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Almost multiplicative functions on commutative Banach algebras

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Abstract. Let A be a complex commutative Banach algebra with unit 1 and $\delta > 0$. A linear map $\phi : A \to \mathbb{C}$ is said to be *δ-almost multiplicative* if

$$|\phi(ab) - \phi(a)\phi(b)| \le \delta\|a\|\,\|b\| \quad \text{for all } a, b \in A.$$

Let $0 < \epsilon < 1$. The *ϵ-condition spectrum* of an element a in A is defined by

$$\sigma_\epsilon(a) := \{\lambda \in \mathbb{C} : \|\lambda - a\|\,\|(\lambda - a)^{-1}\| \ge 1/\epsilon\}$$

with the convention that $\|\lambda - a\|\,\|(\lambda - a)^{-1}\| = \infty$ when $\lambda - a$ is not invertible. We prove the following results connecting these two notions:

(1) If $\phi(1) = 1$ and ϕ is δ-almost multiplicative, then $\phi(a) \in \sigma_\delta(a)$ for all a in A.
(2) If ϕ is linear and $\phi(a) \in \sigma_\epsilon(a)$ for all a in A, then ϕ is δ-almost multiplicative for some δ.

The first result is analogous to the Gelfand theory and the last result is analogous to the classical Gleason–Kahane–Żelazko theorem.

1. Introduction. Let A be a complex commutative Banach algebra with unit 1. The classical Gelfand theory implies that the usual spectrum of an element a in A, denoted by $\sigma(a)$, consists of the values $\phi(a)$ where ϕ is a non-zero multiplicative linear functional (a character) on A. The set of all characters of A, denoted by $\mathrm{Car}(A)$, is called the *carrier space* of A. In this note, we study a possible similar relation between the condition spectrum $\sigma_\epsilon(a)$ and almost multiplicative linear functionals. Let $\mathrm{Inv}(A)$ and $\mathrm{Sing}(A)$ denote respectively the set of all invertible and singular elements of A.

Definition 1. Let $\delta > 0$. A linear map $\phi : A \to \mathbb{C}$ is said to be *δ-almost multiplicative* if

$$|\phi(ab) - \phi(a)\phi(b)| \le \delta\|a\|\,\|b\| \quad \text{for all } a, b \in A.$$

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The study of almost multiplicative linear functions originated with the study of deformation theory of Banach algebras. The multiplicative functions and almost multiplicative functions on certain algebras have interesting properties and applications. There is an almost multiplicative functional near to every multiplicative functional. The investigation of the converse part leads to the study of a class of Banach algebras, known as AMNM algebras (see [2, 3, 6, 5]).

Another notion featuring in the main theorem of this article is *condition spectrum*. It is a generalization of spectrum (similar to pseudospectrum), recently studied by the authors in [4]. Though it can be defined in a wider context, we define it here for Banach algebras.

Definition 2. Let A be a Banach algebra. For $0 < \epsilon < 1$, the ϵ-*condition spectrum* of an element a in A is defined by,

$$\sigma_\epsilon(a) := \{\lambda \in \mathbb{C} : \|\lambda - a\| \, \|(\lambda - a)^{-1}\| \geq 1/\epsilon\}$$

with the convention that $\|\lambda - a\| \, \|(\lambda - a)^{-1}\| = \infty$ when $\lambda - a$ is not invertible.

Since the condition spectrum is a special case of the spectrum defined by Ransford [7], it shares some of the properties of the usual spectrum, like non-emptiness, compactness etc. On the other hand, it has some properties that distinguish it from the usual spectrum, such as: having no isolated points and having a finite number of connected components.

The following two simple properties, mentioned without proof, are necessary to establish the results that follow. The proofs are given in [4].

(i) For every $a \in A$ and for every $\epsilon > 0$, $\sigma(a) \subseteq \sigma_\epsilon(a)$. The two sets coincide if and only if a is a scalar multiple of the identity. Hence, to avoid trivial situations, from now on, in all following results, by a we mean an element which is not a scalar multiple of the identity.

(ii) If $\lambda \in \sigma_\epsilon(a)$ then
$$|\lambda| \leq \frac{1+\epsilon}{1-\epsilon} \, \|a\|.$$

2. Main results. It is known that, for every multiplicative functional ϕ, the value of ϕ, at any element of A, belongs to the spectrum of that element. Similarly, the value of an almost multiplicative functional at an element belongs to the condition spectrum of that element.

Theorem 3. *Let A be a complex Banach algebra with unit* 1 *and let ϕ be a δ-almost multiplicative linear functional on A with $\phi(1) = 1$. Then $\phi(a) \in \sigma_\delta(a)$ for every element a in A.*

Proof. Let $a \in A$ and $\phi(a) = \lambda$. If $\lambda - a$ is not invertible, then $\lambda \in \sigma(a) \subseteq \sigma_\delta(a)$. Thus the conclusion follows from property (i).

Next assume that $\lambda - a$ is invertible. Then

$$1 = |\phi(1)| = |\phi(1) - \phi(\lambda - a)\phi((\lambda - a)^{-1})| \leq \delta\|\lambda - a\|\,\|(\lambda - a)^{-1}\|.$$

That is,

$$\|\lambda - a\|\,\|(\lambda - a)^{-1}\| \geq 1/\delta,$$

which implies $\lambda\ (= \phi(a)) \in \sigma_\delta(a)$. ■

The following lemma gives a sufficient condition for a linear function to be almost multiplicative from its behaviour on the unit sphere. The main idea of the proof can be found in [3].

LEMMA 4. *Let A be a commutative Banach algebra and $\phi : A \to \mathbb{C}$ be a linear map. If*

$$|\phi(a^2) - (\phi(a))^2| \leq \delta_1 \quad \textit{for all } a \in A \textit{ with } \|a\| = 1$$

then ϕ is $2\delta_1$-almost multiplicative.

Proof. Noting that the inequality holds trivially if $a = 0$ and replacing a non-zero a by $a/\|a\|$, we obtain

$$|\phi(a^2) - (\phi(a))^2| \leq \delta_1\|a\|^2 \quad \text{for all } a \in A.$$

Next note that for all $a, b \in A$, we have

$$4(\phi(ab) - \phi(a)\phi(b)) = \phi((a+b)^2) - (\phi(a+b))^2 - \phi((a-b)^2) + (\phi(a-b))^2.$$

Hence for all $a, b \in A$ with $\|a\| = 1 = \|b\|$, we get

$$\begin{aligned} |\phi(ab) - \phi(a)\phi(b)| &\leq \tfrac{1}{4}\delta_1(\|a+b\|^2 + \|a-b\|^2) \\ &\leq \tfrac{1}{4}\delta_1(4+4) = 2\delta_1. \end{aligned}$$

Hence for arbitrary $a, b \in A$, we get

$$|\phi(ab) - \phi(a)\phi(b)| \leq 2\delta_1\|a\|\,\|b\|. \ ■$$

The following theorem that connects condition spectrum and almost multiplicative linear functionals can be considered as an approximate version of the Gleason–Kahane–Żelazko Theorem. The proof is similar to the proof of Theorem 8.7 in [3].

THEOREM 5. *Let A be a complex commutative Banach algebra with unit 1, $0 < \epsilon < 1/3$ and $\phi : A \to \mathbb{C}$ be a linear function. If $\phi(a) \in \sigma_\epsilon(a)$ for every a in A, then ϕ is δ-almost multiplicative, where*

$$\delta = \frac{4}{\ln(1/\epsilon)}\left(1 + \frac{2}{(\ln(2)/3)^2}\right).$$

Proof. Note that, since $\sigma_\epsilon(1) = \{1\}$, we have $\phi(1) = 1$. Also, it follows from property (ii) that ϕ is continuous and

$$\|\phi\| \leq \frac{1+\epsilon}{1-\epsilon}.$$

Next, let $a \in A$ with $\|a\| = 1$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) := \phi(\exp(za)), \qquad \forall z \in \mathbb{C}. \tag{1}$$

Then f is an entire function. Also, since for all $z \in \mathbb{C}$,

$$|f(z)| \leq \|\phi\| \, \|\exp(za)\| \leq \frac{1+\epsilon}{1-\epsilon} \exp(|z| \, \|a\|) \leq \frac{1+\epsilon}{1-\epsilon} \exp(|z|),$$

the function f is entire and of exponential type of order less than or equal to one.

From equation (1), by linearity and continuity of ϕ, f also has the form

$$f(z) = \sum_{n=0}^{\infty} \frac{\phi(a^n) z^n}{n!}. \tag{2}$$

Let α_j, $j = 1, 2, \ldots$, denote the zeros of f arranged in such a way that

$$|\alpha_1| \leq |\alpha_2| \leq \cdots .$$

Claim.

$$\phi(a^2) - (\phi(a))^2 = -\sum_j \frac{1}{\alpha_j^2}. \tag{3}$$

The right hand side above becomes a finite sum if the number of zeros of f is finite, and reduces to 0 *if f has no zero.*

By Hadamard's theorem [1], the genus of f is either zero or one, as its order is one.

Case 1: *Genus of $f(z)$ is* 1. By the Hadamard factorization theorem [1], there exists a polynomial g of degree less than or equal to 1 such that

$$f(z) = \exp(g(z)) \prod_j \left(1 - \frac{z}{\alpha_j}\right) \exp\left(\frac{z}{\alpha_j}\right), \qquad \forall z \in \mathbb{C}. \tag{4}$$

Since $f(0) = \phi(1) = 1$, we have $g(0) = 0$. Hence we may assume $g(z) = \beta z$ for some $\beta \in \mathbb{C}$. Next, each term in the product can be written as

$$\left(1 - \frac{z}{\alpha_j}\right) \exp\left(\frac{z}{\alpha_j}\right) = \left(1 - \frac{z}{\alpha_j}\right)\left(1 + \frac{z}{\alpha_j} + \frac{1}{2}\frac{z^2}{\alpha_j^2} + \cdots\right) = 1 - \frac{1}{2}\frac{z^2}{\alpha_j^2} + \cdots .$$

Thus,

$$f(z) = \left(1 + \beta z + \frac{1}{2}\beta^2 z^2 + \cdots\right) \prod_j \left(1 - \frac{1}{2}\frac{z^2}{\alpha_j^2} + \cdots\right). \tag{5}$$

Comparing the coefficients of z and z^2 in the two expressions, (2) and (5), of $f(z)$, we get

$$\phi(a) = \beta, \qquad \frac{1}{2}\phi(a^2) = \frac{1}{2}\beta^2 - \frac{1}{2}\sum_j \frac{1}{\alpha_j^2}.$$

Thus,

$$\phi(a^2) - (\phi(a))^2 = -\sum_j \frac{1}{\alpha_j^2}. \tag{6}$$

CASE 2: *Genus of $f(z)$ is* 0. By the Hadamard factorization theorem [1], there exists a polynomial g of degree zero such that

$$f(z) = \exp(g(z)) \prod_j \left(1 - \frac{z}{\alpha_j}\right), \quad \forall z \in \mathbb{C}. \tag{7}$$

Since $f(0) = \phi(1) = 1$, we have $g \equiv 0$. Thus,

$$f(z) = \prod_j \left(1 - \frac{z}{\alpha_j}\right). \tag{8}$$

Comparing the coefficients of z and z^2 in (2) and (8), we get

$$\phi(a) = -\sum_j \frac{1}{\alpha_j}, \qquad \frac{1}{2}\,\phi(a^2) = \sum_{i<j} \frac{1}{\alpha_i \alpha_j}.$$

Thus,

$$\phi(a^2) - (\phi(a))^2 = -\sum_j \frac{1}{\alpha_j^2}. \tag{9}$$

Thus we end up with the same expression as (6) and that proves the claim.

To get the bound for the right hand side of (3), we estimate $|\alpha_j|$ in two ways. First, since $0 = f(\alpha_j) = \phi(\exp(\alpha_j a))$, we have $0 \in \sigma_\epsilon(\exp(\alpha_j a))$. Hence

$$\frac{1}{\epsilon} \leq \|\exp(\alpha_j a)\| \, \|\exp(-\alpha_j a)\| \leq \exp(2|\alpha_j| \, \|a\|).$$

Since $\|a\| = 1$, we obtain

$$|\alpha_j| \geq \frac{1}{2} \ln\left(\frac{1}{\epsilon}\right).$$

The second estimate is obtained from Jensen's formula [8]. For this, let $r > 0$, $n(r)$ denote the number of zeros of f in the closed disc with centre at the origin and radius r, and

$$M(r) := \sup\{|f(r\exp(i\theta))| : 0 \leq \theta < 2\pi\} \leq \frac{1+\epsilon}{1-\epsilon} \exp(r).$$

Then by Jensen's formula,

$$n(r)\ln(2) \leq \ln(M(2r)) \leq \ln\left(\frac{1+\epsilon}{1-\epsilon}\right) + 2r.$$

Putting $r = |\alpha_j|$ in the above inequality, we get

$$j\ln(2) \leq \ln\left(\frac{1+\epsilon}{1-\epsilon}\right) + 2|\alpha_j|$$

Suppose ϵ is such that $\ln\left(\frac{1+\epsilon}{1-\epsilon}\right) \le \frac{1}{2}\ln(1/\epsilon)$. (This is satisfied if $0<\epsilon<1/3$.) Then by using $|\alpha_j| \ge \frac{1}{2}\ln(1/\epsilon)$, we get $j\ln(2) \le 3|\alpha_j|$, that is, $|\alpha_j| \ge (\ln(2)/3)j$.

Let $\gamma := \frac{1}{2}\ln(1/\epsilon)$ and $\eta := \ln(2)/3$. Then $|\alpha_j| \ge \gamma$ as well as $|\alpha_j| \ge \eta j$ for all j. Consider $k = [\gamma]$, the integral part of γ. Then $k \le \gamma \le k+1$. Recall

$$\phi(a^2) - (\phi(a))^2 = -\sum_j \frac{1}{\alpha_j^2}.$$

To estimate $|\phi(a^2) - (\phi(a))^2|$, we split the right hand side sum into two parts and use the first inequality for $1 \le j \le k$ and the second inequality for $j > k$. Hence

$$\begin{aligned}
|\phi(a^2) - (\phi(a))^2| &\le \sum_{j=1}^{k} \left|\frac{1}{\alpha_j^2}\right| + \sum_{j=k+1}^{\infty} \left|\frac{1}{\alpha_j^2}\right| \\
&\le \frac{k}{\gamma^2} + \frac{1}{\eta^2}\sum_{j=k+1}^{\infty}\frac{1}{j^2} \\
&\le \frac{1}{\gamma} + \frac{1}{\eta^2}\left(\frac{1}{(k+1)^2} + \int_{k+1}^{\infty}\frac{1}{x^2}\,dx\right) \\
&\le \frac{1}{\gamma} + \frac{1}{\eta^2}\left(\frac{1}{(k+1)^2} + \frac{1}{k+1}\right) \\
&\le \frac{1}{\gamma} + \frac{1}{\eta^2}\left(\frac{1}{\gamma^2} + \frac{1}{\gamma}\right) \le \frac{1}{\gamma}\left(1 + \frac{2}{\eta^2}\right).
\end{aligned}$$

We have proved that

$$|\phi(a^2) - (\phi(a))^2| \le \delta_1 \quad \text{for all } a \in A \text{ with } \|a\| = 1,$$

where $\delta_1 := (1/\gamma)(1 + 2/\eta^2)$. Thus the conclusion follows from Lemma 4 with

$$\delta := 2\delta_1 = \frac{2}{\gamma}\left(1 + \frac{2}{\eta^2}\right) = \frac{4}{\ln(1/\epsilon)}\left(1 + \frac{2}{(\ln(2)/3)^2}\right). \ \blacksquare$$

Using Theorem 5, we can deduce the classical Gleason–Kahane–Żelazko Theorem for commutative Banach algebras.

COROLLARY 6 (GKZ Theorem). *Let A be a complex commutative unital Banach algebra and $\phi : A \to \mathbb{C}$ be a linear function. If $\phi(a) \in \sigma(a)$ for every a in A, then ϕ is multiplicative.*

Proof. Since $\sigma(a) \subseteq \sigma_\epsilon(a)$ for every $0<\epsilon<1$ (by property (i)),

$$\phi(a) \in \sigma_\epsilon(a), \quad \forall a \in A,\ 0<\epsilon<1/3.$$

Applying Theorem 5, we see that ϕ is δ-almost multiplicative. Note that $\delta \to 0^+$ as $\epsilon \to 0^+$. Hence ϕ is multiplicative. $\blacksquare$

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