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A study on the local convergence of a Steffensen-King-type iterative method

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Abstract. We study the local convergence of a Steffensen-King-type method to approximate a locally unique solution of a nonlinear equation. Earlier studies such as [14, 15, 17] show convergence under hypotheses on the third derivative or even higher. The convergence in this study is shown under hypotheses on the first derivative. Hence, the applicability of the method is expanded. Finally, numerical examples are also provided in this study.

1 Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq S \to S$ is a nonlinear function, D is a convex subset of S and S is $\mathbb{R}$ or $\mathbb{C}$. Newton-like methods are used for finding solutions of (1.1). These methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [1]–[19].

We present the local convergence analysis of the Steffensen-King-type method defined [14] for each $n = 0, 1, 2, \cdots$ by

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$$
\begin{aligned}
w_n &= x_n + \gamma F(x_n), \\
y_n &= x_n - \frac{F(x_n)}{F[x_n, w_n]} \\
x_{n+1} &= y_n - \varphi(\frac{F(y_n)}{F(x_n)}) \frac{F(x_n) - \beta F(y_n)}{F(x_n) + (\beta - 2)F(y_n)} \frac{F(y_n)}{F[w_n, y_n]},
\end{aligned} \tag{1.2}
$$

where x_0 is an initial point, $\beta, \gamma \in S$, $F[x,y]$ is a divided difference of order one for function F at the points x, y satisfying

$$
\begin{aligned}
F[x,y] &= \frac{F(x) - F(y)}{x - y} \ \text{ if } x \neq y \\
F[x,x] &= F'(x) \ \text{ for each } \ x, y \in D,
\end{aligned}
$$

if F is differentiable function and $\varphi : S \to S$ is a weight function. Method (1.2) is a useful alternative to the method [17] defined for each $n = 0, 1, 2, \dots$ by

$$
\begin{aligned}
w_n &= x_n + \gamma_n F(x_n), \\
y_n &= x_n - \frac{F(x_n)}{F[x_n, w_n]} \\
x_{n+1} &= y_n - \varphi(\frac{F(y_n)}{F(x_n)}) \frac{F(x_n) - \beta F(y_n)}{F(x_n) + (\beta - 2)F(y_n)} \frac{F(y_n)}{F[w_n, y_n]},
\end{aligned} \tag{1.3}
$$

where $\gamma_0 \in S$ and for each $n = 1, 2, \cdots$

$$N_3'(x_n) = F[x_n, y_{n-1}] + F[x_n, y_{n-1}, w_{n-1}](x_n - y_{n-1}) + F[x_n, y_{n-1}, w_{n-1}, x_{n-1}](x_n - y_{n-1})(x_n - w_{n-1}),$$

$F[x,t,z], F[x,y,z,w]$ are divided differences of order two and three, respectively and $\gamma_n = -\frac{1}{N_3'(x_n)}$. The convergence of the preceding methods has been shown under hypotheses on higher order derivatives which limits the applicability of these methods. As a motivational example, let us define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$
f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, \ x \neq 0 \\ 0, \ x = 0 \end{cases}
$$

Choose $x^* = 1$. We have that

$$
\begin{aligned}
f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \ f'(1) = 3, \\
f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\
f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22.
\end{aligned}
$$

Then, obviously, function f''' is unbounded on D. In the present paper we only use hypotheses on the first Fréchet derivative. This way we expand the applicability of method (1.2) and method (1.3).

The rest of the paper is organized as follows. In Section 2 the local convergence analysis of method (1.2) and method (1.3) is given. The numerical examples are presented in the concluding Section 3.

2 Local convergence analysis

This Section contains the local convergence analysis of method (1.2) and method (1.3). Let $L_0 > 0, L > 0, L_1 > 0, L_2 > 0, M_0 > 0, M \geq 1, \beta, \gamma \in S$ be parameters and $\varphi : S \to S$ be a given function. The study of the local convergence analysis of methods (1.2) and (1.3) requires the introduction of some parameters and functions. Define parameters

$$L_3 = L_1 + L_2(1 + |\gamma| M_0),$$

$$L_4 = (L_0 + L_1 + L_2(1 + |\gamma| M))M,$$

$$r_1 = \frac{2}{2L_0 + L}$$

and

$$N = \min\{\frac{1}{L_0}, \frac{1}{L_3}\}.$$

Define functions on the interval $[0, N)$ by

$$\begin{aligned}
g_1(t) &= \frac{Lt}{2(1 - L_0 t)}, \\
g_2(t) &= g_1(t) + \frac{L_4 t}{2(1 - L_0 t)(1 - L_3 t)}, \\
&= \frac{1}{2(1 - L_0 t)}[L + \frac{L_4}{1 - L_3 t}]t, \\
g(t) &= (L_1(1 + |\gamma| M) + L_2 g_2(t))t, \\
g_0(t) &= \frac{L_0 t}{2} + |\beta - 2| M g_2(t), \\
h_1(t) &= g_1(t) - 1, \\
h_2(t) &= g_2(t) - 1, \\
h(t) &= g(t) - 1
\end{aligned}$$

and

$$h_0(t) = g_0(t) - 1.$$

Notice that r_1 is the zero of function h_1 on the interval $(0, \frac{1}{L_0})$. We also have $h_2(0) = h(0) = h_0(0) = -1 < 0$ and $h_2(t) \to +\infty, h(t) \to \infty, h_0(t) \to \infty$ as $t \to N^-$. It follows from the Intermediate Value Theorem that functions h_2, h and h_0 have zeros in the interval $(0, N)$. Denote by r_2, r and r_0 the smallest such zeros, respectively. Set

$$\lambda = \min\{r_0, r_1, r_2, r\}.$$

Define function on the interval $[0, \lambda)$ by

$$g_3(t) = [1 + |\varphi\left(\frac{M g_2(t)}{(1 - \frac{L_0}{2} t)}\right)| \frac{M^2(1 + |\beta| g_2(t))}{(1 - g_0(t))(1 - g(t))}] g_2(t)$$

and

$$h_3(t) = g_3(t) - 1.$$

Then, we have that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to \lambda^-$. Hence, function h_3 has a smallest zero denoted by $r*$. It follows that for each $t \in [0, r^*)$

$$0 \le g(t) < 1, \tag{2.1}$$

$$0 \le g_0(t) < 1, \tag{2.2}$$

$$0 \le g_1(t) < 1, \tag{2.3}$$

$$0 \le g_2(t) < 1. \tag{2.4}$$

and

$$0 \le g_3(t) < 1. \tag{2.5}$$

We denote by $U(v,\rho), \bar{U}(v,\rho)$ stand for the open and closed balls in S, respectively, with center $v \in S$ and of radius $\rho > 0$. Using the preceding notation we can show the following local convergence result for method (1.2).

THEOREM 2.1. Let $F : D \subseteq S \to S$ be a differentiable function. Suppose there exist a divided difference of order one $F[.,.] : D \times D \to S$, a continuous function $\varphi : S \to S$, a point $x^* \in D$, parameters $L_0 > 0, L > 0, L_1 > 0, L_2 > 0, M_0 > 0, M \ge 1, \beta, \gamma \in S$ such that for each $x, y \in D$

$$F(x^*) = 0,\ F'(x^*) \neq 0, \tag{2.6}$$

$$|F'(x^*)^{-1}(F'(x) - F'(x^*))| \le L_0|x - x^*|, \tag{2.7}$$

$$|F'(x^*)^{-1}(F'(x) - F'(y))| \le L|x - y|, \tag{2.8}$$

$$|F'(x^*)^{-1}(F[x,y] - F'(x^*))| \le L_1|x - x^*| + L_2|y - x^*|, \tag{2.9}$$

$$|F'(x)| \le M_0, \tag{2.10}$$

$$|F'(x^*)^{-1}F'(x)| \le M, \tag{2.11}$$

$$|\varphi(t)| \le |\varphi(|t|)| \le |\varphi(u)| \text{ for each } t \in D, u \in [0,+\infty) \text{ such that } |t| \le u \tag{2.12}$$

and

$$\bar{U}(x^*, (1 + |\gamma|M_0)r^*) \subseteq D, \tag{2.13}$$

where the radius r^* is defined above Theorem 2.1. Then, sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r^*) - \{x^*\}$ by method (1.2) is well defined, remains in $U(x^*, r^*)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold

$$|w_n - x^*| \le (1 + |\gamma|M_0)|x_n - x^*|, \tag{2.14}$$

$$|y_n - x^*| \le g_2(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r^*, \tag{2.15}$$

and

$$|x_{n+1} - x^*| \le g_3(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|, \tag{2.16}$$

where the "g" functions are defined above Theorem 2.1. Furthermore, for $R \in [r^*, \frac{2}{L_0})$ the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, R) \cap D$.

Proof. We use mathematical induction to show estimates (2.14)–(2.16). First, to show (2.14) notice that by hypothesis $x_0 \in U(x^*, r^*) - \{x^*\}$ and we can write

$$\begin{aligned} w_0 - x^* &= x_0 - x^* + \gamma(F(x_0) - F(x^*)) \\ &= x_0 - x^* + \gamma \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*) d\theta \end{aligned} \tag{2.17}$$

and $|x^* + \theta(x_0 - x^*) - x^*| = \theta|x_0 - x^*| < r^*$ for each $\theta \in [0,1]$. Then, using (2.10), (2.13) and (2.17), we get that

$$\begin{aligned} |w_0 - x^*| &\le |x_0 - x^* + \gamma \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*) d\theta| \\ &\le |x_0 - x^*| + |\gamma| |\int_0^1 F'(x^* + \theta(x_0 - x^*)) d\theta| |x_0 - x^*| \\ &= (1 + |\gamma| M_0)|x_0 - x^*|, \end{aligned}$$

which shows (2.14) and $w_0 \in \bar{U}(x^*, (1 + |\gamma| M_0) r) \subseteq D$. Next, we shall show that $F'(x_0)$ and $F[x_0, w_0]$ are invertible. Using (2.7), we get that

$$|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \le L_0|x_0 - x^*| < L_0 r^* < 1. \tag{2.18}$$

It follows from (2.18) and the Banach Lemma on invertible functions [5, 7, 16, 19] that $F'(x_0)$ is invertible and and

$$|F'(x_0)^{-1}F'(x^*)| \le \frac{1}{1 - L_0|x_0 - x^*|}. \tag{2.19}$$

By (2.9) and (2.14) (for $n = 0$) we obtain that

$$\begin{aligned} &|F'(x^*)^{-1}(F[x_0, w_0] - F'(x^*))| \\ &\le L_1|x_0 - x^*| + L_2|w_0 - x^*| \\ &\le L_1|x_0 - x^*| + L_2(1 + |\gamma| M_0)|x_0 - x^*| \\ &\le L_3|x_0 - x^*| < L_3 r^* < 1. \end{aligned} \tag{2.20}$$

So $F[x_0, w_0]$ is invertible and

$$|F[x_0, w_0]^{-1}F'(x^*)| \le \frac{1}{1 - L_3|x_0 - x^*|}. \tag{2.21}$$

Hence, y_0 is well defined by the second sub-step of method (1.2) for $n = 0$. We can write

$$\begin{aligned} y_0 - x^* &= [x_0 - x^* - \frac{F(x_0)}{F'(x_0)}] + [\frac{F(x_0)}{F'(x_0)} - \frac{F(x_0)}{F[x_0, w_0]} \\ &= F'(x_0)^{-1}F'(x^*) \int_0^1 F'(x^*)^{-1} \\ &\quad \times [F(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*) d\theta] \\ &\quad + \frac{F'(x^*)^{-1}((F[x_0, w_0] - F'(x^*)) + (F'(x^*) - F'(x_0)))F'(x^*)^{-1}F(x_0)}{F'(x^*)^{-1}F'(x_0)F'(x^*)^{-1}F[x_0, w_0]}. \end{aligned} \tag{2.22}$$

Then, using (2.4), (2.8), (2.9), (2.11), (2.19), (2.21), (2.22) and the triangle inequality, we get in turn that

$$\begin{aligned}
|y_0-x^*| \leq & \frac{L|x_0-x^*|^2}{2(1-L_0|x_0-x^*|)} \\
& +|F'(x_0)^{-1}F(x^*)||F[x_0,w_0]^{-1}F(x^*)| \\
& \times(|F'(x^*)^{-1}(F[x_0,w_0]-F'(x^*))|+|F'(x^*)^{-1}(F'(x_0)-F'(x^*))|) \\
& \leq g_1(|x_0-x^*|)|x_0-x^*|+\frac{(L_1|x_0-x^*|+L_2|w_0-x^*|+L_0|x_0-x^*|)M|x_0-x^*|}{(1-L_0|x_0-x^*|)(1-L_3|x_0-x^*|)} \\
& \leq g_1(|x_0-x^*|)|x_0-x^*|+\frac{(L_1+L_2(1+|\gamma|M_0)+L_0)M|x_0-x^*|^2}{(1-L_0|x_0-x^*|)(1-L_3|x_0-x^*|)} \\
& = g_1(|x_0-x^*|)|x_0-x^*|+\frac{L_4|x_0-x^*|^2}{(1-L_0|x_0-x^*|)(1-L_3|x_0-x^*|)} \\
& = g_2(|x_0-x^*|)|x_0-x^*| < |x_0-x^*| < r^*,
\end{aligned}$$

which shows (2.15) for $n=0$ and $y_0 \in U(x^*,r^*)$. We shall show that $F[w_0,y_0]$ and $F(x_0)+(\beta-2)F(y_0)$ are invertible. Using (2.9), (2.14) (for $n=0$) and (2.1) we get that

$$\begin{aligned}
& |F'(x^*)^{-1}((F[w_0,y_0]-F'(x^*))| \\
& \leq L_1|w_0-x^*|+L_2|y_0-x^*| \\
& \leq L_1(1+|\gamma|M_0)|x_0-x^*|+L_2g_2(|x_0-x^*|)|x_0-x^*| \\
& = g(|x_0-x^*|) < g(r^*) < 1.
\end{aligned} \tag{2.23}$$

It follows from (2.23) that $F[w_0,y_0]$ is invertible and

$$|F[w_0,y_0]^{-1}F'(x^*)| \leq \frac{1}{1-g(|x_0-x^*|)}. \tag{2.24}$$

Using (2.2), (2.6), (2.7), (2.15)(for $n=0$), (2.11), $x_0 \neq x^*$ we get in turn that

$$\begin{aligned}
& |(F'(x^*)(x_0-x^*))^{-1}[F(x_0)-F(x^*)+(\beta-2)F(y_0)-F'(x^*)(x_0-x^*)]| \\
\leq & |x_0-x^*|^{-1}[|\int_0^1 F'(x^*)^{-1}(F'(x^*+\theta(x_0-x^*))-F'(x^*))d\theta||x_0-x^*| \\
& +|\beta-2||\int_0^1 F'(x^*)^{-1}F'(x^*+\theta(y_0-x^*)d\theta||y_0-x^*| \\
\leq & |x_0-x^*|^{-1}[\frac{L_0|x_0-x^*|^2}{2}+|\beta-2|Mg_2(|x_0-x^*|)|x_0-x^*|] \\
= & g_0(|x_0-x^*|) < g_0(r^*) < 1.
\end{aligned} \tag{2.25}$$

It follows from (2.25) that $F(x_0)+(\beta-2)F(y_0)$ is invertible and

$$|(F(x_0)+(\beta-2)F(y_0))^{-1}F'(x^*)| \leq \frac{1}{|x_0-x^*|(1-g_0(|x_0-x^*|))}. \tag{2.26}$$

Using (2.26) in particular for $\beta=2$, we get that

$$|F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{|x_0-x^*|(1-\frac{L_0|x_0-x^*|}{2})}. \tag{2.27}$$

Then, by (2.12), (2.11), (2.15) (for $n = 0$,) and (2.17) we obtain that

$$\begin{aligned}|\varphi(\frac{F(y_0)}{F(x_0)})| &= |\varphi(\frac{F'(x^*)^{-1}F(y_0)}{F'(x^*)^{-1}F(x_0)})| \\ &\leq |\varphi(|\frac{F'(x^*)^{-1}F(y_0)}{F'(x^*)^{-1}F(x_0)}|)| \\ &\leq |\varphi(\frac{M|y_0-x^*|}{|x_0-x^*|(1-\frac{L_0|x_0-x^*|}{2})})| \\ &\leq |\varphi(\frac{Mg_2(|x_0-x^*|)}{(1-\frac{L_0|x_0-x^*|}{2})})|. \end{aligned} \tag{2.28}$$

It follows that x_1 is well defined by the third sub-step of method (1.2) for $n = 0$. Then,using (2.5), (2.11), (2.15) (for $n=0$), (2.24), (2.26), (2.28) the third step of method (1.2) for $n = 0$, and the identity

$$\begin{aligned}x_1 - x^* &= y_0 - x^* - \varphi(\frac{F(y_0)}{F(x_0)}) \\ &\quad\times \frac{[F'(x^*)^{-1}(F(x_0)-F(x^*)) - \beta F'(x^*)^{-1}(F(y_0)-F(x^*))]F'(x^*)^{-1}F(y_0)}{F'(x^*)^{-1}(F(x_0)+(\beta-2)F(y_0))F'(x^*)^{-1}F[w_0,y_0]},\end{aligned}$$

we get that

$$\begin{aligned}|x_1-x^*| &\leq |y_0-x^*| + |(F'(x_0)+(\beta-2)F(y_0))^{-1}F'(x^*)| \\ &\quad\times|F[w_0,y_0]^{-1}F'(x^*)||\varphi(\frac{F(y_0)}{F(x_0)})| \\ &\quad\times[|F'(x^*)^{-1}(F(x_0)-F(x^*))| + |\beta-2||F'(x^*)^{-1}(F(y_0)-F(x^*))|)M|y_0-x^*|] \\ &\leq g_2(|x_0-x^*|)|x_0-x^*| + |\varphi\left(\frac{Mg_2(|x_0-x^*|)}{(1-\frac{L_0|x_0-x^*|}{2})}\right)| \\ &\quad\times\frac{(M|x_0-x^*|+|\beta|M|y_0-x^*|)M|y_0-x^*|}{|x_0-x^*|(1-g_0(|x_0-x^*|)(1-g(|x_0-x^*|))} \\ &\leq [1+|\varphi\left(\frac{Mg_2(|x_0-x^*|)}{(1-\frac{L_0|x_0-x^*|}{2})}\right)| \\ &\quad\times\frac{M^2(1+|\beta|g_2(|x_0-x^*|))}{(1-g_0(|x_0-x^*|)(1-g(|x_0-x^*|))}g_2(|x_0-x^*|)|x_0-x^*| \\ &= g_3(|x_0-x^*|)|x_0-x^*| < |x_0-x^*| < r^*,\end{aligned}$$

which shows (2.16) for $n = 0$ and $x_1 \in U(x^*, r^*)$. If we simply replace x_0, w_0, y_0, x_1 by x_k, w_k, y_k, x_{k+1} in the preceding estimates we obtain (2.14)– (2.16) which complete the induction for these estimates. Then, from the estimate $|x_{k+1}-x^*| < |x_k - x^*| < r^*$, we deduce that $x_{k+1} \in U(x^*, r^*)$ and $\lim_{k\to\infty} x_k = x^*$. Finally, to show the uniqueness part, let $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Set $A = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$. Then, we get by (2.7) that

$$\begin{aligned}|F'(x^*)^{-1}(A-F'(x^*))| &\leq \int_0^1 L_0|y^*+\theta(x^*-y^*)-x^*|d\theta \\ &\leq \int_0^1 (1-\theta)|x^*-y^*|d\theta \leq \frac{L_0}{2}T < 1.\end{aligned} \tag{2.29}$$

Then, it follows from (2.29) that A is invertible and from the identity $0 = F(x^*) - F(y^*) = A(x^* - y^*)$, we deduce that $x^* = y^*$.

□

Similar results can be obtained for method (1.3) as follows: Suppose that there exist divided difference of order two $F[x,y,z]$ on $D \times D \times D$ and divided difference of order three $F[x,y,z,w]$ on $D \times D \times D \times D$ and parameters $\alpha_1 > 0, \alpha_2 > 0$ such that for all $x,y,z,w \in D$

$$|F'(x^*)^{-1}F[x,y,z]| \leq \alpha_1 \tag{2.30}$$

and

$$|F'(x^*)^{-1}F[x,y,z,w]| \leq \alpha_2. \tag{2.31}$$

Define

$$\bar{r} = \frac{-(L_1+L_2+2\alpha_1) + \sqrt{(L_1+L_2+2\alpha_1)^2 + 16\alpha_2}}{8\alpha_2}.$$

Then, $\bar{r}$ is the only positive solution of quadratic equation

$$4\alpha_2 t^2 + (L_1+L_2+2\alpha_1)t - 1 = 0.$$

$$r^* < \bar{r} \tag{2.32}$$

and set

$$\gamma = \max\{|\gamma_0|, \frac{1}{|F'(x^*)|[1-(L_1+L_2+2\alpha_1+4\alpha_2 r^*)r^*]}\} \tag{2.33}$$

Moreover, suppose that sequence $\{x_n\}$ generated by method (1.3) for $x_0 \in U(x^*,r^*) - \{x^*\}$ is well defined and remains in $U(x^*,r^*)$. Then, using (2.9) and (2.30)–(2.32), we get that

$$\begin{aligned}
& |F'(x^*)^{-1}(N_3'(x_k) - F'(x^*))| \\
\leq\ & |F'(x^*)^{-1}(F[x_k,y_{k-1}] - F'(x^*))| \\
& + |F'(x^*)^{-1}F[x_k,y_{k-1},w_{k-1}]||x_k - y_{k-1}| \\
& + |F'(x^*)^{-1}F[x_k,y_{k-1},w_{k-1},x_{k-1}]||x_k - y_{k-1}||x_k - w_{k-1}| \\
\leq\ & L_1|x_k - x^*| + L_2|y_{k-1} - x^*| + \alpha_1(|x_k - x^*| + |y_{k-1} - x^*|) \\
& + \alpha_2(|x_k - x^*| + |y_{k-1} - x^*|)(|x_k - x^*| + |w_{k-1} - x^*|) \\
<\ & (L_1+L_2)r^* + 2\alpha_1 r^* + 4\alpha_2 (r^*)^2 \\
<\ & (L_1+L_2+2\alpha_1)\bar{r} + 4\alpha_2 \bar{r}^2 = 1.
\end{aligned} \tag{2.34}$$

It follows from (2.33) that $N_3(x_k)$ is invertible and

$$|\gamma_k| = |N_3(x_k)^{-1}| \leq \frac{1}{1-(L_1+L_2+2\alpha_1+4\alpha_2 r^*)r^*} \leq \gamma. \tag{2.35}$$

Hence, we arrive at the following local convergence result for method (1.3).

THEOREM 2.2. Let $F: D \subseteq S \to S$ be a differentiable function. Suppose that the hypotheses of Theorem 2.1 and (2.30)–(2.32) hold with γ defined by (2.33). Then, sequence $\{x_n\}$ generated for $x_0 \in U(x^*,r^*) - \{x^*\}$ by method (1.3) is well defined, remains in $U(x^*,r^*)$ for each $n = 0,1,2,\cdots$ and converges to x^*. Moreover, the following estimates hold

$$|w_n - x^*| \leq (1 + |\gamma|M_0)|x_n - x^*|,$$

$$|y_n - x^*| \leq g_2(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r^*,$$

and

$$|x_{n+1} - x^*| \leq g_3(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|.$$

Furthermore, for $R \in [r^*, \frac{2}{L_0})$ the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, R) \cap D$.

REMARK 2.3. 1. In view of (2.7) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\| \end{aligned}$$

condition (2.11) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t$$

or by $M(t) = M = 2$ since $t \in [0, \frac{1}{L_0})$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [5] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. The radius r_1 was shown by us to be the convergence radius of Newton's method [5, 7]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.36}$$

under the conditions (2.6)–(2.8). The convergence radius r^* of the method (1.2) cannot be larger than the convergence radius r_1 of the second order Newton's method (2.36). As already noted in [5, 7] r_1 is at least as large as the convergence ball given by Rheinboldt [18]

$$r_R = \frac{2}{3L}. \tag{2.37}$$

In particular, for $L_0 < L$ we have that

$$r_R < r$$

and

$$\frac{r_R}{r_1} \to \frac{1}{3} \; as \; \frac{L_0}{L} \to 0.$$

That is our convergence ball r is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [19].

4. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [14, 15, 17]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{|x_{n+1} - x^*|}{|x_n - x^*|}\right) / \ln\left(\frac{|x_n - x^*|}{|x_{n-1} - x^*|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}\right) / \ln\left(\frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

3 Numerical Examples

We present numerical examples in this section. All the three examples we have taken $\varphi(t) = 1$ and $\gamma = 1$.

EXAMPLE 3.1. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073$, $M = 2$, $M_0 = \frac{M}{F'(x^*)}$, $L_1 = L_2 = \frac{L_0}{2}$, $\beta = 2.5$. Then the parameters are

$$r_1 = 0.0045,\ r_2 = 0.0012 = \lambda,\ r = 0.0023,\ r_0 = 0.0188,\ r^* = 0.0013.$$

EXAMPLE 3.2. Let $D = [-1, 1]$. Define function f of D by

$$f(x) = e^x - 1. \tag{3.1}$$

Using (3.1) and $x^* = 0$, we get that $L_0 = e - 1 < L = e$, $M = 2$, $M_0 = \frac{M}{F'(x^*)}$, $L_1 = L_2 = \frac{L_0}{2}$, $\beta = 2.5$. Then the parameters are

$$r_1 = 0.3249,\ r_2 = 0.0936,\ r = 0.1758,\ r_0 = 0.0865 = \lambda,\ r^* = 0.0971.$$

EXAMPLE 3.3. Let $D = (-\infty, +\infty)$. Define function f of D by

$$f(x) = \sin(x). \tag{3.2}$$

Then we have for $x^* = 0$ that $L_0 = L = 1, M = 1$, $M_0 = \frac{M}{F'(x^*)}$, $L_1 = L_2 = \frac{L_0}{2}$, $\beta = 2.5$. Then the parameters are

$$r_1 = 0.6667,\ r_2 = 0.2755 = \lambda,\ r = 0.4186,\ r_0 = 0.2774,\ r^* = 0.2992.$$

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