ and $A^* = W_2(C_2 \oplus 0)S_2$.*

Proof. $(1 \Rightarrow 2)$: The proof follows from Theorem 5.2.5.
$(2 \Rightarrow 1)$: Assume (a) holds. $A = V_1(B_1 \oplus 0)S_1$ and V_1 and B_1 are injective, we get

$$\mathcal{N}(A) = S_1^{-1}(\{0\} \oplus \mathcal{L}_1)$$

and $A^* = W_1(C_1 \oplus 0)S_1$ gives

$$S_1^{-1}(\{0\} \oplus L_1) \subseteq \mathcal{N}(A^*).$$

Therefore $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$. By (b) we get $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$. Hence A is EP. $\square$

The above statement may look clumsy, but it tells us the effectiveness of B_1 is isomorphism. The next theorem we are going to show that effectiveness of the assumption that B_1 is isomorphism.

Theorem 5.2.8. *Let $A \in \mathcal{B}(\mathcal{K})$ be regular. Then the following are equivalent:*

1. *A is EP ;*

2. *There exist Krein spaces $\mathcal{H}_1, \mathcal{L}_1, U \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{L}_1, \mathcal{K})$ isomorphism and $B \in \mathcal{B}(\mathcal{H}_1)$ isomorphism and $C \in \mathcal{B}(\mathcal{H}_1)$ such that $A = U(B \oplus 0)U^{-1}$ and $A^* = U(C \oplus 0)U^{-1}$.*

Proof. $(1 \Rightarrow 2)$: The proof follows from Theorem 5.2.5.
$(2 \Rightarrow 1)$: From the proof of $(2 \Rightarrow 1)$ in Theorem 5.2.7 we get $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$. Taking adjoint in expressions given by A and A^*, we get

$$A^* = (U^*)^{-1}(B^* \oplus 0)U^*, A = (U^*)^{-1}(C^* \oplus 0)U^*.$$

In the same argument, we get $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$. Hence A is EP. □

5.3 SIMULTANEOUS FACTORIZATION OF AA^* AND A^*A

In the section, we use simultaneous factorization of AA^* and A^*A to characterize EP operators on Krein spaces.

Proposition 5.3.1. *Let $A \in \mathcal{B}(\mathcal{K})$ be regular. Then the following are equivalent:*

1. *A is EP ;*

2. *There exist Krein spaces $\mathcal{H}_1, \mathcal{L}_1, U_1 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{L}_1, \mathcal{K})$ unitary and $B_1 \in \mathcal{B}(\mathcal{H}_1)$ isomorphism such that $A^*A = U_1(B_1^* B_1 \oplus 0)U_1^*$ and $AA^* = U_1(B_1 B_1^* \oplus 0)U_1^*$;*

3. (a) *There exist Krein spaces $\mathcal{H}_2$ and $\mathcal{L}_2$, $V_2 \in \mathcal{B}(\mathcal{H}_2 \oplus \mathcal{L}_2, \mathcal{K})$ injective, $W_2 \in \mathcal{B}(\mathcal{H}_2 \oplus \mathcal{L}_2, \mathcal{K}), S_2 \in \mathcal{B}(\mathcal{K}, \mathcal{H}_2 \oplus \mathcal{L}_2), B_2 \in \mathcal{B}(\mathcal{H}_2)$ injective and $C_2 \in \mathcal{B}(\mathcal{H}_2)$ such that $A^*A = V_2(B_2 \oplus 0)S_2$ and $AA^* = W_2(C_2 \oplus 0)S_2$;*

   (b) *There exist Krein spaces $\mathcal{H}_3$ and $\mathcal{L}_3$, $V_3 \in \mathcal{B}(\mathcal{H}_3 \oplus \mathcal{L}_3, \mathcal{K})$, $W_3 \in \mathcal{B}(\mathcal{H}_3 \oplus \mathcal{L}_3, \mathcal{K})$ injective, $S_3 \in \mathcal{B}(\mathcal{K}, \mathcal{H}_3 \oplus \mathcal{L}_3), B_3 \in \mathcal{B}(\mathcal{H}_3)$ and $C_3 \in \mathcal{B}(\mathcal{H}_3)$ injective such that $AA^* = V_3(B_3 \oplus 0)S_3$ and $AA^* = W_3(C_3 \oplus 0)S_3$;*

4. *There exist Krein spaces $\mathcal{H}_4, \mathcal{L}_4, U_4 \in \mathcal{B}(\mathcal{H}_4 \oplus \mathcal{L}_4, \mathcal{K})$ isomorphism and $B_4 \in \mathcal{B}(\mathcal{H}_4)$ isomorphism and $C_4 \in \mathcal{B}(\mathcal{H}_4)$ such that $A^*A = U_4(B_4 \oplus 0)U_4^{-1}$ and $AA^* = U_4(C_4 \oplus 0)U_4^{-1}$.*

Proof. $(1 \Rightarrow 2)$: By Theorem 5.2.5. $(2 \Rightarrow 3), (2) \Rightarrow (4)$ and $(3 \Rightarrow 1)$ are obvious. $(2 \Rightarrow 1)$: As in proof of $(2 \Rightarrow 1)$ in Theorem 5.2.7, we get $\mathcal{N}(AA^*) = \mathcal{N}(A^*A)$. But we know that $\mathcal{N}(AA^*) = \mathcal{N}(A^*)$ and $\mathcal{N}(A^*A) = \mathcal{N}(A)$. Therefore $\mathcal{N}(A) = \mathcal{N}(A^*)$. Hence A is EP. $\square$

Remark 5.3.2. *If we assume that one of the conditions in (2) holds, then A is not in general EP. Consider the Krein space $\mathcal{K}$ in Example 5.1.3. Let $A \in \mathcal{B}(\mathcal{K})$ be defined by $A(x_1, x_2, x_3, \ldots) = (x_2, 0, 0, \ldots)$. Then $A^*(x_1, x_2, x_3, \ldots) = (0, -x_1, 0, \ldots)$. Here $AA^*(x_1, x_2, x_3, \ldots) = (-x_1, 0, 0, \ldots)$ and $A^*A = 0$. Since A is not EP, factorization of AA^* exists whereas factorization of A^*A does not exist.*

Proposition 5.3.3. *Let $A \in \mathcal{B}(\mathcal{K})$ be regular. Then the following are equivalent:*

1. *A is EP ;*
2. *There exists an isomorphism $N_1 \in \mathcal{B}(\mathcal{K})$ such that $A^* = N_1A$;*
3. *There exists $N_2 \in \mathcal{B}(\mathcal{K})$ injective such that $A^* = N_2A$;*
4. *There exist $S_1, S_2 \in \mathcal{B}(\mathcal{K})$ such that $A^* = S_1A$ and $A = S_2A^*$.*

Proof. $(1 \Rightarrow 2)$: By Theorem 5.2.5, we have $A = U(B_1 \oplus 0)U^*$ with $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{L}, \mathcal{K})$ unitary and $B \in \mathcal{B}(\mathcal{H})$ an isomorphism. If we take $N_1 = U(B^*B^{-1} \oplus I)U^* : \mathcal{K} \to \mathcal{K}$, then N_1 is an isomorphism with $A^* = N_1A$. $(2 \Rightarrow 3)$ is direct and $(2 \Rightarrow 4)$ follows from $A = N_1^{-1}A^*$.

$(3 \Rightarrow 1)$: As $A^* = N_1A$, we get $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$. But N_1 is injective implies $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$. Hence A is EP. $(4 \Rightarrow 1)$: By $A^* = S_1A$, we have $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$ and by $A = S_2A^*$ we get that $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$. $\square$

Proposition 5.3.4. *Let $A \in \mathcal{B}(\mathcal{K})$ be regular. Then the following are equivalent:*

1. *A is EP ;*

2. *There exists an isomorphism* $N_1 \in \mathcal{B}(\mathcal{K})$ *such that* $A^\dagger = N_1 A = A N_1$ *;*

3. *There exists* $N_2 \in \mathcal{B}(\mathcal{K})$ *injective such that* $A^\dagger = N_2 A$ *;*

4. *There exist* $S_1, S_2 \in \mathcal{B}(\mathcal{K})$ *such that* $A^\dagger = S_1 A$ *and* $A = S_2 A^\dagger$*.*

Proof. $(1 \Rightarrow 2)$: By Theorem 5.2.5, we have $A = U(B_1 \oplus 0)U^*$ with $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{L}, \mathcal{K})$ unitary and $B \in \mathcal{B}(\mathcal{H})$ an isomorphism. If we take $N_1 = U(B^{-2} \oplus I)U^* : \mathcal{K} \to \mathcal{K}$, then N_1 is an isomorphism with $A^\dagger = N_1 A = A N_1$. As $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$, the rest follows from the proof of Theorem 5.3.3. □

Proposition 5.3.5. *Let* $A \in \mathcal{B}(\mathcal{K})$ *be regular. Then the following are equivalent:*

1. *A is EP ;*

2. *There exists an isomorphism* $N_1 \in \mathcal{B}(\mathcal{K})$ *such that* $A^*A = N_1 AA^*$ *;*

3. *There exists* $N_2 \in \mathcal{B}(\mathcal{K})$ *injective such that* $A^*A = N_2 AA^*$ *;*

4. *There exist* $S_1, S_2 \in \mathcal{B}(\mathcal{K})$ *such that* $A^*A = S_1 AA^*$ *and* $AA^* = S_2 A^*A$*.*

Proof. $(1 \Rightarrow 2)$: By Theorem 5.2.5, we have $A = U(B_1 \oplus 0)U^*$ with $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{L}, \mathcal{K})$ unitary and $B \in \mathcal{B}(\mathcal{H})$ an isomorphism. If we take $N_1 = U(B^*B(B^*)^{-1}B^{-1} \oplus I)U^* : \mathcal{K} \to \mathcal{K}$, then N_1 is an isomorphism with $A^*A = N_1 AA^*$. As $\mathcal{N}(AA^*) = \mathcal{N}(A^*)$ and $\mathcal{N}(A^*A) = \mathcal{N}(A)$. The rest follows from the proof of Theorem 5.3.3. □

CHAPTER 6

UNBOUNDED *EP* AND HYPO-*EP* OPERATORS ON HILBERT SPACES

6.1 INTRODUCTION

The theory of unbounded operators developed in the late 1920s and early 1930s as part of developing a rigorous mathematical framework for quantum mechanics. They are called unbounded observables in quantum mechanics. This type of operators arise in boundary value problems and they are not everywhere defined on Hilbert spaces. Moreover, they are not continuous on their domains of definition.

The basic difference between bounded and unbounded operators is the domain on which they are defined. Domains of unbounded operators on a Hilbert space $\mathcal{H}$ are always proper subspaces of $\mathcal{H}$. Because of this fact, many aspects of the theory of unbounded operators are somewhat counter-intuitive. For example, the algebraic rules for sums and products break down. Hence, one has to be careful while dealing with unbounded operators. Nevertheless the techniques of bounded operators may fail to hold in the case of unbounded operators; in some cases, they work for a certain class of unbounded operators.

Definition 6.1.1. *(Akhiezer and Glazman, 1993) Let A be a linear operator from a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(A)$ to a Hilbert space $\mathcal{K}$. If the graph of A defined by*

$$\mathcal{G}(A) = \{(x, Ax) : x \in \mathcal{D}(A)\}$$

*is closed in $\mathcal{H} \times \mathcal{K}$, then A is called a **closed operator**. Equivalently, A is a closed operator if $x_n \in \mathcal{D}(A)$ such that $x_n \to x$ and $Ax_n \to y$ for some $x \in \mathcal{H}, y \in \mathcal{H}$, then $x \in \mathcal{D}(A)$ and $Ax = y$.*

The set of all closed operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{C}(\mathcal{H}, \mathcal{K})$ and we write $\mathcal{C}(\mathcal{H}, \mathcal{H}) = \mathcal{C}(\mathcal{H})$.

Theorem 6.1.2. *(Riesz and Sz.-Nagy, 1955) Let A be a linear operator on $\mathcal{H}$ with domain $\mathcal{D}(A)$. Then the following are true.*

1. *If A is closed and everywhere defined, then A is bounded.*

2. *If A is bounded, then A is closed if and only if $\mathcal{D}(A)$ is a closed subspace of $\mathcal{H}$.*

Theorem 6.1.3. *(Goldberg, 1966) Let $A \in \mathcal{C}(\mathcal{H}, \mathcal{K})$. Then the following statements are true.*

1. *$\mathcal{N}(A)$ is a closed subspace of $\mathcal{H}$.*

2. *If A^{-1} exists, then A^{-1} is closed. In this case,*

$$\mathcal{G}(A^{-1}) = \{(Ax, x) : x \in \mathcal{D}(A)\}.$$

The denseness of domain is necessary and sufficient for existence of the adjoint. That is, A^* exists if and only if $\mathcal{D}(A)$ is dense in $\mathcal{H}$. Given any densely defined operator A (not necessarily closed), the adjoint of A is always closed. We call $\mathcal{D}(A) \cap \mathcal{N}(A)^{\perp}$, the **carrier** of A and it is denoted by $C(A)$.

Theorem 6.1.4. *(Ben-Israel and Greville, 2003) Let $A \in \mathcal{C}(\mathcal{H})$ be densely defined. Then the following are true.*

1. $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp, \quad \mathcal{N}(A^*A) = \mathcal{N}(A)$.
2. $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp, \quad \mathcal{N}(AA^*) = \mathcal{N}(A^*)$.
3. $\overline{\mathcal{R}(A)} = \mathcal{N}(A^*)^\perp, \quad \overline{\mathcal{R}(A)} = \overline{\mathcal{R}(AA^*)}$.
4. $\overline{\mathcal{R}(A^*)} = \mathcal{N}(A)^\perp, \quad \overline{\mathcal{R}(A^*)} = \overline{\mathcal{R}(A^*A)}$.

Definition 6.1.5. *(Rudin, 1991) Let A be a densely defined linear operator with domain $\mathcal{D}(A)$. The operator A is said to be*

1. ***normal*** *if $AA^* = A^*A$,*
2. ***symmetric*** *if $A \subset A^*$,*
3. ***self-adjoint*** *if $A = A^*$,*
4. ***positive*** *if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{D}(A)$.*

Definition 6.1.6. *(Nashed, 1976) Let $A \in \mathcal{C}(\mathcal{H})$ be densely defined. The **Moore-penrose inverse** of A is the linear operator $A^\dagger$ defined on the dense subspace $\mathcal{D}(A^\dagger) := \mathcal{R}(A) + \mathcal{R}(A)^\perp$ on $\mathcal{H}$ and taking values in $\mathcal{N}(A)^\perp \cap \mathcal{D}(A)$ with $\mathcal{N}(A^\dagger) = \mathcal{R}(A)^\perp$ and*

$$A^\dagger Ax = Px \text{ for } x \in \mathcal{D}(A),$$

where P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(A)^\perp$.

From the definition it follows that for $y \in \mathcal{D}(A^\dagger)$, $A^\dagger y$ is the unique element of $\mathcal{N}(A)^\perp \cap \mathcal{D}(A)$ satisfying

$$AA^\dagger y = Qy,$$

where Q is the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(A)}$.

The Moore-Penrose inverse $A^\dagger$ of A is closed and densely defined. Moreover, $A^\dagger$ is bounded if and only if $\mathcal{R}(A)$ is closed.

Theorem 6.1.7. *(Ben-Israel and Greville, 2003) Let $A \in \mathcal{C}(\mathcal{H})$ be densely defined. Then the following are true.*

1. $\mathcal{D}(A^\dagger) = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$.
2. $\mathcal{N}(A^\dagger) = \mathcal{R}(A)^\perp = \mathcal{N}(A^*)$.
3. $\mathcal{R}(A^\dagger) = C(A)$.

Theorem 6.1.8. *(Nashed, 1976) Let $A \in \mathcal{C}(\mathcal{H})$ be densely defined. Then each of the following set of conditions characterizes the Moore-Penrose inverse of A.*

1. (a) $A^\dagger A A^\dagger y = A^\dagger y$ *for all* $y \in \mathcal{D}(A^\dagger)$,

   (b) $A^\dagger A x = P_{\overline{\mathcal{R}(A^\dagger)}} x$ *for all* $x \in \mathcal{D}(A)$,

   (c) $AA^\dagger y = P_{\overline{\mathcal{R}(A)}} y$ *for all* $y \in \mathcal{D}(A^\dagger)$.

2. (a) $AA^\dagger A x = Ax$ *for all* $x \in \mathcal{D}(A)$,

   (b) $A^\dagger A A^\dagger y = A^\dagger y$ *for all* $y \in \mathcal{D}(A^\dagger)$,

   (c) $A^\dagger A$ *and* $AA^\dagger$ *are symmetric operators.*

Definition 6.1.9. *(Ben-Israel and Greville, 2003) Let $A \in \mathcal{C}(\mathcal{H})$ be densely defined. Then the number*

$$\gamma(A) = \inf\{\|Ax\| : x \in C(A), \|x\| = 1\}$$

is called the ***reduced minimum modulus*** *of A. Moreover, $\gamma(A) = \gamma(A^*)$.*

Theorem 6.1.10. *(Ben-Israel and Greville, 2003) Let A be a densely defined closed operator on $\mathcal{H}$. Then the following statements are equivalent:*

1. $\mathcal{R}(A)$ *is closed ;*
2. $\mathcal{R}(A^*)$ *is closed ;*
3. $\mathcal{R}(A^*A)$ *is closed ;*
4. $\mathcal{R}(AA^*)$ *is closed ;*

5. $A|_{C(A)}$ *has a bounded inverse ;*

6. $\gamma(A) > 0$ *;*

7. $A^\dagger$ *is bounded.*

Theorem 6.1.11. *(Ben-Israel and Greville, 2003) Let $A \in \mathcal{C}(\mathcal{H})$ be densely defined. Then the following are true.*

1. $A^{\dagger\dagger} = A$.

2. $A^{*\dagger} = A^{\dagger *}$.

3. $\mathcal{N}(A^{*\dagger}) = \mathcal{N}(A)$.

4. A^*A *and* $A^\dagger A^{*\dagger}$ *are non-negative and* $(A^*A)^\dagger = A^\dagger A^{*\dagger}$.

5. AA^* *and* $A^{*\dagger}A^\dagger$ *are non-negative and* $(AA^*)^\dagger = A^{*\dagger}A^\dagger$.

6. *A is bounded if and only if $\mathcal{R}(A^\dagger)$ is closed.*

Theorem 6.1.12. *(Douglas, 1966) Let A and B be densely defined operators in $\mathcal{C}(\mathcal{H})$. Then the following are true:*

1. *If $AA^* \leq BB^*$, there exists a contraction C so that $A \subseteq BC$.*

2. *If C is an operator so that $A \subseteq BC$, then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.*

3. *If $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, then there exists a densely defined operator C so that $A = BC$ and a number $k > 0$ so that $\|Cx\|^2 \leq k\left\{\|x\|^2 + \|A^*x\|^2\right\}$ for $x \in \mathcal{D}(C)$.*

6.2 UNBOUNDED EP OPERATORS ON HILBERT SPACES

Definition 6.2.1. *Let $A \in \mathcal{C}(\mathcal{H})$ be densely defined. The operator A is said to be an EP **operator** if $\mathcal{R}(A)$ is closed and $\mathcal{R}(A) = \mathcal{R}(A^*)$.*

Example 6.2.2. *(Huang et al., 2012) Let $t : [0,1] \to \mathbb{C}$ by*

$$t(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ \frac{1}{\sqrt{x}} & \text{if} \quad 0 < x \leq 1. \end{cases}$$

Define

$$Af = tf$$

for all f in the domain

$$\mathcal{D}(A) = \left\{ f \in L^2[0,1] : tf \in L^2[0,1] \right\}.$$

Then A is a densely defined closed operator. As $|t(x)| \geq 1$ for all $x \in [0,1]$, we have $\mathcal{R}(A) = L^2[0,1]$ and A has bounded inverse $A^{-1} : L^2[0,1] \to L^2[0,1]$ defined by $A^{-1}g = t_1 g$ for all $g \in L^2[0,1]$ where

$$t_1(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ \sqrt{x} & \text{if} \quad 0 < x \leq 1. \end{cases}$$

Hence A is a closed EP operator on $L^2[0,1]$.

Example 6.2.3. *Let $\mathcal{H} = L^2[0,1]$. Let*

$$\mathcal{AC}[0,1] = \Big\{ f \in \mathcal{H} : f : [0,1] \to \mathbb{C} \quad \text{is absolutely continuous and } f' \in \mathcal{H} \Big\}.$$

Let $\mathcal{D}(A) = \Big\{ f \in \mathcal{AC}[0,1] : f(0) = f(1) \Big\}$.
Define $A : \mathcal{D}(A) \to \mathcal{H}$ by

$$Af = if' \quad \text{for all } f \in \mathcal{D}(A).$$

We claim that A is self-adjoint. Let $f \in \mathcal{D}(A)$ and $g \in \mathcal{D}(A)$. Then

$$\begin{aligned} \langle Af, g \rangle = \int_0^1 if'(t)\overline{g(t)}dt &= i[(fg)(t)]_0^1 - i \int_0^1 g(t)\overline{f'(t)}dt \\ &= i \int_0^1 g(t)\overline{f'(t)}dt \quad \text{since } f(0)\bar{g}(0) = f(1)\bar{g}(1) \\ &= \langle g, Af \rangle. \end{aligned}$$

This shows that $A \subset A^*$. It only remains to prove $\mathcal{D}(A^*) \subseteq \mathcal{D}(A)$. Let $g \in \mathcal{D}(A^*)$. Put $\phi = A^*g$ and $\Phi(x) = \int\limits_0^x \phi$. Then, for any $f \in \mathcal{D}(A)$, we have

$$\int\limits_0^1 if\bar{g} = \langle Af, g\rangle = \langle f, \phi\rangle = f(1)\overline{\Phi(1)} - \int\limits_0^1 f'\Phi. \tag{6.2.1}$$

Since $\mathcal{D}(A)$ contains nonzero constants, substituting $f = c \neq 0$ in Equation 6.2.1, we end up with $\Phi(1) = 0$ and $ig - \Phi \in \mathcal{R}(A)^\perp$, where

$$\mathcal{R}(A) = \Big\{u \in \mathcal{H} : \int\limits_0^1 u(t)dt = 0\Big\} = span\{1\}^\perp.$$

Hence $ig - \Phi = \alpha$, for some constant $\alpha \neq 0$. As Φ is absolutely continuous, it follows that g is absolutely continuous. Using the fact that $\Phi(1) = 0 = \Phi(0)$ and $ig - \Phi = \alpha$, we can conclude that $g(0) = g(1)$. Hence $g \in \mathcal{D}(A)$. This proves that $A = A^*$. Hence A is a closed EP operator.

Theorem 6.2.4. *(Kulkarni and Ramesh, 2011) If A is a densely defined closed operator on $\mathcal{H}$, then $\overline{C(A)} = \mathcal{N}(A)^\perp$.*

Theorem 6.2.5. *Let $A \in \mathcal{C}(\mathcal{H})$ be a densely defined with closed range. Then the following are equivalent:*

1. *A is an EP operator ;*
2. *$AA^\dagger = A^\dagger A$ on $\mathcal{D}(A)$;*
3. *$\mathcal{N}(A) = \mathcal{N}(A^\dagger)$;*
4. *$\mathcal{N}(A) = \mathcal{N}(A^*)$;*
5. *$\overline{C(A)} = \mathcal{R}(A)$;*
6. *$\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{R}(A)$;*
7. *If $\mathcal{D}(A) = \mathcal{D}(A^*)$, then $A^* = PA$, where P is a bijective linear operator on $\mathcal{H}$.*

Proof. Assume that A is an EP operator. Let $x \in \mathcal{D}(A) = \mathcal{N}(A) \oplus C(A)$. Then $x = x_1 + x_2, x_1 \in \mathcal{N}(A), x_2 \in C(A) = \mathcal{R}(A^\dagger)$. Hence $A^\dagger Ax = A^\dagger A(x_1 + x_2) = A^\dagger Ax_2 = x_2$. As $C(A) = \mathcal{R}(A^\dagger) \subseteq \mathcal{R}(A^*) = \mathcal{R}(A)$ and $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp = \mathcal{R}(A)^\perp = \mathcal{N}(A^\dagger)$, $AA^\dagger x = AA^\dagger(x_1 + x_2) = AA^\dagger x_1 + AA^\dagger x_2 = AA^\dagger x_2 = x_2$. Therefore $AA^\dagger = A^\dagger A$ on $\mathcal{D}(A)$.

Now assume $A^\dagger A = AA^\dagger$ on $\mathcal{D}(A)$. Then $\overline{\mathcal{R}(A^\dagger)} = \overline{\mathcal{R}(A)}$ and hence $\mathcal{R}(A^*) = \mathcal{R}(A)$. Therefore A is EP.

The following set of equations will prove the implications ($1 \Leftrightarrow \ 3 \Leftrightarrow 4$).

$$\begin{aligned} &\mathcal{R}(A) = \mathcal{R}(A^*) \\ \Leftrightarrow\ &\mathcal{R}(A)^\perp = \mathcal{R}(A^*)^\perp \\ \Leftrightarrow\ &\mathcal{N}(A^\dagger) = \mathcal{N}(A) \\ \Leftrightarrow\ &\mathcal{N}(A^*) = \mathcal{N}(A). \end{aligned}$$

Assume $\mathcal{R}(A) = \mathcal{R}(A^*)$. Then

$$\begin{aligned} \mathcal{H} &= \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \\ &= \mathcal{R}(A) \oplus \mathcal{R}(A^*)^\perp \\ &= \mathcal{R}(A) \oplus \mathcal{N}(A). \end{aligned}$$

Assume $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{R}(A)$. But we have $\mathcal{H} = \mathcal{R}(A)^\perp \oplus \mathcal{R}(A)$. Hence we get $\mathcal{N}(A) = \mathcal{R}(A)^\perp = \mathcal{N}(A^*)$. Therefore by (4), $\mathcal{R}(A) = \mathcal{R}(A^*)$. Hence A is EP. (1 $\Leftrightarrow$ 6)is trivial from the fact that $\mathcal{R}(A) = \mathcal{R}(A^*) = \mathcal{N}(A)^\perp = \overline{C(A)}$.

Assume $\mathcal{N}(A) = \mathcal{N}(A^*)$. Let $x \in \mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Then $x = x_1 + x_2, x_1 \in \mathcal{R}(A), x_2 \in \mathcal{N}(A)$. Since $x_1 \in \mathcal{R}(A)$ and A is bijective from $C(A)$ to $\mathcal{R}(A)$, there exists $u \in C(A)$ such that $Au = x_1$. Define $Px = A^*u + x_2$. If $x \in \mathcal{R}(A)$, then $Px = A^*u$ where $u \in C(A)$ and $Au = x$.

First we prove that P is linear. Let $x, y \in \mathcal{H}$. Then $x = x_1 + x_2, y = y_1 + y_2$ with $x_1, y_1 \in \mathcal{R}(A), x_2, y_2 \in \mathcal{N}(A)$. Let $u, v \in C(A)$ such that $Au = x_1, Av = x_2$. Then $Px = A^*u + x_2, Py = A^*v + y_2$. For $\alpha, \beta \in \mathbb{C}$, $\alpha x + \beta y = (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2)$

and $A(\alpha u + \beta v) = \alpha x_1 + \beta y_1$. Therefore

$$\begin{aligned} P(\alpha x + \beta y) &= A^*(\alpha u + \beta v) + (\alpha x_2 + \beta y_2) \\ &= \alpha(A^* u + x_2) + \beta(A^* v + y_2) \\ &= \alpha P x + \beta P y. \end{aligned}$$

Hence P is linear.

Now we prove that $PA = A^*$. Let $x \in \mathcal{D}(A) = C(A) \oplus \mathcal{N}(A)$ and $x = x_1 + x_2$ where $x_1 \in C(A), x_2 \in \mathcal{N}(A)$. Then

$$\begin{aligned} PAx &= PA(x_1 + x_2) \\ &= PAx_1. \end{aligned}$$

As $Ax_1 \in \mathcal{R}(A)$, $PAx_1 = A^* u$, where $u \in C(A)$ and $Au = Ax_1$. Since A is bijective from $C(A)$ to $\mathcal{R}(A)$ and $u, x_1 \in C(A)$, $Au = Ax_1$ implies $u = x_1$. Hence $PAx = PAx_1 = A^* u = A^* x_1$. But $A^* x = A^* x_1 + A^* x_2 = A^* x_1$. Hence $PAx = A^* x$.

We now claim that P is injective. Take $x \in \mathcal{H}$ such that $Px = 0$, where $x = x_1 + x_2, x_1 \in \mathcal{R}(A), x_2 \in \mathcal{N}(A)$. Then $A^* u + x_2 = 0$ where $Au = x_1$ and $u \in C(A)$. Hence $A^* u = -x_2$. But $A^* u \in \mathcal{R}(A^*) = \mathcal{N}(A)^\perp$ and $x_2 \in \mathcal{N}(A)$. Therefore $A^* u = x_2 = 0$. As $u \in C(A) = C(A^*)$ and A^* is a bijective map from $C(A^*)$ to $\mathcal{R}(A^*)$, $u = 0$. Hence $x_1 = Au = 0$ and $x = x_1 + x_2 = 0$. Therefore P is injective.

Finally we prove that P is surjective. Let $y = y_1 + y_2 \in \mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, where $y_1 \in \mathcal{R}(A) = \mathcal{R}(A^*), y_2 \in \mathcal{N}(A)$. As $y_1 \in \mathcal{R}(A^*)$, there exists $u \in \mathcal{D}(A^*)$ such that $A^* u = y_1$. Let $u = u_1 + u_2$ with $u_1 \in C(A)$ and $u_2 \in \mathcal{N}(A) = \mathcal{N}(A^*)$. Therefore $y_1 = A^* u = A^* u_1 + A^* u_2 = A^* u_1$. Take $x = Au_1 + y_2 \in \mathcal{R}(A) \oplus \mathcal{N}(A)$. Then

$$\begin{aligned} Px &= A^* u_1 + y_2 \\ &= y_1 + y_2 \\ &= y. \end{aligned}$$

Therefore P is surjective. Hence $A^* = PA$.

Assume $A^* = PA$ for some bijective operator $P \in \mathcal{L}(\mathcal{H})$. Then $\mathcal{N}(A) \subseteq \mathcal{N}(PA) = \mathcal{N}(A^*)$. Also $A = P^{-1}A^*$. Then we have $\mathcal{N}(A^*) \subseteq \mathcal{N}(P^{-1}A^*) = \mathcal{N}(A)$. Therefore $\mathcal{N}(A) = \mathcal{N}(A^*)$. □

If we drop the assumption that $\mathcal{R}(A)$ is closed, then the part (4) of Theorem 6.2.5 can be restated as "$AA^\dagger \subseteq A^\dagger A$ if and only if $\mathcal{N}(A) = \mathcal{N}(A^*)$ and $\mathcal{D}(A^\dagger) \subseteq \mathcal{D}(A)$." Indeed, if $AA^\dagger \subseteq A^\dagger A$, then $\mathcal{D}(A^\dagger) \subseteq \mathcal{D}(A)$ and

$$\begin{aligned}
AA^\dagger &= A^\dagger A \quad \text{on } \mathcal{D}(A^\dagger) \\
P_{\overline{\mathcal{R}(A)}} &= P_{\overline{\mathcal{R}(A^\dagger)}} \\
\overline{\mathcal{R}(A)} &= \overline{\mathcal{R}(A^\dagger)} \\
\overline{\mathcal{R}(A)}^\perp &= \overline{\mathcal{R}(A^\dagger)}^\perp \\
\mathcal{N}(A^*) &= \mathcal{N}(A).
\end{aligned}$$

Conversely, if $\mathcal{N}(A) = \mathcal{N}(A^*)$ and $\mathcal{D}(A^\dagger) \subseteq \mathcal{D}(A)$, then

$$\begin{aligned}
\mathcal{N}(A) &= \mathcal{N}(A^\dagger) \\
\mathcal{N}(A)^\perp &= \mathcal{N}(A^\dagger)^\perp \\
\overline{C(A)} &= \overline{\mathcal{R}(A)} \\
\overline{\mathcal{R}(A^\dagger)} &= \overline{\mathcal{R}(A)}.
\end{aligned}$$

Then by Theorem 6.1.8, we have $AA^\dagger x = A^\dagger Ax$ for all $x \in \mathcal{D}(A^\dagger)$. Hence $AA^\dagger \subseteq A^\dagger A$. Similarly, we can prove that $A^\dagger A \subseteq AA^\dagger$ if and only if $\mathcal{N}(A) = \mathcal{N}(A^*)$ and $\mathcal{D}(A) \subseteq \mathcal{D}(A^\dagger)$.

Example 6.2.6. *(Kulkarni and Ramesh, 2010) Let $\mathcal{H}$ be the real space $L^2[0, \pi]$ of real valued functions and $\mathcal{H}' = \Big\{\phi \in \mathcal{AC}[0, \pi] : \phi' \in \mathcal{H}\Big\}$. Let A be the operator $\dfrac{d}{dt}$ with*

$$\mathcal{D}(A) = \{x \in \mathcal{H}' : x(0) = x(\pi) = 0\}.$$

It can be shown using the fundamental theorem of integral calculus that $A \in \mathcal{C}(\mathcal{H})$. Since $\{\sin nt : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$ and is contained in $\mathcal{D}(A)$, A is densely defined. Also $C(A) = \mathcal{D}(A)$. i.e., A is one-to-one. It can be shown that $\mathcal{R}(A) = \{y \in \mathcal{H} : \int_0^\pi y(t)\, dt = 0\} = span\{1\}^\perp$. Hence in this case

$\mathcal{D}(A^\dagger) = \mathcal{H}$. *Let* $z \in \mathcal{H}$. *Then* $z = y + c$, *where* $y \in \mathcal{R}(A)$ *and* $0 \neq c \in \mathcal{R}(A)^\perp$. *Hence*

$$z = y + c \Rightarrow y = z - c \Rightarrow 0 = \int_0^\pi y(t)dt = \int_0^\pi (z(t) - c)dt$$
$$\Rightarrow c = \frac{1}{\pi}\int_0^\pi z(t)dt.$$

Hence $y(t) = z(t) - \frac{1}{\pi}\int_0^\pi z(t)dt$.

Since $y \in \mathcal{R}(A)$, *we have* $A^\dagger y = A^\dagger z$. *Thus*

$$A^\dagger z = A^\dagger y = A^{-1}y = \int_0^s y(t)dt = \int_0^s z(t)dt - \frac{1}{\pi}\int_0^s\int_0^\pi z(u)dudt.$$

Hence $A^\dagger z = \int_0^s z(t)dt - \frac{s}{\pi}\int_0^\pi z(u)du, \quad 0 \leq s \leq \pi$. *Also* $AA^\dagger = A^\dagger A$. *Hence* A *is a closed* EP *operator.*

6.3 UNBOUNDED HYPO-EP OPERATORS ON HILBERT SPACES

Definition 6.3.1. *Let* $A \in \mathcal{C}(\mathcal{H})$ *be densely defined. The operator* A *is said to be a* ***hypo-EP operator*** *if* $\mathcal{R}(A)$ *is closed and* $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$.

Example 6.3.2. *Let* $\mathcal{H} = \ell_2$ *and*

$$\mathcal{D}(A) = \{(x_1, x_2, \ldots) \in \mathcal{H} : (x_1, 2x_2, 3x_3, \ldots) \in \mathcal{H}\}.$$

Define $A(x_1, x_2, x_3, \ldots) = (0, x_1, 2x_2, 3x_3, \ldots)$ *for all* $(x_1, x_2, x_3, \ldots) \in \mathcal{D}(A)$. *The operator* A *is a hypo-EP operator, but not EP.*

Theorem 6.3.3. *Let* $A \in \mathcal{C}(\mathcal{H})$ *be closed range. Then* A *is hypo-EP if and only if* $A^\dagger A^2 A^\dagger = AA^\dagger$.

Proof. Suppose $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$ and $\mathcal{R}(A)$ is closed. Then $AA^\dagger x \in \mathcal{R}(A)$ for each $x \in \mathcal{H}$ and hence $AA^\dagger x \in \overline{\mathcal{R}(A^\dagger)} = \mathcal{R}(A^*)$. As $A^\dagger A$ is a projection onto $\overline{\mathcal{R}(A^\dagger)}$, we have $A^\dagger A(AA^\dagger x) = AA^\dagger x$. Hence $A^\dagger A^2 A^\dagger = AA^\dagger$. □

Theorem 6.3.4. *Let $A \in \mathcal{C}(\mathcal{H})$ be densely defined. Then each of the following statements implies the next statement:*

1. *A is hypo-EP ;*
2. *$A(A^\dagger)^2 A = AA^\dagger$ on $\mathcal{D}(A)$;*
3. *$AA^\dagger \leq A^\dagger A$ on $\mathcal{D}(A)$;*
4. *$\|AA^\dagger x \leq \|A^\dagger Ax\|$ for all $x \in \mathcal{D}(A)$.*

Proof. Assume that A is hypo-EP and $x \in \mathcal{D}(A)$. Then

$$\begin{aligned}
\left\langle AA^\dagger A^\dagger Ax, x\right\rangle &= \left\langle (AA^\dagger)^* A^\dagger Ax, x\right\rangle \\
&= \left\langle A^\dagger Ax, AA^\dagger x\right\rangle \\
&= \left\langle (A^\dagger A)^* x, AA^\dagger x\right\rangle \\
&= \left\langle x, A^\dagger A^2 A^\dagger x\right\rangle \\
&= \left\langle x, AA^\dagger x\right\rangle \\
&= \left\langle AA^\dagger x, x\right\rangle .
\end{aligned}$$

Hence $A(A^\dagger)^2 A = AA^\dagger$ on $\mathcal{D}(A)$.

Assume that $A(A^\dagger)^2 A = AA^\dagger$ on $\mathcal{D}(A)$. Let $x \in \mathcal{D}(A)$. Then

$$\begin{aligned}
\left\langle AA^\dagger x, x\right\rangle &= \left\langle AA^\dagger AA^\dagger x, x\right\rangle \\
&= \left\langle (AA^\dagger)^* AA^\dagger x, x\right\rangle \\
&= \|AA^\dagger x\|^2 \\
&= \|A(A^\dagger)^2 Ax\|^2 \\
&\leq \|AA^\dagger\|^2 \|A^\dagger Ax\|^2 \\
&= \|A^\dagger Ax\|^2 \\
&= \left\langle A^\dagger Ax, A^\dagger Ax\right\rangle \\
&= \left\langle A^\dagger Ax, x\right\rangle .
\end{aligned}$$

Hence $AA^\dagger \leq A^\dagger A$ on $\mathcal{D}(A)$.

Assume that $AA^\dagger \leq A^\dagger A$ on $\mathcal{D}(A)$. Let $x \in \mathcal{D}(A)$. Then

$$\begin{aligned}
&\left\langle AA^\dagger x, x\right\rangle \leq \left\langle A^\dagger Ax, x\right\rangle \\
\Rightarrow &\left\langle AA^\dagger AA^\dagger x, x\right\rangle \leq \left\langle A^\dagger AA^\dagger Ax, x\right\rangle \\
\Rightarrow &\left\langle AA^\dagger x, AA^\dagger x\right\rangle \leq \left\langle A^\dagger Ax, A^\dagger Ax\right\rangle \\
\Rightarrow &\|AA^\dagger x\|^2 \leq \|A^\dagger Ax\|^2.
\end{aligned}$$

Thus $\|AA^\dagger x \leq \|A^\dagger Ax\|$ for all $x \in \mathcal{D}(A)$. □

Remark 6.3.5. *If $\mathcal{R}(A) \subseteq \mathcal{D}(A)$, all the necessary conditions for hypo-EP in Theorem 6.3.4 become sufficient conditions for hypo-EP operators.*

Theorem 6.3.6. *Let $A \in \mathcal{C}(\mathcal{H})$ be densely defined. If A is hypo-EP, then there exists $k > 0$ such that $|\langle Ax, y\rangle| \leq k\|Ay\|$, for all $y \in \mathcal{D}(A)$.*

Proof. Suppose A is hypo-EP. If $x \in \mathcal{N}(A)$, then the result is trivial. Let $x \in \mathcal{D}(A)$ such that $Ax \neq 0$. Then $Ax \in \mathcal{R}(A) \subseteq \mathcal{R}(A^*)$. Therefore there exists a non-zero $z \in \mathcal{D}(A)$ such that $A^*z = Ax$. Then for all $y \in \mathcal{D}(A)$,

$$|\langle Ax, y\rangle| = |\langle A^*z, y\rangle| = |\langle z, Ay\rangle| \leq \|z\|\|Ay\|.$$

Taking $k = \|z\|$, we get

$$|\langle Ax, y\rangle| \leq k\|Ay\|,$$

for all $y \in \mathcal{D}(A)$. □

The converse of Theorem 6.3.6 has been proved for bounded hypo-EP operators on Hilbert spaces in Chapter 2 in which Douglas' theorem for bounded operators was used. Unlike the bounded operators, Douglas' theorem for densely defined closed operators does not guarantee the equivalance of the notions of majorization, range inclusion and factorization.

CHAPTER 7

CONCLUSION AND FUTURE WORK

7.1 CONCLUSION

Bounded EP operators behave better than closed range operators on Hilbert spaces because of the additional "range-Hermitian" condition. For instance, there are operators A and B on a Hilbert space $\mathcal{H}$ such that A, B and AB have closed ranges but BA does not have closed range. However when A and B are EP operators, the closed rangeness of AB implies the closed rangeness of BA and vice-versa. Also, there are operators A on $\mathcal{H}$ such that A has closed range but A^2 does not have closed range. However it has been observed that if A is EP, then A^2 has closed range always. Moreover, any natural power of A has closed range.

Because of the Pearl's characterization, several characterizations came out in terms of Moore-Penrose inverses. There are at least 60 characterizations for EP matrices available in literature and most of them are extended to infinite-dimensional settings. However, much attention has not been paid to study unbounded EP operators which would be quite useful to know the properties which resemble those of normal/hyponormal operators. In the case of finite dimensional settings, EP and hypo-EP are the same. There are few more types of EP matrices (k-EP, Cen-EP, Con-S-K-EP, Co-EP, Core-EP) being studied by mathematicians.

In this thesis we have discussed the following for EP and hypo-EP operators :

1. Algebraic and analytic characterizations for bounded and unbounded operators on Hilbert spaces ;
2. Algebraic sum, product and restriction for bounded operators on Hilbert spaces ;
3. Factorization of operators on Krein spaces ;
4. As an application, Fuglede-Putnam type theorems for bounded operators on Hilbert spaces.

7.2 FUTURE WORK

Moore-Penrose inverses of matrices have important roles in theoretical and numerical methods of linear algebra. The most significant fact is that we can use Moore-Penrose inverse of matrices, in the case when ordinary inverses do not exist, in order to solve some matrix equations. Similar reasoning can be applied to linear (bounded or unbounded) operators on Hilbert spaces. Then, it is interesting to consider Moore-Penrose inverses of elements in Banach and C^*-algebras, more generally, in rings with or without involution.

Rakocevic (Rakočević, 1988) introduced the notion of Moore-Penrose inverse to elements of a Banach algebra, which led to study EP Banach space operators and EP Banach algebra elements by Boasso (Boasso, 2008). Well-known results obtained in the frame of Hilbert space operators and C^*-algebra elements are extended for EP Banach space operators and Banach algebra elements.

In rings with involution, EP elements are elements for which the Drazin and the Moore-Penrose inverse exist and coincide. Dijana et al. (Mosić and Djordjević, 2012) introduced and investigated generalized normal and generalized Hermitian elements in rings. As a consequence, several new characterizations for elements in rings with involution to be normal and Hermitian elements are presented. Moreover, Dijana et al. have investigated EP elements in Banach algebras (Mosić and

Djordjević, 2011a) and weighted-EP elements in C^*-algebras (Mosić and Djordjević, 2011b).

EP modular operators on Hilbert C^*-modules have been first studied by Kamran Sharifi (Sharifi, 2014) and necessary and sufficient conditions are provided for the product of two EP modular operators to be EP. These results are extension of results by Koliha (Koliha, 2000) for an arbitrary C^*-algebra and the C^*-algebras of compact operators. We have seen theoretical developments of EP operators from finite dimensional spaces to Hilbert C^*-modules.

Our future plan is to analyze algebraic and topological structures of those collection of operators in a much more general settings, such as C^*-algebras and Hilbert C^*-modules.

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